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PRINCIPLE OF CONSERVATION OF MOMENTUM, ANGULAR MOMENTUM AND ENERGY FOR A DEFORMABLE CONTINUUM IN AN ELECTROMAGNETIC FIELD

1. Introduction. The question of exchange of energy, momentum and angular momentum for a material deformable continuum has not been satisfactorily solved yet. Various authors do not agree to define the mechanical quantities for an electromagnetic field [4] or the electromagnetic ones for a material continuum (see, e.g., [8]–[10]). This is due to the fact that in the relativistic theory of electromagnetic field there is no criterion uniquely defining the energy and momentum tensor of an electromagnetic field. The choice of a suitable tensor remains an open question. Various authors favour different forms of the so-called Maxwell stress tensor of an electromagnetic field [4], [5].

Among the attempts of explanation of difficulties due to the problem of energy and momentum of an electromagnetic field one should mention Fock's paper [2], in which the author completes the electromagnetic field equations with an additional postulate and shows that then the momentum-energy tensor of an electromagnetic field is uniquely defined up to two constants.

For the case of material continua (and all the more for the deformable ones) placed in an electromagnetic field the definition of a quantity such as the four-tensor of momentum-energy (three-dimensional stress tensor) requires great care, for discrimination between the electromagnetic and mechanical quantities is to some extent a matter of convention [4].

In the present paper, making use of the four-dimensional tensor notation, we deal with the problem of formulating the principle of momentum-energy conservation for a deformable continuum in an external electromagnetic

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field. We should separate the conditions which relate to all coupled mechanical and electromagnetic fields from those pertinent to the chosen model of interaction between the material continuum and the electromagnetic field (generalized constitutive relations).

A generalized form of the four-dimensional momentum-energy tensor of an electromagnetic field is postulated. It is composed of the four-tensor of deformations and stresses of an electromagnetic field defined previously [7]. Our formulae generalize the traditional form of the Lorentz force vector and the form of the momentum-energy tensor.

Assuming moreover linear equations of polarization of a material continuum (dielectrics, diamagnetics) one can obtain the formulae in their usual form.

The traditional stress tensor of a deformable continuum has to be generalized as well since the presence of an electromagnetic field causes the antisymmetric part of this tensor to be not zero also for the case of classical media, as is shown here.

In the present paper a mathematical formalism presented in [7] to describe coupled mechanical and electromagnetic fields by means of linear differential operators is used. The formulation proposed here has its origin in Drobot's paper [1], in which a method of describing purely mechanical interactions for statical problems has been given.

Space-time equilibrium equations obtained in [7] are interpreted here through the use of a projection procedure on a three-dimensional subspace [9], [6], and next the influence of electromagnetic effects on the equations of motion of the deformable continuum as well as on the first principle of thermodynamics for the case of coupled fields is analysed.

2. Principle of conservation of momentum and angular momentum for a deformable medium in an electromagnetic field. The principle of conservation of momentum for a deformable medium is usually (see [3]) written as

$$D\int_{\Omega} P_i\,dv = \int_{\Omega} F_i\,dv = \int_{\partial\Omega} T^{j}_{.i} n_j\,ds\,, \tag{1}$$

where P_i, F_i, $T^{j}_{.i}$ denote the tensors of volume density of momentum, of mechanical mass-forces and the deformable medium stress tensor, resp., and $D(\cdots)$ denotes the so-called material derivative with respect to time.

The condition (1) which holds for any region Ω of variability of the so-called Lagrange coordinates leads to the known equations of motion of a deformable medium in Lagrange's description:

$$\nabla_j T^{j}_{.i} + F_i = DP_i\,. \tag{2}$$

Considering a material deformable continuum in an external electromagnetic field one should write the principle of momentum conservation taking into account the additional forces of electromagnetic nature, or, without introducing any new forces, one has to treat this case as a system composed of a material continuum and an electromagnetic field and write the principle of conservation of momentum for such a system.

Using the stress tensor of an electromagnetic field ${}_{\mathrm{e}}T^{j(\mathrm{M})}_{\cdot i}$, which is to be interpreted as the Maxwell tensor, and the momentum of an electromagnetic field ${}_{\mathrm{e}}P_i$ we replace equation (2) by

$$\nabla_j(T^j_{\cdot i} + {}_{\mathrm{e}}T^{j(\mathrm{M})}_{\cdot i}) + F_i = D(P_i + {}_{\mathrm{e}}P_i)\,. \tag{3}$$

Therefore the principle of conservation of momentum should be fulfilled by the total momentum and the total stress tensor of the system. It is a separate question how to express ${}_{\mathrm{e}}T^{j(\mathrm{M})}_{\cdot i}$ and ${}_{\mathrm{e}}P_i$ in terms of quantities describing the electromagnetic field such as the four-tensor of deformations and stresses [7].

Let us now recall that the principle of conservation of angular momentum for a deformable medium without any electromagnetic field may be written in the form

$$D\int\limits_{\Omega} x_{[j}P_{k]}\,dv = \int\limits_{\Omega} x_{[j}F_{k]}\,dv + \int\limits_{\partial\Omega} x_{[j}T^i_{\cdot k]}n_i\,ds\,, \tag{4}$$

where [ ] denotes the antisymmetric part of a tensor.

Integrating by parts and transforming the surface integral into a volume-type one we obtain

$$\int\limits_{\Omega} x_{[j}DP_{k]}\,dv = \int\limits_{\Omega} (x_{[j}(F_{k]} + \nabla_i T^i_{\cdot k]}) + \delta_{i[j}T^i_{\cdot k]})\,dv\,.$$

Applying the motion equation (2) to the above formula we get a condition equivalent to equation (4) to express the balance of the angular momentum of a material continuum as follows:

$$T_{[j\,k]} = 0\,. \tag{5}$$

Note that our considerations concern classical continua and the condition (5) includes the fact that spins do not appear in the balance (4).

It is easy to guess that formula (4) for the system consisting of a deformable medium and an electromagnetic field should read

$$D\int\limits_{\Omega} x_{[j}(P + {}_{\mathrm{e}}P)_{k]}\,dv = \int\limits_{\Omega} x_{[j}F_{k]}\,dv + \int\limits_{\partial\Omega} x_{[j}(T + {}_{\mathrm{e}}T^{(\mathrm{M})})^i_{\cdot k]}n_i\,ds\,,$$

and together with the motion equation (3) it takes the equivalent form

$$(T + {}_{\mathrm{e}}T^{(\mathrm{M})})_{[j\,k]} = 0\,. \tag{6}$$

The last equation shows that the interaction of an electromagnetic field with a material medium may cause the antisymmetric part of the three-dimensional stress tensor of a deformable continuum to be non-zero.

3. Momentum-energy tensor of an electromagnetic field in a material medium. Now we shall extend the Maxwell stress tensor ${}_{\mathrm{e}}T_{ij}^{(\mathrm{M})}$ of an electromagnetic field to the four-tensor ${}_{\mathrm{e}}T_{\mu\nu}$ of momentum-energy; that is, we shall define additional components T_{00} and T_{0i} which, as will be seen, represent the density of electromagnetic energy and momentum, respectively. As mentioned in the introduction, this topic remains to be controversial in mechanics. Various authors do not agree to define mechanical quantities for an electromagnetic field, particularly for the material medium case.

As an example let us compare different formulae for the electromagnetic field momentum ${}_{\mathrm{e}}\mathbf{P}$ favoured by various authors [5]:

$$(7a)\qquad {}_{\mathrm{e}}\mathbf{P} = \mathbf{D} \times \mathbf{B},$$

$$(7b)\qquad {}_{\mathrm{e}}\mathbf{P} = \mathbf{E} \times \mathbf{H},$$

$$(7c)\qquad {}_{\mathrm{e}}\mathbf{P} = \mathbf{E} \times \mathbf{B}.$$

It should be remarked that the formulae do not differ in fact if we deal with an electromagnetic field in a vacuum. In this case the vectors of electric intensity $\mathbf{E}$ and magnetic intensity $\mathbf{B}$ and the suitable vectors of electric induction $\mathbf{D}$ and magnetic induction $\mathbf{H}$ are coupled by the following relations:

$$(8)\qquad \mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{H} = (1/\mu_0)\mathbf{B},$$

where ε_0, μ_0 are universal constants and represent the electric and magnetic permittivity of the vacuum, respectively [5].

In the case of electromagnetic field in a material medium the question which of the formulae (7a)–(7c) should be chosen is now crucial due to the necessity to add constitutive relations (polarization laws) for the considered material. For dielectrics and diamagnetics polarization laws are linear. However, in the case of ferroelectrics and ferromagnetics the constitutive relations take a more complicated functional form due to the electric or magnetic hysteresis and therefore formulae (7a)–(7c) differ fundamentally.

In our opinion, if one distinguishes the stress tensor of an electromagnetic field from the deformation tensor, then the problem of a unique formula for the electromagnetic momentum-energy tensor can be removed.

As has been seen in [7] the antisymmetric deformation tensor $\mathcal{G}_{\alpha\beta}$ of an electromagnetic field can be built up with the aid of the vectors of electric field intensity E_i and magnetic field intensity B_i as follows:

$$(9) \qquad \mathcal{G}_{\alpha\beta} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_2 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

(the vector B_i is usually called the magnetic induction vector).

In a similar way, with the use of the three-vectors of electric and magnetic induction D^i and H^i we construct an antisymmetric four-tensor $\mathcal{F}^{\alpha\beta}$ called the stress tensor of an electromagnetic field in the following matrix form:

$$(10) \qquad \mathcal{F}^{\alpha\beta} = \begin{bmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3 & H^2 \\ D^2 & H^3 & 0 & -H^1 \\ D^3 & -H^2 & H^1 & 0 \end{bmatrix}.$$

The four-tensor ${}_{\mathrm{e}}T_{\alpha}^{\cdot\beta}$ of momentum-energy of an electromagnetic field is defined by means of the above tensors in the following way:

$$(11) \qquad {}_{\mathrm{e}}T_{\alpha}^{\cdot\beta} = \tfrac{1}{2}(\mathcal{F}_{\alpha\nu}\mathcal{G}^{\nu\beta} - \widehat{\mathcal{G}}_{\alpha\nu}\widehat{\mathcal{F}}^{\alpha\beta}),$$

where the dual tensors $\widehat{\mathcal{F}}^{\nu\beta}$, $\widehat{\mathcal{G}}_{\nu\beta}$ are given by

$$\widehat{\mathcal{F}}^{\nu\beta} = \tfrac{1}{2}\varepsilon^{\nu\beta\gamma\delta}\widehat{\mathcal{F}}_{\gamma\delta}, \qquad \widehat{\mathcal{G}}_{\alpha\nu} = \tfrac{1}{2}\varepsilon_{\alpha\nu\mu\varrho}\mathcal{G}^{\mu\varrho},$$

and $\varepsilon^{\nu\beta\gamma\delta}$ is the usual four-dimensional permutation symbol.

Raising and lowering of tensorial indices is performed by means of the following metric tensor (pseudoeuclidean metric with index 3):

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

We postulate here that formula (11) is valid regardless of the constitutive relations combining the stresses $\mathcal{F}^{\alpha\beta}$ and the deformations $\mathcal{G}_{\alpha\beta}$.

Now we present arguments for adopting the above form of the electromagnetic field momentum-energy tensor.

In the case of electromagnetic field in a vacuum one may take ${}_{\mathrm{e}}T_{\alpha}^{\cdot\beta} = {}_{\mathrm{e}}\theta_{\alpha}^{\cdot\beta}$, where

$$(12) \qquad {}_{\mathrm{e}}\theta_{\alpha}^{\cdot\beta} = \tfrac{1}{2}(\mathcal{G}_{\alpha\nu}\mathcal{G}^{\nu\beta} - \widehat{\mathcal{G}}_{\alpha\nu}\widehat{\mathcal{G}}^{\alpha\beta}).$$

It is easy to verify that ${}_{\mathrm{e}}\theta^{\alpha\beta} = {}_{\mathrm{e}}\theta^{\beta\alpha}$. Note that for the material medium case the discrimination between the tensors $\mathcal{G}_{\alpha\beta}$, $\mathcal{F}^{\alpha\beta}$ leads in general to the condition

$${}_{\mathrm{e}}T_{[\alpha\,\beta]} \neq 0.$$

Using the formula

$$\varepsilon^{k_1 k_2 \dots r_1 \dots r_{n-m}} \varepsilon_{s_1 s_2 \dots r_1 \dots r_{n-m}} = (n-m)! \delta^{k_1 \dots k_2}_{s_1 \dots s_m} ,$$

where $m < n$, n denotes the dimension of the space ($n = 4$), we can write (11) in the form

$$(13) \qquad {}_{\mathrm{e}}T_{\alpha}{}^{\beta} = \mathcal{F}_{\alpha\nu}\mathcal{G}^{\nu\beta} + \tfrac{1}{4}\delta_{\alpha}{}^{\beta}\mathcal{F}_{\mu\nu}\mathcal{G}^{\mu\nu} .$$

In the case of electromagnetic field in a vacuum formula (12) leads to the known symmetric form of the momentum-energy tensor [5]:

$$(14) \qquad {}_{\mathrm{e}}\theta^{\alpha\beta} = \mathcal{G}^{\alpha}_{\cdot\nu}\mathcal{G}^{\nu\beta} + \tfrac{1}{4}g^{\alpha\beta}\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu} .$$

Now let us compute the four-divergence of the momentum-energy tensor (11). Using the Maxwell equations which may be written in the form [7]

$$(15a) \qquad \nabla_{\alpha}\mathcal{F}^{\alpha}{}_{\nu} = I^{\nu} ,$$

$$(15b) \qquad \nabla_{\alpha}\widehat{\mathcal{G}}^{\alpha\beta} = 0 ,$$

we get

$$(16) \qquad \nabla_{\alpha}\ {}_{\mathrm{e}}T^{\alpha\beta} = \mathcal{G}^{\nu\beta}I_{\nu} + \tfrac{1}{4}G^{\nu\lambda}\nabla^{\beta}\mathcal{F}_{\nu\lambda} - \tfrac{1}{4}\mathcal{F}_{\nu\lambda}\nabla^{\beta}\mathcal{G}^{\nu\lambda} .$$

Note that in the case of electromagnetic field in a vacuum and of the momentum-energy tensor ${}_{\mathrm{e}}\theta^{\alpha\beta}$ we get

$$(17) \qquad \nabla_{\alpha}\ {}_{\mathrm{e}}\theta^{\alpha\beta} = \mathcal{G}^{\nu\beta}I_{\nu} = -\mathcal{G}^{\beta\nu}I_{\nu} ,$$

since the last two components in (16) vanish due to (8).

The right side of (17) presents the known Lorentz force with which an electromagnetic field acts on currents and charges. Therefore we can treat the right side of (16) as a definition of the Lorentz force in the general case i.e., for an electromagnetic field not in the vacuum but in a material medium. Let us admit therefore the formula

$$(18) \qquad {}_{\mathrm{e}}F^{\beta} = \mathcal{G}^{\beta\nu}I_{\nu} + \tfrac{1}{4}\mathcal{F}_{\mu\lambda}\nabla^{\beta}\mathcal{G}^{\mu\lambda} - \tfrac{1}{4}\mathcal{G}^{\mu\lambda}\nabla^{\beta}\mathcal{F}_{\mu\lambda} .$$

Now (16) may be written with the use of the Lorentz force as

$$(19) \qquad \nabla_{\alpha}\ {}_{\mathrm{e}}T^{\alpha\beta} = -{}_{\mathrm{e}}F^{\beta} .$$

This is the principle of conservation of momentum and energy of an electromagnetic field; we shall show that by interpreting the components of the tensor ${}_{\mathrm{e}}T^{\alpha\beta}$. From (19) we get in particular for the covariant space components ${}_{\mathrm{e}}F_i$ of the Lorentz force

$$(20) \qquad \nabla_0\ {}_{\mathrm{e}}T^{0}_{\cdot i} + \nabla_j\ {}_{\mathrm{e}}T^{j}_{\cdot i} = -{}_{\mathrm{e}}F_i .$$

Using the relations

$$\mathcal{F}^{0k} = -D^k, \qquad \mathcal{G}_{0k} = E_k ,$$
$$\mathcal{F}^{ks} = -\varepsilon^{ksr}H_r, \quad \mathcal{G}_{ks} = -\varepsilon_{ksr}B^r ,$$

the components ${}_{\mathrm{e}}F_i$ may be represented in the inertial laboratory frame as

$$(21)\quad {}_{\mathrm{e}}F_i = -E\varrho - \varepsilon_{ikp}B^pI^k - \tfrac{1}{2}(E_k\nabla_iD^k - D^k\nabla_iE_k) - \tfrac{1}{2}(H_k\nabla_iB^k - B^k\nabla_iH_k)\,,$$

where I^k is the three-vector of electric current and ϱ is the charge density.

Note that the traditional formula for the Lorentz force does not include the terms with derivatives of the electromagnetic field vectors. For such continua as dielectrics and diamagnetics, the additional terms vanish.

Let us return to the interpretation of formula (20). The space components ${}_{\mathrm{e}}T^{j}_{\cdot i}$ of the electromagnetic field momentum-energy tensor ${}_{\mathrm{e}}T^{\mu}_{\cdot\nu}$ define the general form of the so-called Maxwell stress tensor which describes the stresses acting on surface elements of a material medium in an electromagnetic field. We take

$$(22a)\qquad {}_{\mathrm{e}}T^{i\ (\mathrm{M})}_{\cdot j} = -{}_{\mathrm{e}}T^{i}_{\cdot j}\,.$$

That implies

$$(22b)\qquad {}_{\mathrm{e}}T^{j(\mathrm{M})}_{\cdot i} = E_iD^j + H_iB^j - \tfrac{1}{2}\delta_i{}^j(E_kD^k + H_kB^k)\,,$$

which is easy to verify using (13).

Note that the Maxwell stress tensor does not satisfy in general the condition of symmetry. This may result in the antisymmetric part of the three-dimensional stress tensor of a deformable continuum being non-zero and at the same time

$$T_{[i\,j]} = -{}_{\mathrm{e}}T^{(\mathrm{M})}_{[i\,j]}\,,$$

as required by the principle of conservation of angular momentum expressed by (6).

In the case of constitutive relations of piezoelectrics the antisymmetric part of the stress tensor is non-zero, as will be shown in a separate paper. Here we only note that constitutive relations for stresses and deformations of a material medium should be formulated in such a way that the principle of conservation of angular momentum represented by (6) should be satisfied.

The components ${}_{\mathrm{e}}T^{0}_{\cdot i}$ of the momentum-energy tensor define the momentum P_i of the electromagnetic field. By (20) we have

$$(23)\qquad {}_{\mathrm{e}}P_i = {}_{\mathrm{e}}T^{0}_{\cdot i} = -{}_{\mathrm{e}}T^{0i} = -(\mathbf{D}\times\mathbf{B})^i = (\mathbf{D}\times\mathbf{B})_i\,,$$

which agrees with formula (7a).

Equation (20) may be interpreted as the principle of conservation of the electromagnetic field momentum because of the above interpretation of the tensorial quantities appearing there.

For the components T^{i0} of the momentum-energy tensor we have

$$(24)\qquad {}_{\mathrm{e}}T^{i0} = {}_{\mathrm{e}}T^{i}_{\cdot 0} = (\mathbf{E}\times\mathbf{H})^i\,,$$

which agrees with the definition of electromagnetic field momentum according to (7b). We interpret here the above components as the Poynting vector, representing the energy flux of the electromagnetic field.

Note that in the case of momentum-energy tensor in a vacuum we have

$$\theta^{0i} = \theta^{i0} = (\mathbf{E} \times \mathbf{H})^i \,, \tag{25}$$

which is consistent with formula (7c).

Now let us consider the explicit form of the component $T^{00} = T^{0}_{\cdot 0} = T_{0}^{\cdot 0}$ of the electromagnetic momentum-energy tensor. That component represents the energy of the electromagnetic field in a material medium, that is,

$$_{\mathrm{e}}T^{00} = -\tfrac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \,. \tag{26}$$

In the case of electromagnetic field in a vacuum we get

$$_{\mathrm{e}}\theta^{00} = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \,. \tag{27}$$

Admitting the above definition of mechanical quantities for an electromagnetic field we may interpret the first equation (19) for $\beta = 0$ as the principle of electromagnetic field energy conservation.

It is easy to verify that the component F^0 of the Lorentz force is of the form

$$F^0 = E_k J^k + \tfrac{1}{2}(E_k \nabla_0 D^k - D^k \nabla_0 E_k) + \tfrac{1}{2}(H_k \nabla_0 B^k - B^k \nabla_0 H_k) \,, \tag{28}$$

and represents the density of the electromagnetic field power. Additional components involving time derivatives of the vectors of the electromagnetic field do not vanish in the case of the constitutive relations of piezoelectricity.

4. Interpretation of the space-time equilibrium equation of a material deformable medium in an electromagnetic field. As has been shown in [7], the space-time equilibrium equations of a deformable medium have the form

$$\nabla_\mu F^{\mu}_{\cdot\nu} = F_\nu + {}_{\mathrm{e}}F_\nu \,, \tag{29}$$

where F_ν, $_{\mathrm{e}}F_\nu$, $F^{\mu}_{\cdot\nu}$ denote the four-vector of external mass force, the Lorentz force and the stress four-tensor of a deformable medium, respectively.

Equations (29) represent the principle of conservation of momentum ($\nu = k$) and energy ($\nu = 0$) for the system consisting of a deformable medium and an electromagnetic field.

With the aid of the electromagnetic momentum-energy tensor $_{\mathrm{e}}T^{\mu}_{\cdot\nu}$ (13) and using (19) system (29) may be written equivalently as

$$\nabla_\mu (F^{\mu}_{\cdot\nu} + {}_{\mathrm{e}}T^{\mu}_{\cdot\nu}) = F_\nu \,. \tag{30}$$

Notice that equations (29) and (30) are coupled with the Maxwell equations (15) by formulas (18) and (13), respectively.

Now we present the considered equilibrium equations in terms of three-tensors commonly used in continuum mechanics.

We adopt here the nonrelativistic point of view when describing interactions between an electromagnetic field and continuous matter. The Galilean transformations and their extension to the group G [6] of space-time transformations preserving absolute time are taken here as the admitted coordinate transformations. The nonrelativistic point of view and nonrelativistic transformations of electromagnetic field are here sufficiently good approximations of the relativistic formulae based on Lorentz transformations for the velocity v of the deformable medium is small compared with the speed c of light in vacuo $(v/c \ll 1)$.

As is known the Maxwell equations are invariant with respect to Lorentz transformations between any two inertial frames. In nonrelativistic formulation, however, the Galilean transformations do not preserve the form of the Maxwell equations and thus prefer some frame—the rest frame in the medium.

The transformation formulae for the electromagnetic field vectors and the vector of electric current $\mathbf{I}$ obtained as approximation from the relativistic formulae for $v/c \ll 1$ (where v denotes the velocity of a material particle in the laboratory frame) are of the form

$$\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}\,, \\ \mathbf{H}' &= \mathbf{H} - \mathbf{v} \times \mathbf{D}\,, \\ \mathbf{D}' &= \mathbf{D}\,, \\ \mathbf{B}' &= \mathbf{B}\,, \\ \mathbf{I}' &= \mathbf{I} - \varrho \mathbf{v}\,, \end{aligned} \tag{31}$$

where the primed quantities relate to the rest frame in a medium whereas the nonprimed ones to the laboratory frame.

The space and time coordinates are connected by means of the Galilean transformations

$$x_i' = x_i - v_i t, \qquad t' = t\,.$$

Hence for the suitable derivatives we obtain

$$\begin{aligned} &\nabla_i'(\ldots) = \nabla_i(\ldots)\,, \\ &\nabla_0'(\ldots) = \nabla_0(\ldots) + v^i \nabla_i(\ldots) = D(\ldots)\,. \end{aligned} \tag{32}$$

It follows from (9), (10) that the Maxwell equations (15a), (15b) in the local rest frame of a material medium described by the three-tensors take the form

$$\begin{aligned} \operatorname{rot} \mathbf{H}' - \partial \mathbf{D}'/\partial t' = \mathbf{I}', &\quad \operatorname{div} \mathbf{D}' = \varrho\,, \\ \operatorname{rot} \mathbf{E}' + \partial \mathbf{B}'/\partial t = 0, &\quad \operatorname{div} \mathbf{B}' = 0\,. \end{aligned} \tag{33}$$

Note that in the case of moving material medium the constitutive relations for an electromagnetic field preserve the usual form in the rest frame of a material particle.

Formulae (21)–(28) relate to the laboratory frame. We have to use the transformation rules (31), (32) to express the suitable quantities in the rest frame of a material medium. For example for the space components ${}_{\mathrm{e}}F_i$ of the Lorentz force (21) we obtain in the local rest frame of a material particle

$$
\begin{aligned}
(34\mathrm{a}) \qquad {}_{\mathrm{e}}F_i = & -E'_i\varrho - (\mathbf{I}' \times \mathbf{B}')_i + (\mathbf{B}' \times \mathbf{D}')_k \nabla_i v^k \\
& - \tfrac{1}{2}(E'_k \nabla'_i D'^k - D'^k D E'_k) - \tfrac{1}{2}(H'_k \nabla'_i B'^k - B'^k \nabla'_i H'_k) .
\end{aligned}
$$

Note that the additional term ${}_{\mathrm{e}}P'_k \nabla'_i v^k$ vanishes when the material medium moves as a rigid body.

For later use, observe that by (21), (28) and (31), the invariant ${}_{\mathrm{e}}F_\mu v^\mu$, where $v^\mu = (1, v^i)$, satisfies

$$
\begin{aligned}
(34\mathrm{b}) \qquad {}_{\mathrm{e}}F_\mu v^\mu = & E'_k I'_k + (\varrho \mathbf{v} \times \mathbf{B}')_k v^k + \tfrac{1}{2}(E'_k D D'^k - D'^k D E'_k) \\
& + \tfrac{1}{2}(H'_k D B'_k - B'^k D H'_k) + (\mathbf{B}' \times \mathbf{D}')_k D v^k .
\end{aligned}
$$

Note that the additional term ${}_{\mathrm{e}}P_k D v^k$ vanishes when the material point moves with a constant velocity, as required by Galilean transformations.

In [9] a projection procedure on a three-dimensional subspace has been presented. In this way it is possible to obtain three-quantities in the local rest frame of a material particle interpreted as three-tensors commonly used in continuum mechanics. Making use of the decomposition scheme of four-tensors we now write (29) in terms of three-tensors. We obtain the following form of the equation of continuum motion ($\nu = k$), which represents the principle of conservation of momentum for a deformable medium in an electromagnetic field:

$$
(35) \qquad \nabla_j T^j_{.i} + F_i + {}_{\mathrm{e}}F_i = D P_i ,
$$

where $T^j_{.i}$ denotes the three-dimensional stress-tensor of a material medium, P_i denotes the momentum vector of a body, and the Lorentz force ${}_{\mathrm{e}}F_i$ in the rest frame of a material particle is given by (34a).

We obtain the principle of conservation of energy for the considered system through the use of the projection procedure and putting $\nu = 0$ in (29). We get

$$
(36) \qquad P_i D v^i - T^j_{.i} \nabla_j v^j + \nabla_j q^j + D\varepsilon - F + {}_{\mathrm{e}}F_\mu v^\mu = 0 ,
$$

where ε denotes the density of internal energy, F the density of thermal power, q^j the heat flux vector. The invariant ${}_{\mathrm{e}}F^\mu v_\mu$ is defined in the local rest frame by formula (34b).

Equations (35), (36) coupled with the Maxwell equations (33) are valid for any deformable medium in an electromagnetic field. A separate question is the problem of formulating the constitutive relations for stresses and deformations of the deformable medium and of the electromagnetic field and thus of adopting some interaction models between electromagnetic field and continuous matter. For the piezoelectric case these topics will be discussed in a separate paper.

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