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P. GRZEGORZEWSKI (Warszawa)

THE INFINITESIMAL ROBUSTNESS OF TESTS AGAINST DEPENDENCE

Abstract. It is shown that tests are infinitesimally robust against dependence. A new tool called Rüschendorf's ε-neighbourhoods for investigations of dependence is proposed.

1. Introduction. Situations with some kind of dependencies for a certain test were investigated by Hollander, Pledger, Lin [1], Serfling [5], Pettit and Syskind [2], Zieliński [6], [7]. Concluding his papers Zieliński states that it would be interesting to study the performance of a test under small dependencies described nonparametrically. In this paper we propose such a description of dependence called Rüschendorf's ε-neighbourhoods (Section 3). Section 6 contains motivations for such a description. We take advantage of our new tool to investigate the infinitesimal robustness of tests against dependence (Section 4).

2. Problem and notation. Consider a random sample $X_1, \ldots, X_n$ from a distribution with distribution function $F \in \mathcal{F}$. Let $\mathcal{X}$ denote a sample space and F^* the corresponding product measure. Denote by $\phi : \mathcal{X} \to [0,1]$ a test of size α, i.e. $\int_{\mathcal{X}} \phi \, dF^* \le \alpha$ $(\forall F \in \mathcal{F}_H)$ where $\mathcal{F}_H \subset \mathcal{F}$ is the family of distribution functions for which the tested hypothesis H holds.

Moreover, let $\mathcal{P}(F) = \{P : P(X_i \le t) = F(t),\ i = 1, \ldots, n\}$ describe all possible violations of independence. We denote $\mathcal{P}(F)$ briefly by $\mathcal{P}$. Let $\mathcal{P}_{\mathrm{C}} \subset \mathcal{P}$ be the subfamily of all continuous distribution functions.

We shall consider robustness of a test ϕ against dependence, specifically the influence of dependence on the size of ϕ. So we shall investigate the functional $\int_{\mathcal{X}} \phi dP$ for those $P \in \mathcal{P}_{\mathrm{C}}$ which correspond to infinitesimally small departures from independence (i.e. to departures which tend to zero).

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Without loss of generality we assume that F is the uniform distribution on the interval $[0,1]$.

3. Rüschendorf's ε-neighbourhoods. Let $\mathcal{M}_n$ be the set of all measures on $[0,1]^n$ with uniform marginals and continuous w.r.t. the Lebesgue measure $d\mu$ on $[0,1]^n$.

THEOREM 1 (Rüschendorf [3]). *h is the density of a measure $P \in \mathcal{M}_n$ if and only if $h = 1 + Sf$ where $f \in L^1([0,1]^n)$ and $S : L^1 \to L^1$ is the linear operator given by*

$$Sf = f - \sum_{i=1}^{n} \int\limits_{[0,1]^{n-1}} f\,dx_1 \ldots \widehat{dx_i} \ldots dx_n + (n-1) \int\limits_{[0,1]^n} f\,dx_1 \ldots dx_n\,,$$

where the hat denotes omission. If, moreover, $Sf \geq -1$ then P is a probability measure.

By this theorem, the function f which is identically zero gives us the product measure, and so do all $f \in L^1([0,1]^n)$ such that $Sf = 0$ (for more details see Section 5.2).

One may ask whether functions close to zero (in L^1 norm) lead to probability measures which describe small violations of independence. The answer to this question is positive. This enables us to construct a nonparametric family of distributions corresponding to the joint distribution functions of a sample with small departures from independence.

Precisely, let $\mathcal{L}_n = \{f \in L^1([0,1]^n) : Sf \geq -1\}$ and $[0,u]^n = \{x \in [0,1]^n : 0 \leq x_i \leq u_i, i = 1, \ldots, n\}$.

DEFINITION 1. A *Rüschendorf ε-neighbourhood* is $R_\varepsilon = \{f \in \mathcal{L}_n : \|Sf\| \leq \varepsilon\}$, where $\|\cdot\|$ is the L^1 norm.

DEFINITION 2. Set $\mathcal{C}_\varepsilon = \{C : C(u) = \int_{[0,u]^n} (1 + Sf)\,d\mu,\ f \in R_\varepsilon\}$.

COROLLARY 1. $\bigcup_{\varepsilon>0} \mathcal{C}_\varepsilon = \mathcal{P}_{\mathrm{C}}$.

The following lemmas will be useful:

LEMMA 1.

$$(\forall f \in \mathcal{L}_n) \qquad \int\limits_{[0,1]^n} Sf\,d\mu = 0.$$

Proof. By Theorem 1, $h = 1 + Sf$, $\int_{[0,1]^n} h\,d\mu = 1$, thus $\int_{[0,1]^n} Sf\,d\mu = 0$. ■

LEMMA 2. *If $f \in R_\varepsilon$, then $(\forall u \in [0,1]^n) \int_{[0,u]^n} Sf d\mu \leq \varepsilon/2$.*

The proof follows immediately from Lemma 1.

4. The infinitesimal robustness of tests. The notions introduced in the previous section enable us to investigate robustness of tests against infinitesimally small departures from independence:

THEOREM 2. *Statistical tests are infinitesimally robust against dependence (in the sense of stability of size) provided that we restrict the class of joint distribution functions to those continuous w.r.t. Lebesgue measure.*

Proof. Take a test ϕ and a sample with a joint distribution function $C \in \mathcal{P}_{\mathrm{C}}$ (we recall that C^* denotes the corresponding product measure). By the lemmas above we get

$$\begin{aligned}\Big|\int_{[0,1]^n} \phi\, dC\Big| &= \Big|\int_{[0,1]^n} \phi\, dC^* + \int_{[0,1]^n} \phi\, Sf\, d\mu\Big| \\ &\le \Big|\int_{[0,1]^n} \phi\, dC^*\Big| + \Big|\int_{[0,1]^n} \phi\, Sf\, d\mu\Big| \\ &\le \Big|\int_{[0,1]^n} \phi\, dC^*\Big| + \int_{[0,1]^n} |Sf|\, d\mu \le \Big|\int_{[0,1]^n} \phi\, dC^*\Big| + \varepsilon\,.\end{aligned}$$

Hence

$$\Big|\int_{[0,1]^n} \phi\, dC\Big| \to \Big|\int_{[0,1]^n} \phi\, dC^*\Big| \quad \text{as } \varepsilon \to 0\,.$$

So we get the stability of size, which completes the proof. ∎

5. Additional remarks

5.1. Let $\int_{\mathcal{X}} \Phi\, dP$ be a functional which characterizes a certain statistic (e.g. power of test, bias etc.).

COROLLARY 2. *The statistic given above is infinitesimally robust against dependencies in a sample provided that we restrict the class of joint distribution functions to those continuous w.r.t. Lebesgue measure and Φ is bounded.*

The proof follows immediately from Theorem 2.

5.2. It was shown in Section 3 that all functions $f \in L^1([0,1]^n)$ such that $Sf = 0$ lead to the product measure. The following theorem gives us the form of such functions.

THEOREM 3. *Let $f \in L^1([0,1]^n)$. Then $Sf = 0$ if and only if $f = \sum_{i=1}^n \psi_i(x_i) + C$, where $\psi_i \in L^1([0,1])$, $i = 1, \ldots, n$, and C is a constant.*

Proof. Suppose $Sf = 0$. Then

$$\begin{aligned} f(x_1,\ldots,x_n) &= \int\limits_{[0,1]^{n-1}} f\,dx_2\ldots dx_n + \int\limits_{[0,1]^{n-1}} f\,dx_1 dx_3\ldots dx_n \ldots \\ &\quad + \int\limits_{[0,1]^{n-1}} f\,dx_1\ldots dx_{n-1} - (n-1)\int\limits_{[0,1]^n} f\,dx_1\ldots dx_n \\ &= \psi_1(x_1) + \ldots + \psi_n(x_n) + C\,. \end{aligned}$$

Conversely, if $f = \sum_{i=1}^n \psi_i(x_i) + C$ then

$$\begin{aligned} Sf &= \sum_{i=1}^n \psi_i(x_i) + C - \Big[\psi_1(x_1) + \int\limits_{[0,1]^{n-1}} \Big(\sum_{i=2}^n \psi_i(x_i) + C\Big)\,dx_2\ldots dx_n + \ldots \\ &\quad + \psi_n(x_n) + \int\limits_{[0,1]^{n-1}} \Big(\sum_{i=1}^{n-1} \psi_i(x_i) + C\Big)\,dx_1\ldots dx_{n-1}\Big] \\ &\quad + (n-1)\int\limits_{[0,1]^n} \Big(\sum_{i=1}^n \psi_i(x_i) + C\Big)\,dx_1\ldots dx_n \\ &= \sum_{i=1}^n \psi_i(x_i) + C - \sum_{i=1}^n \psi_i(x_i) - \Big[(n-1)\int_0^1 \psi_1(x_1)\,dx_1 + \ldots \\ &\quad + (n-1)\int_0^1 \psi_n(x_n)\,dx_n\Big] - nC + (n-1)C \\ &\quad + (n-1)\Big[\int_0^1 \psi_1(x_1)\,dx_1 + \ldots + \int_0^1 \psi_n(x_n)dx_n\Big] = 0\,, \end{aligned}$$

which proves the theorem. ■

6. Rüschendorf's ε-neighbourhoods and dependence. Many measures of dependence have been proposed and studied in the literature. The most familiar are (see [4]):

$$\varrho(C) = 12\int_0^1\int_0^1 |C(u,v) - uv|\,du\,dv\,,$$

$$\gamma(C) = \Big(90\int_0^1\int_0^1 [C(u,v) - uv]^2\,du\,dv\Big)^{1/2},$$

$$\kappa(C) = 4\sup_{u,v\in[0,1]} |C(u,v) - uv|\,,$$

$$\tau(C) = 4 \int_0^1 \int_0^1 C(u,v)\, dC(u,v) - 1 \quad \text{(Kendall's } \tau),$$

where C is a joint distribution function.

In this section we are interested in connections between these measures of dependence and Rüschendorf's ε-neighbourhoods.

THEOREM 4. *If $C \in \mathcal{C}_\varepsilon$, then $\varrho(C) = O(\varepsilon)$. Conversely, if $\varrho(C) = 0$, then $C \in \mathcal{C}_{\varepsilon=0}$.*

Proof. Suppose that $C \in \mathcal{C}_\varepsilon$. Then by Lemma 2 we get

$$\begin{aligned}\varrho(C) &= 12 \int_0^1 \int_0^1 |C(u,v) - uv|\, du\, dv \\ &= 12 \int_0^1 \int_0^1 \Big| \int_0^u \int_0^v Sf(x,y)\, dx\, dy \Big|\, du\, dv \le 12 \frac{\varepsilon}{2} = 6\varepsilon,\end{aligned}$$

which establishes the first part of the theorem.

Now suppose that $\varrho(C) = 0$. Then $\int_0^1 \int_0^1 | \int_0^u \int_0^v Sf(x,y)\, dx\, dy| du\, dv = 0$. Thus $| \int_0^u \int_0^v Sf(x,y)\, dx\, dy| = 0$ a.e. $(u,v) \in [0,1]^2$. Therefore if $g(x,v) = \int_0^v Sf(x,y)\, dy$ we have, for any fixed v, $\int_0^u g(x,v)\, dx = 0$ a.e. u.

Let g^+ and g^- be the positive and negative parts of g. Hence $\int_0^u g^+(x,v) dx = \int_0^u g^-(x,v)\, dx$ a.e. u. It follows that $\int_A g^+(x,v)\, dx = \int_A g^-(x,v)\, dx$ for all Borel sets $A \subset [0,1]$, and hence $g(x,v) = 0$ a.e. x for a fixed v. So $g(x,v) = 0$ except on a set N_v of x values which has Lebesgue measure zero and which may depend on v. Since $g(x,v)$ is a continuous function of v for any fixed x, we get $g(x,v) = 0$ a.e. x for all v.

Thus finally $Sf = 0$ a.e. (u,v), which completes the proof. ∎

The same reasoning applies to the case of remaining measures of dependence and to other measures of dependence which satisfy the modified Rényi axioms (see [4]).

COROLLARY 3. *The family of distributions C_ε built on Rüschendorf's ε-neighbourhoods corresponds to the distributions of samples with departures from independence.*

This motivates the application of Rüschendorf's ε-neighbourhoods in the investigations of robustness against dependence.

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PRZEMYSŁAW GRZEGORZEWSKI
INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
PL. POLITECHNIKI 1
00-661 WARSZAWA, POLAND

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