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LATENT ROOTS AND OFF-DIAGONAL ELEMENTS OF THE ASSOCIATION MATRIX IN PBIB DESIGNS AND THE NUMBER OF ASSOCIATE CLASSES

Abstract. This paper contains a proof that any PBIB design with association matrix having two different off-diagonal elements and two different latent roots (not including the latent root corresponding to the latent vector composed of ones) has two associate classes. This generalizes an analogous result established by Kageyama ([5], p. 548).

1. Introduction. We recall several basic properties of PBIB designs, which can be found e.g. in [1].

Let v, b, r, k, n_i, λ_i and p^i_{jk} $(i,j,k = 1,\ldots,m)$ be the parameters of a PBIB design with m associate classes which is referred to as PBIB(m) for short. If $\mathbf{N}$ is the incidence matrix of a PBIB(m) design, then association matrix $\mathbf{NN}'$ takes the form

$$\mathbf{NN}' = \sum_{i=0}^{m} \lambda_i \mathbf{A}_i \tag{1.1}$$

where $\lambda_0 = r$, $\mathbf{A}_i$ are association matrices, and $\mathbf{A}_0$ is the unit matrix $\mathbf{I}$. The values λ_i, $i = 1,\ldots,m$, are off-diagonal elements of the matrix $\mathbf{NN}'$, since association matrices are binary matrices satisfying the condition $\sum_{i=0}^{m} \mathbf{A}_i = \mathbf{11}'$, where $\mathbf{1}$ is the column vector of ones.

Let z_{ij} denote the ith latent root of the matrix $\mathbf{P}_j = [p^{s'}_{sj}]$, $i,s,s',j = 0,1,\ldots,m$, where $p^0_{jk} = n_j\delta_{jk}$, $n_0 = 1$, $p^i_{0k} = p^i_{k0} = \delta_{ik}$ and δ_{jk} denotes the Kronecker delta. It is known that

$$\sum_{j=1}^{m} z_{ij} = -1, \quad i = 1,\ldots,m. \tag{1.2}$$

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The matrix (1.1) can be equivalently written in the form

$$\mathbf{NN}' = \sum_{i=0}^{m} \varrho_i \mathbf{X}_i \tag{1.3}$$

where $\mathbf{X}_i$ and ϱ_i are orthogonal idempotent matrices and the latent roots of the matrix $\mathbf{NN}'$, respectively, and

$$\varrho_i = \sum_{j=0}^{m} \lambda_j z_{ij} \,. \tag{1.4}$$

The equality $\mathbf{P}_j \mathbf{1} = n_j \mathbf{1}$, together with (1.4), gives $\varrho_0 = rk$ $(= \sum_{j=0}^{m} \lambda_j n_j)$.

The matrices $\mathbf{A}_i$ and $\mathbf{P}_i$ have the same set of latent roots and therefore

$$\mathbf{A}_i = \sum_{j=0}^{m} z_{ji} \mathbf{X}_j \,. \tag{1.5}$$

Now, the matrix $\mathbf{Z} = [z_{ij}]'$, $i, j = 0, 1, \ldots, m$, is nonsingular and so

$$\mathbf{X}_i = \sum_{j=0}^{m} z^{ji} \mathbf{A}_j \tag{1.6}$$

where z^{ji} is the (i,j)th element of the matrix $\mathbf{Z}^{-1}$.

2. Results. It can happen that the incidence matrix $\mathbf{N}$, besides conditions (1.1) and (1.3), also satisfies

$$\mathbf{NN}' = \sum_{i=0}^{l} \lambda_i^* \mathbf{A}_i^* \tag{2.1}$$

and

$$\mathbf{NN}' = \sum_{i=0}^{l} \varrho_i^* \mathbf{X}_i^* \tag{2.2}$$

where $l < m$. If the matrices $\mathbf{A}_i^*$ and $\mathbf{X}_i^*$ $(i = 0, 1, \ldots, l)$ satisfy conditions similar to those for $\mathbf{A}_i$ and $\mathbf{X}_i$ $(i = 1, \ldots, m)$ from the previous section, then $\mathbf{N}$ is the incidence matrix of both a PBIB design with m associate classes and of one with l associate classes. We then say that *reduction of associate classes* from m to l has occurred. In this section we present necessary and sufficient conditions for reduction of associate classes.

We assume that each of the matrices $\mathbf{A}_i^*$ is either one of the matrices $\mathbf{A}_i$ or the sum of a certain number of the matrices $\mathbf{A}_i$. Thus we can write $\mathbf{A}_0^* = \mathbf{A}_0 = \mathbf{I}$ and

$$\mathbf{A}_i^* = \sum_{j=1}^{s_i} \mathbf{A}_{w_{ij}} \tag{2.3}$$

where $\{w_{ij} : 1 \le i \le l,\ 1 \le j \le s_i\} = \{1, \dots, m\}$. We assume that the w_{ij} are all distinct, i.e. in (2.3) each $\mathbf{A}_j$, $j = 1, \dots, m$, occurs exactly once.

Reduction of associate classes is dealt with in Theorem 3.1 of [2], p. 473:

THEOREM 2.1. *Let* $\mathbf{N}$ *be the incidence matrix of a PBIB*(m) *design with matrices* $\mathbf{NN}'$ *and* $\mathbf{A}_j$ *having latent roots* ϱ_i *and* z_{ij}, $i, j = 0, 1, \dots, m$, *respectively. Reduction of associate classes from* m *to* l $(< m)$ *occurs if and only if*

(i) $$R_{i1} = \ldots = R_{it_i}\ (= \varrho_i^*), \qquad i = 1, \dots, l,$$

(ii) $$Z_{i1j} = \ldots = Z_{it_ij}\ (= z_{ij}^*), \qquad i, j = 1, \dots, l,$$

where $\{w'_{ij} : 1 \le i \le l,\ 1 \le j \le t_i\} = \{1, \dots, m\}$ *and*

$$R_{ij} = \varrho_{w'_{ij}}, \qquad Z_{ii'j} = \sum_{k=1}^{s_i} z_{w'_{ii'} w_{jk}}\,.$$

The above theorem yields

COROLLARY 2.1. *The association matrix of a PBIB design has at most as many different latent roots (not including the single latent root equal to* rk*) as there are associate classes.*

In Corollary 2.1, the words "at most" cannot be omitted because of condition (ii) of Theorem 2.1.

Theorem 2.2 of [4] also concerns reduction of associate classes. Assuming that the classes with numbers w_{ij} $(i = 1, \dots, l;\ j = 1, \dots, s_i)$ are grouped into a new ith association, one can formulate it as

THEOREM 2.2. *Reduction of associate classes from* m *to* l *occurs if and only if*

(i) $$\Lambda_{i1} = \ldots = \Lambda_{is_i}\ (= \lambda_i^*), \qquad i = 1, \dots, l,$$

(ii) $$P_{i1jk} = \ldots = P_{is_ijk}\ (= p_{ij}^{*k}), \qquad i, j, k = 1, \dots, l,$$

where

$$\Lambda_{ij} = \lambda_{w_{ij}}, \qquad P_{ii'jk} = \sum_{u=1}^{s_i} \sum_{u'=1}^{s_k} p^{w_{ii'}}_{w_{ju} w_{ku'}}\,.$$

Theorem 2.2 implies

COROLLARY 2.2. *The association matrix of a PBIB design has at most as many off-diagonal elements as there are associate classes.*

As in Corollary 2.1, one cannot omit the words "at most" since condition (ii) is essential. However, the following question arises: do conditions (i) in Theorems 2.1 and 2.2 imply that the corresponding PBIB design has l associate classes? An affirmative answer for $l = 2$ is contained in

THEOREM 2.3. *Let* $\mathbf{N}$ *be the incidence matrix of a PBIB*(m) *design with off-diagonal elements and latent roots (not including the latent root corresponding to the latent vector composed of ones) of the matrix* $\mathbf{NN}'$ *equal to* λ_i *and* ϱ_i, $i = 1, \ldots, m$, *respectively. If*

$$\text{(i)} \qquad \Lambda_{i1} = \ldots = \Lambda_{is_i} = \lambda_i^*, \quad i = 1, 2, \ \lambda_1^* \neq \lambda_2^*,$$

$$\text{(ii)} \qquad R_{i1} = \ldots = R_{it_i} = \varrho_i^*, \quad i = 1, 2,$$

then this is a PBIB(2) *design.*

Proof. From (1.4) and (i) we have

$$\begin{aligned} \varrho_1 &= r + z'_{11}\lambda_1^* + z'_{12}\lambda_2^*, \\ \varrho_2 &= r + z'_{21}\lambda_1^* + z'_{22}\lambda_2^*, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \varrho_m &= r + z'_{m1}\lambda_1^* + z'_{m2}\lambda_2^*, \end{aligned}$$

where $z'_{ij} = \sum_{k=1}^{s_j} z_{iw_{jk}}$ for $i, j = 1, 2$. Hence and from (ii) it follows that

$$\varrho_e - \varrho_f = (z'_{e1} - z'_{f1})\lambda_1^* + (z'_{e2} - z'_{f2})\lambda_2^* = 0, \tag{2.4}$$

for $e, f \in \{w'_{1j} : 1 \leq j \leq t_1\}$ and for $e, f \in \{w'_{2j} : 1 \leq j \leq t_2\}$. Applying formula (1.2) to (2.4) we obtain $(z'_{e1} - z'_{f1})(\lambda_1^* - \lambda_2^*) = 0$. Since by assumption $\lambda_1^* \neq \lambda_2^*$, we have

$$z'_{e1} = z'_{f1}, \tag{2.5}$$

and by (1.2)

$$z'_{e2} = z'_{f2}, \tag{2.6}$$

for e, f as above. The equalities (2.5) and (2.6) together are equivalent to condition (ii) of Theorem 2.1, which completes the proof.

An analogous theorem for $s_1 = t_1$ and $s_2 = t_2$ is given in [5], p. 548. It remains an open problem whether there is an analogue of Theorem 2.3 for $l > 2$.

3. Examples. To illustrate Theorem 2.3 let us use a block design with incidence matrix

$$\mathbf{N} = \mathbf{N}_1 \otimes \mathbf{N}_2, \tag{3.1}$$

where $\otimes$ is the Kronecker product, and $\mathbf{N}_i$, $i = 1, 2$, are the incidence matrices of BIB designs with parameters v_i, b_i, r_i, k_i and $\lambda_{(i)}$. We do not exclude BIB designs with $k_i = 1$ and $\lambda_{(i)} = 0$ and ones with $r_i = \lambda_{(i)}$. We exclude two situations: $k_1 = k_2 = 1$, $\lambda_{(1)} = \lambda_{(2)} = 0$ and $r_2 = \lambda_{(2)}$, $r_1 = \lambda_{(1)}$. It is known (see e.g. [3]) that (3.1) is the incidence matrix of a PBIB(3) design, where $v = v_1v_2$, $b = b_1b_2$, $r = r_1r_2$, $k = k_1k_2$, $\lambda_1 = r_1\lambda_{(2)}$, $\lambda_2 = r_2\lambda_{(1)}$, $\lambda_3 = \lambda_{(1)}\lambda_{(2)}$, $\varrho_0 = r_1k_1r_2k_2$, $\varrho_1 = r_1k_1(r_2 - \lambda_{(2)})$, $\varrho_2 = (r_1 - \lambda_{(1)})r_2k_2$

and $\varrho_3 = (r_1 - \lambda_{(1)})(r_2 - \lambda_{(2)})$. The question arises of when (3.1) is also the incidence matrix of a PBIB(2) design. According to Theorem 2.3 this can only happen in the following cases:

(i) $\lambda_1 = \lambda_2$ and $\varrho_1 = \varrho_2$, which occurs when $v_1 = v_2$ and $k_1 = k_2$. Such a situation is discussed by Kageyama ([3]); we then obtain a design with association scheme L_2.

(ii) $\lambda_1 = \lambda_3$ and $\varrho_2 = \varrho_3$, which occurs when $r_1 = \lambda_{(1)}$, $k_2 = 1$ and $\lambda_{(2)} = 0$. An example of the incidence matrix of such a PBIB(2) design is $\mathbf{N} = \mathbf{11}' \otimes \mathbf{I}$. This is a so-called design of divisible groups with $n = v_1$ and $m = v_2$.

(iii) A design of divisible groups with $n = v_2$ and $m = v_1$ is obtained when $\lambda_2 = \lambda_3$ and $\varrho_1 = \varrho_3$. This occurs when $k_1 = 1$, $\lambda_{(1)} = 0$, and when $r_1 = \lambda_{(1)}$.

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