ZASTOSOWANIA MATEMATYKI
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W. ZIELIŃSKI (Warszawa)

A ROBUST TEST FOR VARIANCE

1. Introduction. Consider the problem of testing the hypothesis

$$H : \sigma^2 = \sigma_0^2 \quad \text{vs} \quad K : \sigma^2 > \sigma_0^2$$

in the normal distribution. The most powerful test based on the sample $X_1, \ldots, X_n$ is the standard chi-square test with the test statistic

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 .$$

It is well known that the distribution of many real features is not normal. Skewness and kurtosis are two measures of nonnormality (for the normal distribution both are zero). In [3] a map for those two parameters for many empirical distributions of real features is shown. For more than 70% of them, the skewness and kurtosis are significantly different from zero.

Scheffé [4] showed that the size of the standard test depends on the kurtosis of the underlying distribution:

kurtosis	0	0.5	1	2	4	7
size	0.05	0.08	0.11	0.17	0.26	0.36

(asymptotical results; see [2] for finite samples results). Our aim is to find a test whose size is more stable than that of the standard test. The test we will construct will be referred to as the robust test.

Formally, let $\alpha_S(\gamma_2)$ be the size of the standard test when the kurtosis of the model distribution is γ_2. Of course, $\alpha_S(0) = \alpha$ is the chosen level of significance in the normal model. Now we would like to find a test whose size $\alpha_R(\gamma_2)$ is such that

$$(*) \qquad \max\{\alpha_R(\gamma_2) - \alpha, 0\} < \max\{\alpha_S(\gamma_2) - \alpha, 0\} \quad \text{for all } \gamma_2 .$$

2. New test. It seems that one of the reasons of nonstability of the size of the standard test is the dependence of the variance of s^2 on the kurtosis of the underlying distribution:

$$\mathrm{Var}(s^2) = \sigma^4((n-1)\gamma_2 + 1)/n.$$

The variance of a quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ equals ([1])

$$\mathrm{Var}(\mathbf{x}'\mathbf{A}\mathbf{x}) = \sigma^4(\gamma_2 \mathbf{a}'\mathbf{a} + \mathrm{tr}(\mathbf{A}^2)),$$

where $\mathbf{a}' = (a_{11}, \ldots, a_{nn})$ is the vector of diagonal elements of the matrix $\mathbf{A}$ ($\mathbf{x}'\mathbf{A}\mathbf{x}$ is assumed to be shift-invariant, i.e. the value of $\mathbf{x}'\mathbf{A}\mathbf{x}$ does not depend on the mean vector of $\mathbf{x}$). Hence, to have the variance independent of the kurtosis, one must put $a_{11} = \ldots = a_{nn} = 0$. The test with such a test statistic seems to be more robust than the standard one.

The distribution of $\mathbf{x}'\mathbf{A}\mathbf{x}$, even in the normal model, is too complicated to be dealt with (it is a linear combination of independent chi-squares with positive and negative coefficients). Hence the size $\alpha_R(\gamma_2)$ of the robust test was estimated in a Monte-Carlo experiment.

For the Monte-Carlo investigation two tests were taken. For $n = 4$ the test based on the quadratic form with the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

was considered. In the normal model with variance σ^2 this quadratic form is distributed as $\sigma^2(\chi^2(1) - \chi^2(1))$, i.e. as the difference of two independent random variables, each distributed as chi-square with one degree of freedom. The hypothesis H is rejected for large absolute values of the test statistic. For $n = 5$ the test with the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

was considered. In the normal model the test statistic is distributed as $\sigma^2(\chi^2(2) - \chi^2(2))$. We put $\sigma_0^2 = 1$ and take a family of nonnormal distributions

$$(1-\varepsilon)N(0, \tau_1^2) + \varepsilon N(0, \tau_2^2),$$

where ε, τ_1^2 and τ_2^2 are chosen to satisfy $(1-\varepsilon)\tau_1^2 + \varepsilon\tau_2^2 = 1$. The kurtosis of that distribution equals $3\{(1-\varepsilon)\tau_1^4 + \varepsilon\tau_2^4 - 1\}$. The kurtosis of the normal distribution is 0.

TABLE 1
$n = 4$, epsilon $= 0.05$

kur	alpha .01 rob	.01 sta	.05 rob	.05 sta	.10 rob	.10 sta
3	.0105	.0105	.0484	.0521	.0942	.1027
4	.0115	.0173	.0482	.0581	.0912	.1065
5	.0124	.0232	.0461	.0618	.0871	.1049
6	.0125	.0280	.0440	.0637	.0853	.1008
12	.0128	.0457	.0376	.0705	.0704	.0933
15	.0124	.0510	.0360	.0737	.0633	.0918
18	.0123	.0550	.0341	.0768	.0577	.0909
30	.0106	.0654	.0269	.0847	.0436	.0966
33	.0102	.0675	.0254	.0866	.0393	.0976
60	.0049	.0796	.0063	.0964	.0072	.1061

TABLE 2
$n = 4$, epsilon $= 0.10$

kur	alpha .01 rob	.01 sta	.05 rob	.05 sta	.10 rob	.10 sta
3	.0105	.0105	.0484	.0521	.0942	.1027
4	.0117	.0204	.0478	.0575	.0895	.1080
5	.0113	.0262	.0465	.0636	.0850	.1054
6	.0120	.0314	.0441	.0708	.0815	.1077
12	.0140	.0559	.0361	.0913	.0625	.1170
15	.0135	.0629	.0327	.0994	.0552	.1216
18	.0133	.0711	.0297	.1063	.0475	.1278
30	.0112	.0888	.0154	.1215	.0187	.1407

TABLE 3
$n = 4$, epsilon $= 0.20$

kur	alpha .01 rob	.01 sta	.05 rob	.05 sta	.10 rob	.10 sta
3	.0105	.0105	.0484	.0521	.0942	.1027
4	.0128	.0219	.0476	.0635	.0909	.1093
5	.0129	.0309	.0466	.0728	.0858	.1167
6	.0137	.0389	.0449	.0825	.0809	.1255
12	.0166	.0671	.0370	.1200	.0554	.1580
15	.0182	.0774	.0346	.1327	.0470	.1678

TABLE 4
$n = 5$, epsilon $= 0.05$

kur	alpha .01 rob	.01 sta	.05 rob	.05 sta	.10 rob	.10 sta
3	.0103	.0104	.0493	.0504	.0985	.0970
4	.0122	.0171	.0484	.0590	.0909	.1043
5	.0139	.0249	.0461	.0643	.0864	.1065
6	.0143	.0310	.0435	.0690	.0836	.1039
12	.0157	.0523	.0385	.0795	.0647	.1018
15	.0161	.0575	.0367	.0840	.0592	.1038
18	.0162	.0627	.0347	.0880	.0552	.1054
30	.0142	.0767	.0288	.0994	.0444	.1098
33	.0137	.0791	.0283	.1012	.0406	.1121
60	.0082	.0932	.0114	.1113	.0137	.1225

TABLE 5
$n = 5$, epsilon $= 0.10$

kur	alpha .01 rob	.01 sta	.05 rob	.05 sta	.10 rob	.10 sta
3	.0103	.0104	.0493	.0504	.0985	.0970
4	.0129	.0200	.0507	.0593	.0901	.1068
5	.0132	.0283	.0478	.0673	.0858	.1104
6	.0135	.0345	.0466	.0750	.0828	.1130
12	.0167	.0616	.0451	.1052	.0653	.1296
15	.0172	.0727	.0429	.1125	.0610	.1366
18	.0183	.0827	.0399	.1184	.0577	.1436
30	.0183	.1028	.0290	.1392	.0352	.1590

TABLE 6
$n = 5$, epsilon $= 0.20$

kur	alpha .01 rob	.01 sta	.05 rob	.05 sta	.10 rob	.10 sta
3	.0103	.0104	.0493	.0504	.0985	.0970
4	.0117	.0220	.0490	.0645	.0947	.1085
5	.0131	.0325	.0495	.0760	.0889	.1159
6	.0148	.0414	.0482	.0851	.0852	.1276
12	.0197	.0746	.0475	.1290	.0686	.1695
15	.0225	.0872	.0490	.1444	.0698	.1817

Simulations were performed on the IBM PC using a multiplicative random number generator. The generator was programmed by the author himself. Firstly, random numbers from the uniform distribution ($U(0,1)$) were generated: if r_n is the nth random number then $r_{n+1} = cr_n - [cr_n]$, where c is a constant and $[x]$ denotes the integer part of x. Next, as soon as r_n and r_{n+1} were generated, they were transformed into two independent normally distributed ($N(0,1)$) numbers:

$$\xi_n = (-2\log(r_n))^{1/2}\cos(2\pi r_{n+1}), \qquad \eta_n = (-2\log(r_n))^{1/2}\sin(2\pi r_{n+1}).$$

The results of 10000 runs are shown in Tables 1–6. The results of Tables 4 and 6 are ilustrated in Figures 1 and 2, respectively. On the vertical axis, the size of the test multiplied by 10000 is given. Here "alpha" $= \alpha_R(0) = \alpha_S(0)$ means the significance level of the test in the normal model and "kur" is kurtosis. One observes that the size of the new test is more robust than that of the standard test.

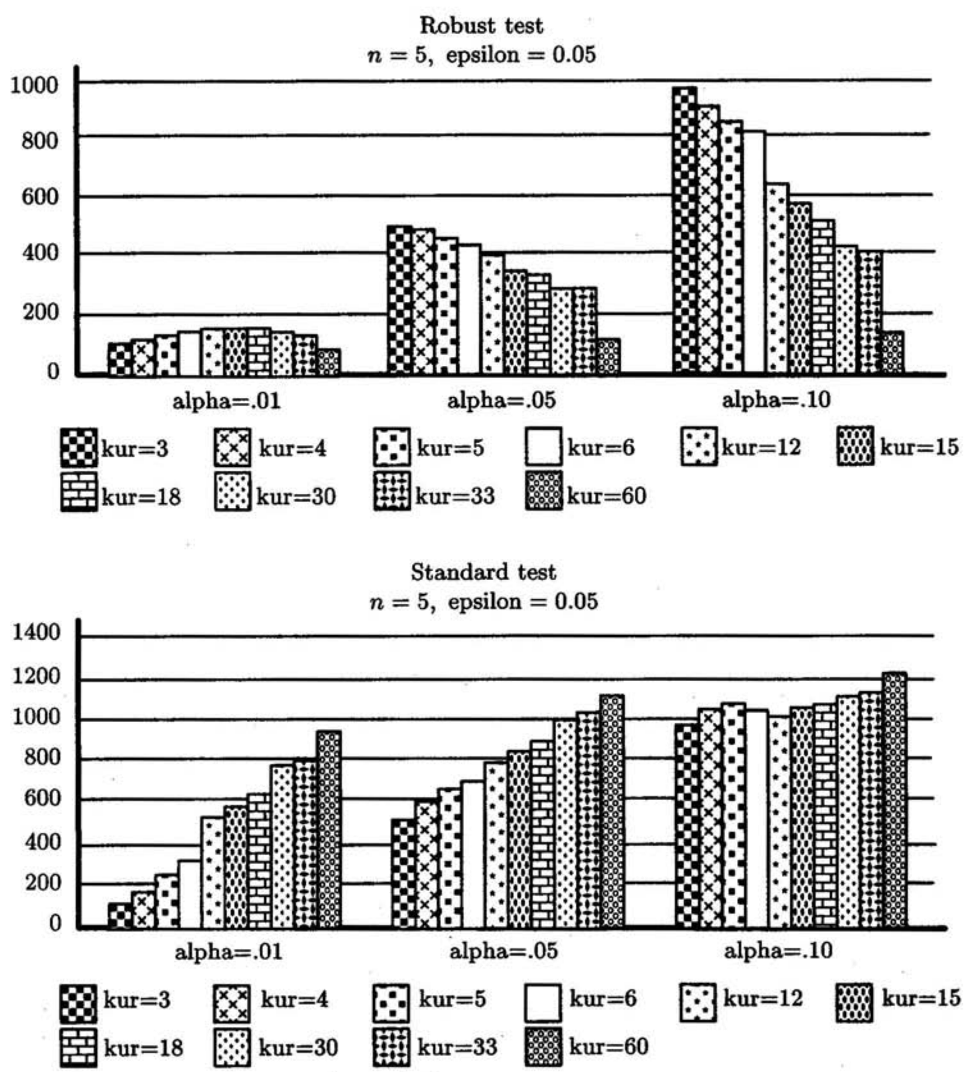

Fig. 1

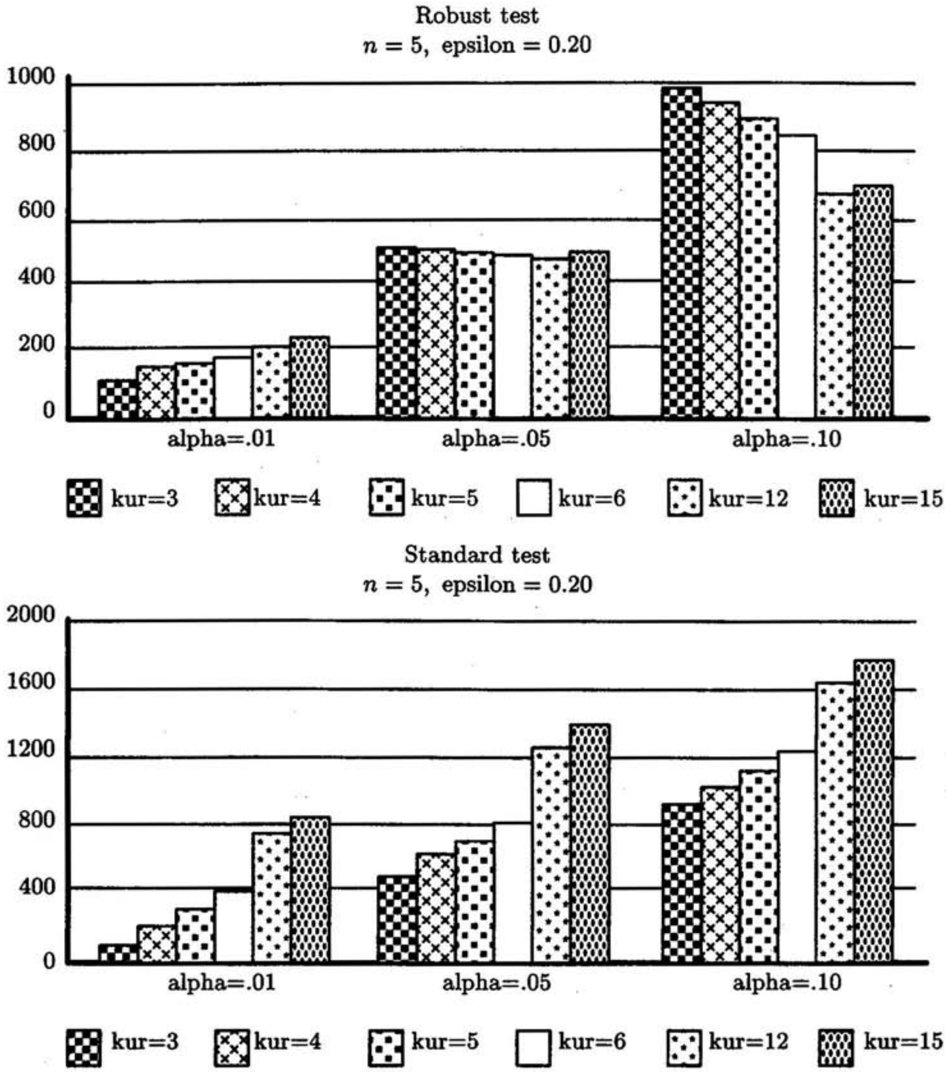

Fig. 2

3. Power comparison. In Figures 3 and 4 the powers of the standard and the robust tests are shown. Of course, the power of the robust test is less than the power of the standard test, but this is the price for the stabilization of the size of that test.

4. Concluding remarks

1. It seems that for any n the statistic of the robust test may be chosen such that its distribution in the normal model is $\chi^2(\frac{n-1}{2}) - \chi^2(\frac{n-1}{2})$ for odd n and $\chi^2(\frac{n}{2}-1) - \chi^2(\frac{n}{2}-1)$ for even n. Explicit general formulas for such

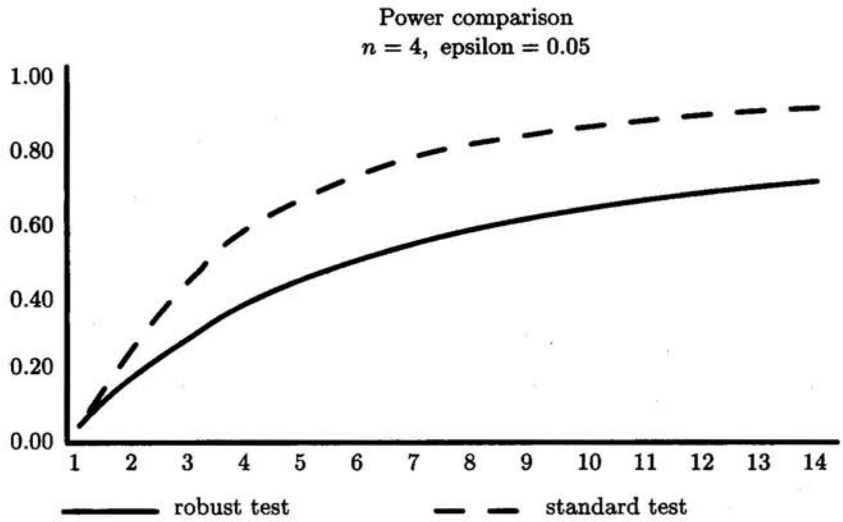

Fig. 3

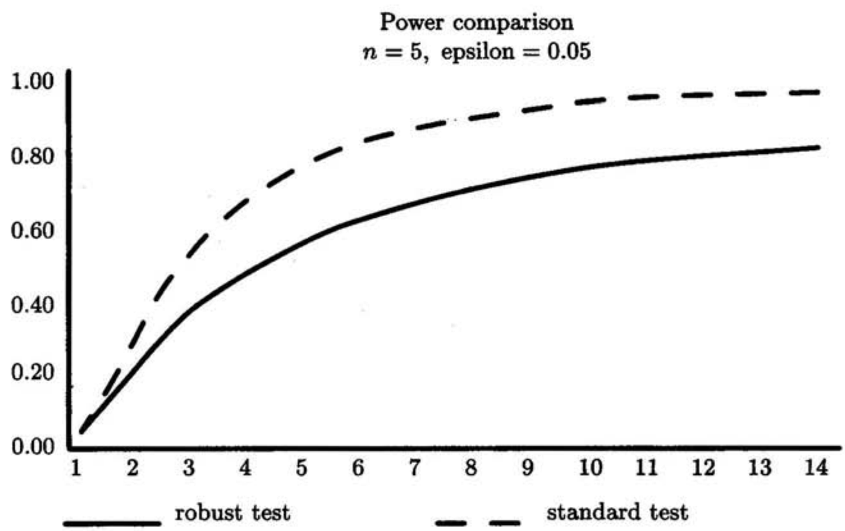

Fig. 4

test statistics are not available. However, for any particular n the matrix of the robust test can be found with the aid of a computer in the following manner.

Let the matrix $\mathbf{A}$ of the robust test have eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 = \ldots = \lambda_{(n-1)/2} = 1$, $\lambda_{(n+1)/2} = \ldots = \lambda_{n-1} = -1$, $\lambda_n = 0$ for n odd, and $\lambda_1 = \ldots = \lambda_{n/2-1} = 1$, $\lambda_{n/2+1} = \ldots = \lambda_{n-2} = -1$, $\lambda_{n-1} = \lambda_n = 0$ for n

even. The vectors $\mathbf{x}_i' = (x_{1i}, \ldots, x_{ni})$, in order to be normalized eigenvectors of $\mathbf{A}$, must satisfy the following system of equations:

n odd :

$$\sum_{i=1}^{(n-1)/2} x_{ki}^2 = \sum_{i=(n+1)/2}^{n-1} x_{ki}^2, \quad k = 1, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ik}^2 = 1, \quad k = 1, \ldots, n-1,$$

$$\sum_{i=1}^{n} x_{ik} x_{il} = 0, \quad k, l = 1, \ldots, n-1, \; k > l,$$

$$\sum_{i=1}^{n} x_{ik} = 0, \quad k = 1, \ldots, n-1,$$

$$x_{n1} = \ldots = x_{nn} = 1/\sqrt{n};$$

n even :

$$\sum_{i=1}^{n/2-1} x_{ki}^2 = \sum_{i=n/2}^{n-2} x_{ki}^2, \quad k = 1, \ldots, n,$$

$$\sum_{i=1}^{n} x_{ik}^2 = 1, \quad k = 1, \ldots, n-1,$$

$$\sum_{i=1}^{n} x_{ik} x_{il} = 0, \quad k, l = 1, \ldots, n-2, \; k > l,$$

$$\sum_{i=1}^{n} x_{ik} = 0, \quad k = 1, \ldots, n-2,$$

$\mathbf{x}_{n-1}$ and $\mathbf{x}_n$ are such that the vector $(1, \ldots, 1)'$ is their linear combination.

Let $\mathbf{X}' = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and let Λ be the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. Then $\mathbf{A} = \mathbf{X}'\Lambda\mathbf{X}$ and the distribution of $\mathbf{x}'\mathbf{A}\mathbf{x}$ in the normal model is $\chi^2(\nu) - \chi^2(\nu)$, where $\nu = (n-1)/2$ for n odd and $\nu = n/2 - 1$ for n even.

2. What are the critical values of the distribution $\chi^2(\nu) - \chi^2(\nu)$ for $n > 2$?

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WOJCIECH ZIELIŃSKI
DEPARTMENT OF MATHEMATICAL STATISTICS AND EXPERIMENTATION
UNIVERSITY OF AGRICULTURE
UL. RAKOWIECKA 26/30
02-528 WARSZAWA, POLAND

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