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**e-LOCALLY ACYCLIC GRAPHS**

Abstract. A graph $G$ is e-locally acyclic if the neighbourhood of any edge (i.e. the subgraph of $G$ induced by the set of all vertices adjacent to at least one vertex of this edge) is an acyclic graph. An upper bound for the number of edges in e-locally acyclic graphs is given in this article.

All graphs considered in this article are finite undirected graphs without loops and multiple edges.

Let $G$ be a graph, let $x$ be its vertex. By the *neighbourhood of $x$* (or *v-neighbourhood*) in $G$ we mean the subgraph of $G$ induced by all vertices adjacent to $x$ and denote it by $N_G(x)$.

Analogously by the *neighbourhood of an edge $f=xy$* (or *e-neighbourhood*) in $G$ we mean the subgraph of $G$ (denoted by $N_G(xy)$) induced by all vertices adjacent to at least one of the vertices $x$, $y$ but different from them.

If the neighbourhood of any vertex (edge) of $G$ is an acyclic graph then $G$ is a called a *v-locally (e-locally) acyclic graph*.

Erdős and Simonovits [1] found the maximal number of edges in v-locally acyclic graphs. Kowalska [4], Zelinka [5], [6] and the author [2], [3] found the maximal number or an upper bound for the number of edges of some special classes of v-locally acyclic graphs.

**Theorem 1** (Erdős and Simonovits). Let $G$ be a v-locally acyclic graph with $n$ vertices and $m$ edges. Then

$$m \leq n(n+1)/4.$$ 

We shall determine an upper bound for the number of edges in e-locally acyclic graphs. First we prove some simple lemmas.

**Lemma 1.** Let $G$ be an e-locally acyclic graph and $x_0$ be its vertex. Then $G_0 = G - x_0$ is also an e-locally acyclic graph.

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Proof. If $xy$ is an edge not incident to $x_0$ then either $N_{G_0}(xy) \cong N_G(xy)$ or $N_{G_0}(xy) \cong N_G(xy) - x_0$, which is also acyclic.

It is clear that each induced subgraph of an e-locally acyclic graph is also e-locally acyclic.

**Lemma 2.** Let $G$ be an e-locally acyclic graph, each edge of which belongs to at most one triangle. Let $G$ contain a subgraph $H = \langle M \rangle \cong K_{2,6}$. Then each vertex not belonging to $M$ is adjacent to at most two vertices of $M$. (The symbol $\langle M \rangle$ denotes a graph induced by the vertex set $M$.)

**Proof.** Let $H = \langle u_1, u_2, v_1, v_2, \ldots, v_6 \rangle$ contain all edges $u_iv_j$. If a vertex $y$ is adjacent to both $u_1, u_2$, then it cannot be adjacent to any $v_j$ ($1 \leq j \leq 6$) —in this case the edge $yv_j$ belongs to two triangles. If $y$ is adjacent to $v_j$ and $v_k$, it cannot be adjacent to any $u_i$ for the same reason.

If $y$ is adjacent to three vertices from $\{v_1, v_2, \ldots, v_6\}$, say $v_1, v_2, v_3$, then $\langle y, u_1, u_2, v_1, v_2, v_3 \rangle \cong K_{3,3}$ and $N_G(u_1v_1) \cong C_4$, which is a contradiction.

Now we consider graphs without triangles. It is clear that a neighbourhood of any vertex in such a graph consists of isolated vertices.

**Lemma 3.** Let $G$ be an e-locally acyclic graph without triangles containing a subgraph $H = \langle M \rangle \cong C_4$ and no subgraph isomorphic to $K_{2,6}$. Then there exist at most 6 vertices not belonging to $M$ which are adjacent to two vertices of $M$.

**Proof.** Let $H = \langle u_1, u_2, v_1, v_2 \rangle$ contain edges $u_iv_j$. According to the assumption at most 3 vertices not belonging to $M$ can be adjacent to both $u_1, u_2$ and another at most 3 vertices to both $v_1, v_2$. Because $G$ has no triangle there is no vertex adjacent to both $u_i, v_j$.

**Lemma 4.** Let $G$ be an e-locally acyclic graph without triangles different from a disjoint union of stars. Let $G$ contain no subgraph isomorphic to $C_4$. Then $G$ has an induced subgraph $H = \langle M \rangle \cong P_4$ and at most one vertex not belonging to $M$ is adjacent to two vertices of $M$.

**Proof.** If $G$ is an acyclic graph, then it is evident that it is either a disjoint union of stars or a graph with $\operatorname{diam} G \geq 3$. If it is not acyclic, then the shortest cycle can be $C_5$ and it contains an induced subgraph $H = \langle u_1, u_2, v_1, v_2 \rangle$ with the edges $u_1v_1, v_1u_2$ and $u_2v_2$. A vertex $y_1$ can be adjacent to two vertices of $M$ only if these vertices are $u_1$ and $v_2$ (in the opposite case $G$ contains $C_3$ or $C_4$). But then there is no other vertex $y_2$ adjacent to both $u_1$ and $v_2$—in this case $\langle y_1, u_1, y_2, v_2 \rangle \cong C_4$, which is a contradiction.

As regards graphs with triangles, we shall proceed similarly. If $G$ contains an edge $x_1x_2$ belonging to 4 triangles $\langle x_1, x_2, y_i \rangle$ ($1 \leq i \leq 4$), we can easily see that $\langle y_1, y_2, y_3, y_4 \rangle$ is an independent set of vertices (if there exists an
edge, say $y_1y_2$, then $N_G(x_1y_3)$ contains the triangle $(x_2, y_1, y_2)$. There is also no vertex $x_i$ ($i \geq 3$) adjacent to two vertices from $\{y_1, y_2, y_3, y_4\}$. If for instance $x_3$ is adjacent to both $y_1, y_2$, then $N_G(x_1y_2)$ contains the triangle $(x_1, x_2, y_1)$ and thus it is clear that $G = \langle y_1, \ldots, y_4, x_1, \ldots, x_n \rangle$ has at most $n + 6$ edges $x_iy_j$ ($1 \leq i \leq n$, $1 \leq j \leq 4$).

If $G$ contains an edge $y_1y_2$ belonging to two triangles $(y_1, y_2, y_3)$ and $(y_1, y_2, y_4)$ but no edge belonging to $4$ triangles, then there exist at most $9$ vertices $x_1, x_2, \ldots, x_9$ which are adjacent to two vertices of $\{y_1, y_2, y_3, y_4\}$ (one vertex can be adjacent to both $y_1, y_2$, no vertex can be adjacent to $y_3, y_4$ and at most two vertices can be adjacent to any other pair). Therefore $G = \langle y_1, \ldots, y_4, x_1, \ldots, x_n \rangle$ contains at most $n + 9$ edges $x_iy_j$.

If each edge belongs to at most one triangle (and a triangle exists), then $G$ contains a triangle $(y_1, y_2, y_3)$ with a hanging edge, say $y_3y_4$. Because there exists at most one vertex adjacent to both $y_3$ and $y_4$ and no vertex adjacent to two vertices of $\{y_1, y_2, y_3\}$ we can see that all other vertices adjacent to two vertices from the set $\{y_1, y_2, y_3, y_4\}$ are adjacent to $y_4$ and either to $y_1$ or to $y_2$. Thus either $G = \langle y_1, \ldots, y_4, x_1, \ldots, x_n \rangle$ contains a subgraph isomorphic to $K_{2,6}$ or there exist at most $4$ vertices adjacent to both $y_1, y_4$ and analogously at most $4$ vertices adjacent to both $y_2, y_4$. Hence $G$ contains at most $n + 9$ edges $x_iy_j$.

Now we have proved the following lemma.

**Lemma 5.** Let $G$ be an e-locally acyclic graph with $n + 4$ vertices containing triangles. Then either $G$ contains an induced subgraph isomorphic to $K_{2,6}$ or there exists a set of vertices $M = \{y_1, y_2, y_3, y_4\}$ such that $G$ has at most $n + 9$ edges $x_iy_j$.

Lemmas 3–5 yield immediately the following

**Corollary.** Let $G = \langle y_1, \ldots, y_4, x_1, \ldots, x_n \rangle$ be an e-acyclic graph with $n + 4$ vertices. Then $G$ contains either an induced subgraph isomorphic to $K_{2,6}$ or an induced subgraph $H = \langle M \rangle = \langle y_1, \ldots, y_4 \rangle$ such that $H$ has at most $5$ edges and $G$ has at most $n + 9$ edges $x_iy_j$.

Now we are able to prove our main result.

**Theorem 2.** Let $G$ be an e-locally acyclic graph with $n$ vertices and $m(n)$ edges. Then

$$m(n) \leq n(n + 24)/8.$$  

**Proof.** We use induction with respect to $n$.

1°. We have

$$n(n - 1)/2 < n(n + 24)/8$$

for each $n \leq 9$ and thus all graphs with at most $9$ vertices have less than $n(n + 24)/8$ edges.
2°. Now suppose that \(m(n) \leq n(n + 24)/8\) for some \(n\). We distinguish two cases:

(i) \(G = \langle y_1, \ldots, y_8, x_1, \ldots, x_n \rangle\) contains \(K_{2,6} = \langle y_1, \ldots, y_8 \rangle\). Then from Lemma 2 it follows that there exist at most \(2n\) edges \(x_i y_j\). Since \(K_{2,6}\) has 12 edges we have from our assumption

\[
m(n + 8) \leq 12 + 2n + m(n) \leq 12 + 2n + \frac{n(n + 24)}{8} = \frac{(n + 8)(n + 32)}{8}.
\]

(ii) \(G = \langle y_1, \ldots, y_4, x_1, \ldots, x_n \rangle\) contains no induced subgraph isomorphic to \(K_{2,6}\). Then by our Corollary there exists a set of vertices \(M = \{y_1, \ldots, y_4\}\) such that \(\langle M \rangle\) contains at most 5 edges and there are at most \(n + 9\) edges \(x_i y_j\). Thus using our induction assumption we see that

\[
m(n + 4) \leq 5 + (n + 9) + m(n) \leq n + 14 + \frac{n(n + 24)}{8} = \frac{(n + 4)(n + 28)}{8}
\]

and the proof is complete.

References


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