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e-LOCALLY ACYCLIC GRAPHS

Abstract. A graph G is e-locally acyclic if the neighbourhood of any edge (i.e. the subgraph of G induced by the set of all vertices adjacent to at least one vertex of this edge) is an acyclic graph. An upper bound for the number of edges in e-locally acyclic graphs is given in this article.

All graphs considered in this article are finite undirected graphs without loops and multiple edges.

Let G be a graph, let x be its vertex. By the *neighbourhood of* x (or *v-neighbourhood*) in G we mean the subgraph of G induced by all vertices adjacent to x and denote it by $N_G(x)$.

Analogously by the *neighbourhood of an edge* $f=xy$ (or *e-neighbourhood*) in G we mean the subgraph of G (denoted by $N_G(xy)$) induced by all vertices adjacent to at least one of the vertices x, y but different from them.

If the neighbourhood of any vertex (edge) of G is an acyclic graph then G is a called a *v-locally* (*e-locally*) *acyclic graph.*

Erdős and Simonovits [1] found the maximal number of edges in v-locally acyclic graphs. Kowalska [4], Zelinka [5], [6] and the author [2], [3] found the maximal number or an upper bound for the number of edges of some special classes of v-locally acyclic graphs.

THEOREM 1 (Erdős and Simonovits). *Let G be a v-locally acyclic graph with n vertices and m edges. Then*

$$m \le n(n+1)/4 .$$

We shall determine an upper bound for the number of edges in e-locally acyclic graphs. First we prove some simple lemmas.

LEMMA 1. *Let G be an e-locally acyclic graph and x_0 be its vertex. Then $G_0 = G - x_0$ is also an e-locally acyclic graph.*

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Proof. If xy is an edge not incident to x_0 then either $N_{G_0}(xy) \cong N_G(xy)$ or $N_{G_0}(xy) \cong N_G(xy) - x_0$, which is also acyclic.

It is clear that each induced subgraph of an e-locally acyclic graph is also e-locally acyclic.

LEMMA 2. *Let G be an e-locally acyclic graph, each edge of which belongs to at most one triangle. Let G contain a subgraph $H = \langle M \rangle \cong K_{2,6}$. Then each vertex not belonging to M is adjacent to at most two vertices of M. (The symbol $\langle M \rangle$ denotes a graph induced by the vertex set M.)*

Proof. Let $H = \langle u_1, u_2, v_1, v_2, \dots, v_6 \rangle$ contain all edges $u_i v_j$. If a vertex y is adjacent to both u_1, u_2, then it cannot be adjacent to any v_j $(1 \le j \le 6)$ —in this case the edge yv_j belongs to two triangles. If y is adjacent to v_j and v_k, it cannot be adjacent to any u_i for the same reason.

If y is adjacent to three vertices from $\{v_1, v_2, \dots, v_6\}$, say v_1, v_2, v_3, then $\langle y, u_1, u_2, v_1, v_2, v_3 \rangle \cong K_{3,3}$ and $N_G(u_1 v_1) \cong C_4$, which is a contradiction.

Now we consider graphs without triangles. It is clear that a neighbourhood of any vertex in such a graph consists of isolated vertices.

LEMMA 3. *Let G be an e-locally acyclic graph without triangles containing a subgraph $H = \langle M \rangle \cong C_4$ and no subgraph isomorphic to $K_{2,6}$. Then there exist at most 6 vertices not belonging to M which are adjacent to two vertices of M.*

Proof. Let $H = \langle u_1, u_2, v_1, v_2 \rangle$ contain edges $u_i v_j$. According to the assumption at most 3 vertices not belonging to M can be adjacent to both u_1, u_2 and another at most 3 vertices to both v_1, v_2. Because G has no triangle there is no vertex adjacent to both u_i, v_j.

LEMMA 4. *Let G be an e-locally acyclic graph without triangles different from a disjoint union of stars. Let G contain no subgraph isomorphic to C_4. Then G has an induced subgraph $H = \langle M \rangle \cong P_4$ and at most one vertex not belonging to M is adjacent to two vertices of M.*

Proof. If G is an acyclic graph, then it is evident that it is either a disjoint union of stars or a graph with $\operatorname{diam} G \ge 3$. If it is not acyclic, then the shortest cycle can be C_5 and it contains an induced subgraph $H = \langle u_1, u_2, v_1, v_2 \rangle$ with the edges $u_1 v_1, v_1 u_2$ and $u_2 v_2$. A vertex y_1 can be adjacent to two vertices of M only if these vertices are u_1 and v_2 (in the opposite case G contains C_3 or C_4). But then there is no other vertex y_2 adjacent to both u_1 and v_2—in this case $\langle y_1, u_1, y_2, v_2 \rangle \cong C_4$, which is a contradiction.

As regards graphs with triangles, we shall proceed similarly. If G contains an edge $x_1 x_2$ belonging to 4 triangles $\langle x_1, x_2, y_i \rangle$ $(1 \le i \le 4)$, we can easily see that $\langle y_1, y_2, y_3, y_4 \rangle$ is an independent set of vertices (if there exists an edge, say y_1y_2, then $N_G(x_1y_3)$ contains the triangle $\langle x_2, y_1, y_2\rangle$). There is also no vertex x_i $(i \geq 3)$ adjacent to two vertices from $\{y_1, y_2, y_3, y_4\}$. If for instance x_3 is adjacent to both y_1, y_2, then $N_G(x_1y_2)$ contains the triangle $\langle x_1, x_2, y_1\rangle$ and thus it is clear that $G = \langle y_1, \dots, y_4, x_1, \dots, x_n\rangle$ has at most $n + 6$ edges x_iy_j $(1 \leq i \leq n,\ 1 \leq j \leq 4)$.

If G contains an edge y_1y_2 belonging to two triangles $\langle y_1, y_2, y_3\rangle$ and $\langle y_1, y_2, y_4\rangle$ but no edge belonging to 4 triangles, then there exist at most 9 vertices $x_1, x_2, \dots, x_9$ which are adjacent to two vertices of $\{y_1, y_2, y_3, y_4\}$ (one vertex can be adjacent to both y_1, y_2, no vertex can be adjacent to y_3, y_4 and at most two vertices can be adjacent to any other pair). Therefore $G = \langle y_1, \dots, y_4, x_1, \dots, x_n\rangle$ contains at most $n + 9$ edges x_iy_j.

If each edge belongs to at most one triangle (and a triangle exists), then G contains a triangle $\langle y_1, y_2, y_3\rangle$ with a hanging edge, say y_3y_4. Because there exists at most one vertex adjacent to both y_3 and y_4 and no vertex adjacent to two vertices of $\{y_1, y_2, y_3\}$ we can see that all other vertices adjacent to two vertices from the set $\{y_1, y_2, y_3, y_4\}$ are adjacent to y_4 and either to y_1 or to y_2. Thus either $G = \langle y_1, \dots, y_4, x_1, \dots, x_n\rangle$ contains a subgraph isomorphic to $K_{2,6}$ or there exist at most 4 vertices adjacent to both y_1, y_4 and analogously at most 4 vertices adjacent to both y_2, y_4. Hence G contains at most $n + 9$ edges x_iy_j.

Now we have proved the following lemma.

LEMMA 5. *Let G be an e-locally acyclic graph with $n + 4$ vertices containing triangles. Then either G contains an induced subgraph isomorphic to $K_{2,6}$ or there exists a set of vertices $M = \{y_1, y_2, y_3, y_4\}$ such that G has at most $n + 9$ edges x_iy_j.*

Lemmas 3–5 yield immediately the following

COROLLARY. *Let $G = \langle y_1, \dots, y_4, x_1, \dots, x_n\rangle$ be an e-acyclic graph with $n + 4$ vertices. Then G contains either an induced subgraph isomorphic to $K_{2,6}$ or an induced subgraph $H = \langle M\rangle = \langle y_1, \dots, y_4\rangle$ such that H has at most 5 edges and G has at most $n + 9$ edges x_iy_j.*

Now we are able to prove our main result.

THEOREM 2. *Let G be an e-locally acyclic graph with n vertices and $m(n)$ edges. Then*

$$m(n) \leq n(n + 24)/8\,.$$

Proof. We use induction with respect to n.

1°. We have

$$n(n - 1)/2 < n(n + 24)/8$$

for each $n \leq 9$ and thus all graphs with at most 9 vertices have less than $n(n + 24)/8$ edges.

2°. Now suppose that $m(n) \le n(n+24)/8$ for some n. We distinguish two cases:

(i) $G = \langle y_1, \ldots, y_8, x_1, \ldots, x_n \rangle$ contains $K_{2,6} = \langle y_1, \ldots, y_8 \rangle$. Then from Lemma 2 it follows that there exist at most $2n$ edges $x_i y_j$. Since $K_{2,6}$ has 12 edges we have from our assumption

$$m(n+8) \le 12 + 2n + m(n) \le 12 + 2n + \frac{n(n+24)}{8} = \frac{(n+8)(n+32)}{8}.$$

(ii) $G = \langle y_1, \ldots, y_4, x_1, \ldots, x_n \rangle$ contains no induced subgraph isomorphic to $K_{2,6}$. Then by our Corollary there exists a set of vertices $M = \{y_1, \ldots, y_4\}$ such that $\langle M \rangle$ contains at most 5 edges and there are at most $n+9$ edges $x_i y_j$. Thus using our induction assumption we see that

$$m(n+4) \le 5 + (n+9) + m(n) \le n + 14 + \frac{n(n+24)}{8} = \frac{(n+4)(n+28)}{8}$$

and the proof is complete.

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