ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,3 (1992), pp. 427–435

M. BROCKMANN (Heidelberg)
J. EICHENAUER-HERRMANN (Darmstadt)

GAMMA-MINIMAX ESTIMATION OF MULTINOMIAL PROBABILITIES

Abstract. Characterizations of a gamma-minimax estimator for the parameters of a multinomial distribution under arbitrary squared error loss are established. It is always assumed that the available vague prior information can be described by a class of priors whose vector of first moments belongs to a suitable convex and compact set. Several known gamma-minimax and minimax results can be obtained from the characterizations in the present paper.

1. Introduction and notation. In the present paper the following estimation problem is considered. The parameter vector $\theta = (\theta_1, \ldots, \theta_k)^T \in \Theta$ of a multinomial distribution $P_\theta = M(n, k; \theta)$ with

$$P_\theta(\{x\}) = \frac{n!}{x_1! \ldots x_k!} \theta_1^{x_1} \ldots \theta_k^{x_k}, \quad x = (x_1, \ldots, x_k)^T \in \mathbb{X},$$

is to be estimated, where the positive integers n and k are assumed to be known,

$$\Theta = \{(\theta_1, \ldots, \theta_k)^T \in [0,1]^k \mid \theta_1 + \ldots + \theta_k = 1\}$$

is the compact parameter space, and

$$\mathbb{X} = \{(x_1, \ldots, x_k)^T \in \{0, \ldots, n\}^k \mid x_1 + \ldots + x_k = n\}$$

denotes the sample space. Let Δ be the set of all (nonrandomized) estimators $\delta : \mathbb{X} \to \Theta$. The *risk function* $R(\cdot, \delta) : \Theta \to \mathbb{R}$ of an estimator δ under squared error loss is defined by

$$R(\theta, \delta) = \sum_{x \in \mathbb{X}} (\theta - \delta(x))^T Q(\theta - \delta(x)) P_\theta(\{x\}), \quad \theta \in \Theta,$$

1991 *Mathematics Subject Classification*: 62C20, 62F10.
Key words and phrases: gamma-minimax estimation, multinomial distribution, arbitrary squared error loss, moment restrictions, saddle point characterization.

with $Q \in \mathbb{R}^{k \times k}$ being a symmetric and positive semidefinite matrix, called the loss matrix. Let Π be the set of all *priors*, i.e. Borel probability measures on the parameter space Θ. For a prior π and an estimator δ

$$r(\pi, \delta) = \int_{\Theta} R(\theta, \delta)\, \pi(d\theta)$$

is called the *Bayes risk* of δ with respect to π. The vector of the first moments

$$\nu_i(\pi) = \int_{\Theta} \theta_i\, \pi(d\theta), \quad i \in \{1, \ldots, k\},$$

of a prior π is denoted by

$$\nu(\pi) = (\nu_1(\pi), \ldots, \nu_k(\pi))^T$$

and belongs to the compact moment space

$$\mathcal{M} = \{(\nu_1, \ldots, \nu_k)^T \in [0,1]^k \mid \nu_1 + \ldots + \nu_k = 1\}.$$

Following Abraham Wald's decision-theoretic approach (cf. [5]) two classical optimality principles, the Bayes and the minimax principle, are frequently used. However, the application of the Bayes principle requires precise prior information on the distribution of the unknown parameter vector θ which can be described by a single prior π. On the other hand, the minimax principle makes it impossible to take into account vague prior information. In order to avoid the disadvantages of both the Bayes and the minimax principle, in the present paper the Γ-minimax principle is used, where Γ denotes a nonvoid subset of Π. In particular, it is assumed that the available vague prior information can be described by a subset of priors of the form

$$\Gamma_{\mathcal{G}} = \{\pi \in \Pi \mid \nu(\pi) \in \mathcal{G}\}$$

with a suitable convex and compact subset $\mathcal{G}$ of the moment space $\mathcal{M}$. A Γ-minimax estimator δ^* minimizes the maximum Bayes risk with respect to the elements of Γ, i.e.

$$\sup_{\pi \in \Gamma} r(\pi, \delta^*) = \inf_{\delta \in \Delta} \sup_{\pi \in \Gamma} r(\pi, \delta).$$

Obviously, a Γ-minimax estimator is a minimax strategy of the second player in the statistical game (Γ, Δ, r). A pair of strategies $(\pi^*, \delta^*) \in \Gamma \times \Delta$ with

$$\sup_{\pi \in \Gamma} r(\pi, \delta^*) = r(\pi^*, \delta^*) = \inf_{\delta \in \Delta} r(\pi^*, \delta)$$

is called a *saddle point* in (Γ, Δ, r). In this case, as is well known, the estimator δ^* is Γ-minimax.

In the present paper characterizations of a saddle point in the statistical game $(\Gamma_{\mathcal{G}}, \Delta, r)$ are established. The second section contains further definitions and some known preliminary results. The main result is stated precisely in the third section. Its proof is given in the fifth section. In the fourth section several interesting special cases are discussed, where the subset $\mathcal{G}$ of the moment space $\mathcal{M}$ is determined by linear inequality restrictions.

2. Preliminary results. Subsequently, a family $(\pi_\nu)_{\nu\in\mathcal{M}}$ of priors is introduced which is indexed by the elements of the moment space $\mathcal{M}$. To that end, put

$$I_0(\nu) = \{i \in \{1,\dots,k\} \mid \nu_i = 0\}, \quad I_+(\nu) = \{1,\dots,k\}\backslash I_0(\nu),$$

and

$$\Theta_\nu = \{(\theta_1,\dots,\theta_k)^T \in \Theta \mid \theta_i = 0 \text{ for } i \in I_0(\nu)\}$$

for $\nu = (\nu_1,\dots,\nu_k)^T \in \mathcal{M}$. If $|I_0(\nu)| = k-1$, then $\Theta_\nu = \{\nu\}$. In this case, let $\pi_\nu = \varepsilon_\nu$ be the Dirac measure on ν. Otherwise, let π_ν denote a Dirichlet distribution which is concentrated on Θ_ν, i.e. $\pi_\nu(\Theta_\nu) = 1$, and has a Lebesgue density f_ν of the form

$$f_\nu(\theta) = \Gamma(\sqrt{n}) \prod_{i\in I_+(\nu)} \frac{\theta_i^{\sqrt{n}\nu_i - 1}}{\Gamma(\sqrt{n}\nu_i)}, \quad \theta = (\theta_1,\dots,\theta_k)^T \in \Theta_\nu,$$

where Γ denotes the usual Γ-function. It is well known (cf. [3], [4]) that $\nu(\pi_\nu) = \nu$ and that the estimator $\delta^\nu = (\delta_1^\nu,\dots,\delta_k^\nu)^T$ with

$$\delta_i^\nu(x) = \frac{x_i + \sqrt{n}\nu_i}{n+\sqrt{n}}, \quad x = (x_1,\dots,x_k)^T \in \mathbb{X},\ i \in \{1,\dots,k\},$$

is a Bayes estimator with respect to the prior π_ν for every $\nu = (\nu_1,\dots,\nu_k)^T \in \mathcal{M}$. The risk function of the estimator δ^ν can be written in the form (cf. [4], [6])

$$R(\theta,\delta^\nu) = \frac{1}{(\sqrt{n}+1)^2}(q^T\theta - 2\nu^T Q\theta + \nu^T Q\nu), \quad \theta \in \Theta,$$

where $q = (q_{11},\dots,q_{kk})^T$ denotes the vector of diagonal elements of the loss matrix Q. Therefore, the Bayes risk of the estimator δ^ν with respect to any prior π is given by

$$r(\pi,\delta^\nu) = \frac{1}{(\sqrt{n}+1)^2}(q^T\nu(\pi) - 2\nu^T Q\nu(\pi) + \nu^T Q\nu).$$

3. Characterization of a saddle point. In this section necessary and sufficient conditions for a pair of strategies (π_ν,δ^ν) with $\nu \in \mathcal{G}$ to be a saddle point in the statistical game $(\Gamma_{\mathcal{G}},\Delta,r)$ are stated. To that end, let $\psi : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ be a function with

$$\psi(\nu,\mu) = q^T\mu - 2\nu^T Q\mu, \quad \nu,\mu \in \mathbb{R}^k,$$

and let $\varphi : \mathbb{R}^k \to \mathbb{R}$ be a concave function with

$$\varphi(\nu) = q^T \nu - \nu^T Q \nu, \quad \nu \in \mathbb{R}^k .$$

The proof of the following main result is given in the fifth section.

THEOREM 1. *There exists a $\nu \in \mathcal{G}$ such that the following three equivalent conditions are satisfied.*

(i) *The pair of strategies (π_ν, δ^ν) is a saddle point in the statistical game $(\Gamma_\mathcal{G}, \Delta, r)$.*

(ii) $\psi(\nu, \nu) = \max_{\mu \in \mathcal{G}} \psi(\nu, \mu)$.

(iii) $\varphi(\nu) = \max_{\mu \in \mathcal{G}} \varphi(\mu)$.

Furthermore, if the matrix Q is positive definite, then there exists exactly one $\nu \in \mathcal{G}$ which satisfies these conditions.

The third condition of Theorem 1 shows that a saddle point in the statistical game $(\Gamma_\mathcal{G}, \Delta, r)$ can be determined by maximizing the concave function φ on the convex and compact subset $\mathcal{G}$ of $\mathbb{R}^k$. In case the loss matrix is positive definite, it follows via the technique of Lagrange multipliers that

$$\tilde{\nu} = \frac{1}{2} Q^{-1} \left(q + \frac{2 - \mathbf{1}^T Q^{-1} q}{\mathbf{1}^T Q^{-1} \mathbf{1}} \mathbf{1} \right)$$

maximizes the function φ on the hyperplane $H = \{\nu \in \mathbb{R}^k \mid \mathbf{1}^T \nu = 1\}$, where $\mathbf{1}$ stands for the vector $(1, \dots, 1)^T \in \mathbb{R}^k$. If $\tilde{\nu} \in \mathcal{G}$, then $\tilde{\nu}$ maximizes φ on $\mathcal{G}$ as well. Otherwise, the point ν on the boundary of $\mathcal{G}$ has to be determined which minimizes $(\nu - \tilde{\nu})^T Q (\nu - \tilde{\nu})$.

4. Linear inequality restrictions. In this section some interesting special cases are considered. It is always assumed that the set $\mathcal{G}$ is determined by linear inequality restrictions, i.e.,

$$\mathcal{G} = \{\nu \in \mathcal{M} \mid g_i^T \nu \geq \gamma_i \text{ for } i \in \{1, \dots, l\}\}$$

with vectors $g_1, \dots, g_l \in \mathbb{R}^k$ and real numbers $\gamma_1, \dots, \gamma_l$. Let G be a matrix with $G = (g_1, \dots, g_l) \in \mathbb{R}^{k \times l}$, and recall that $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^k$. Then the Theorem of Kuhn–Tucker leads to another characterization of a saddle point in the statistical game $(\Gamma_\mathcal{G}, \Delta, r)$. Again, the proof of this result is given in the fifth section.

THEOREM 2. *Let $\mathcal{G}$ be defined as above. Then $\nu = (\nu_1, \dots, \nu_k)^T \in \mathbb{R}^k$ belongs to $\mathcal{G}$ and the pair of strategies (π_ν, δ^ν) is a saddle point in the statistical game $(\Gamma_\mathcal{G}, \Delta, r)$ if and only if there exist subsets K of $\{1, \dots, k\}$ and L of $\{1, \dots, l\}$ as well as a vector $u = (u_1, \dots, u_l)^T \in \mathbb{R}^l$ and a real* *number c such that ν, u, and c form a solution of the $k+l+1$ linear equations*

$$\begin{aligned} (2Q\nu - q - Gu)_i &= c\,, & i &\in K\,,\\ \nu_i &= 0\,, & i &\in K^C\,,\\ g_i^T \nu &= \gamma_i\,, & i &\in L\,,\\ u_i &= 0\,, & i &\in L^C\,,\\ \mathbf{1}^T \nu &= 1 \end{aligned}$$

with $K^C = \{1, \ldots, k\} \setminus K$ and $L^C = \{1, \ldots, l\} \setminus L$ and simultaneously satisfy the $k+l$ inequalities

$$\begin{aligned} (2Q\nu - q - Gu)_i &\geq c\,, & i &\in K^C\,,\\ \nu_i &\geq 0\,, & i &\in K\,,\\ g_i^T \nu &\geq \gamma_i\,, & i &\in L^C\,,\\ u_i &\geq 0\,, & i &\in L\,. \end{aligned}$$

For the unrestricted minimax problem, i.e. for $\mathcal{G} = \mathcal{M}$, Theorem 2 yields the following result of Wilczyński [6].

COROLLARY 1. *The vector $\nu = (\nu_1, \ldots, \nu_k)^T \in \mathbb{R}^k$ belongs to the moment space $\mathcal{M}$ and the pair of strategies (π_ν, δ^ν) is a saddle point in the statistical game (Π, Δ, r) if and only if there exist a subset K of $\{1, \ldots, k\}$ and a real number c such that ν and c form a solution of the $k+1$ linear equations*

$$\begin{aligned} (2Q\nu - q)_i &= c\,, & i &\in K\,,\\ \nu_i &= 0\,, & i &\in K^C\,,\\ \mathbf{1}^T \nu &= 1 \end{aligned}$$

with $K^C = \{1, \ldots, k\} \setminus K$ and simultaneously satisfy the k inequalities

$$\begin{aligned} (2Q\nu - q)_i &\geq c\,, & i &\in K^C\,,\\ \nu_i &\geq 0\,, & i &\in K\,. \end{aligned}$$

Finally, a special choice of the linear inequality restrictions which determine the subset $\mathcal{G}$ of $\mathcal{M}$ is considered (cf. [2]). This form of $\mathcal{G}$ leads to Corollary 2. In the following section its straightforward proof is sketched for the sake of completeness.

COROLLARY 2. *Let $\mathcal{G}$ be defined by*

$$\mathcal{G} = \{(\nu_1, \ldots, \nu_k)^T \in \mathcal{M} \mid \alpha_i \leq \nu_i \leq \beta_i \text{ for } i \in \{1, \ldots, k\}\}\,,$$

where $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ are real numbers with $0 \leq \alpha_i \leq \beta_i \leq 1$ for $i \in \{1, \ldots, k\}$ and $\alpha_1 + \ldots + \alpha_k \leq 1 \leq \beta_1 + \ldots + \beta_k$. Then $\nu = (\nu_1, \ldots, \nu_k)^T \in \mathbb{R}^k$ belongs to $\mathcal{G}$ and the pair of strategies (π_ν, δ^ν) is a saddle point in the statistical game $(\Gamma_\mathcal{G}, \Delta, r)$ if and only if there exist disjoint subsets L_0, L_1, *and L_2 of $\{1,\dots,k\}$ with $L_0 \cup L_1 \cup L_2 = \{1,\dots,k\}$ and a real number c such that ν and c form a solution of the $k+1$ linear equations*

$$\begin{aligned}(2Q\nu - q)_i &= c\,, & i \in L_0\,,\\ \nu_i &= \alpha_i\,, & i \in L_1\,,\\ \nu_i &= \beta_i\,, & i \in L_2\,,\\ \mathbf{1}^T\nu &= 1 & \end{aligned}$$

and simultaneously satisfy the k inequalities

$$\begin{aligned}(2Q\nu - q)_i &\geq c\,, & i \in L_1\,,\\ (2Q\nu - q)_i &\leq c\,, & i \in L_2\,,\\ \alpha_i \leq \nu_i &\leq \beta_i\,, & i \in L_0\,.\end{aligned}$$

In case the loss matrix is simply a diagonal matrix with diagonal elements $q_{11},\dots,q_{kk}$ Corollary 2 yields the following characterization of a saddle point given in [2]. If the real number c satisfies

$$\sum_{i=1}^{k} \operatorname{med}\left(\alpha_i, \frac{1}{2}\left(\frac{c}{q_{ii}} + 1\right), \beta_i\right) = 1\,,$$

then $\nu = (\nu_1,\dots,\nu_k)^T$ with

$$\nu_i = \operatorname{med}\left(\alpha_i, \frac{1}{2}\left(\frac{c}{q_{ii}} + 1\right), \beta_i\right), \quad i \in \{1,\dots,k\}\,,$$

belongs to $\mathcal{G}$ and the pair of strategies (π_ν, δ^ν) is a saddle point in the statistical game $(\Gamma_{\mathcal{G}}, \Delta, r)$.

Known results for the unrestricted minimax problem in [3] are immediate consequences of this characterization (cf. [2], Section 3).

5. Proof of the main results

Proof of Theorem 1. First, the equivalence of conditions (i) and (ii) is shown. Let $\nu \in \mathcal{G}$ be fixed. Then

$$r(\pi_\nu, \delta^\nu) = \frac{1}{(\sqrt{n}+1)^2}(\psi(\nu,\nu) + \nu^T Q \nu)$$

and

$$\begin{aligned}\sup_{\pi\in\Gamma_{\mathcal{G}}} r(\pi, \delta^\nu) &= \frac{1}{(\sqrt{n}+1)^2}(\sup_{\pi\in\Gamma_{\mathcal{G}}} \psi(\nu, \nu(\pi)) + \nu^T Q \nu)\\ &= \frac{1}{(\sqrt{n}+1)^2}(\sup_{\mu\in\mathcal{G}} \psi(\nu, \mu) + \nu^T Q \nu)\end{aligned}$$

follow from $\nu(\pi_\mu) = \mu$ for all $\mu \in \mathcal{G}$. Therefore, $r(\pi_\nu, \delta^\nu) = \sup_{\pi\in\Gamma_{\mathcal{G}}} r(\pi, \delta^\nu)$

is equivalent to

$$\psi(\nu,\nu) = \sup_{\mu\in\mathcal{G}} \psi(\nu,\mu)\,,$$

which yields the desired result, since δ^ν is a Bayes estimator with respect to the prior π_ν.

Now, it will be shown that (ii) implies (iii). It follows from

$$\begin{aligned}\varphi(\nu) &= \psi(\nu,\nu) + \nu^T Q\nu = \sup_{\mu\in\mathcal{G}} \psi(\nu,\mu) + \nu^T Q\nu \\ &= \sup_{\mu\in\mathcal{G}} [\varphi(\mu) + (\mu-\nu)^T Q(\mu-\nu)] \geq \sup_{\mu\in\mathcal{G}} \varphi(\mu)\end{aligned}$$

that $\varphi(\nu) = \sup_{\mu\in\mathcal{G}} \varphi(\mu)$, i.e., condition (iii) is valid.

In order to show that (iii) is sufficient for (ii) assume that

$$\varphi(\nu) = \sup_{\mu\in\mathcal{G}} \varphi(\mu)$$

for some $\nu \in \mathcal{G}$ and that there exists a $\mu \in \mathcal{G}$ with

$$\psi(\nu,\mu) > \psi(\nu,\nu)\,.$$

For $\alpha \in [0,1]$ put $\nu_\alpha = \alpha\mu + (1-\alpha)\nu \in \mathcal{G}$. A short calculation shows that

$$\varphi(\nu_\alpha) = \varphi(\nu) + \alpha[\psi(\nu,\mu) - \psi(\nu,\nu) - \alpha(\mu-\nu)^T Q(\mu-\nu)]\,.$$

The assumption $\psi(\nu,\mu) > \psi(\nu,\nu)$ yields the existence of an $\alpha^* \in (0,1]$ with

$$\psi(\nu,\mu) - \psi(\nu,\nu) > \alpha(\mu-\nu)^T Q(\mu-\nu)$$

for all $\alpha \in [0,\alpha^*)$. Therefore, $\varphi(\nu_\alpha) > \varphi(\nu)$ for all $\alpha \in (0,\alpha^*)$, which is a contradiction to condition (iii).

Finally, the existence of a $\nu \in \mathcal{G}$ which satisfies conditions (i)–(iii) follows from the fact that the continuous function φ attains its maximum on the compact set $\mathcal{G}$. ∎

Proof of Theorem 2. First, put

$$\begin{aligned} g_{l+1} &= (1,0,\ldots,0)^T \in \mathbb{R}^k\,, \\ &\vdots \\ g_{l+k} &= (0,0,\ldots,1)^T \in \mathbb{R}^k\,, \\ g_{l+k+1} &= -g_{l+k+2} = \mathbf{1}\,, \\ \gamma_{l+1} &= \ldots = \gamma_{l+k} = 0\,, \\ \gamma_{l+k+1} &= -\gamma_{l+k+2} = 1\,, \end{aligned}$$

and $m = l+k+2$. Now, the set $\mathcal{G}$ can be written in the form

$$\mathcal{G} = \{\nu \in \mathbb{R}^k \mid g_i^T\nu \geq \gamma_i \text{ for } i \in \{1,\ldots,m\}\}\,.$$

The function $-\varphi$ is convex and differentiable with $\nabla(-\varphi)(\nu) = 2Q\nu - q$ for $\nu \in \mathbb{R}^k$. Then the Theorem of Kuhn–Tucker (cf. [1], Ch. 3.8) shows that

$\nu = (\nu_1, \ldots, \nu_k)^T \in \mathbb{R}^k$ belongs to $\mathcal{G}$ and satisfies

$$\varphi(\nu) = \max_{\mu \in \mathcal{G}} \varphi(\mu)$$

if and only if there exist real numbers $u_1, \ldots, u_m$ with

$$2Q\nu - q = \sum_{i=1}^{m} u_i g_i \,,$$

$$\sum_{i=1}^{m} (g_i^T \nu - \gamma_i) u_i = 0 \,,$$

$$\begin{aligned} \mathbf{1}^T \nu &= 1 \,, \\ \nu_i &\geq 0 \,, \quad i \in \{1, \ldots, k\} \,, \\ g_i^T \nu &\geq \gamma_i \,, \quad i \in \{1, \ldots, l\} \,, \\ u_i &\geq 0 \,, \quad i \in \{1, \ldots, m\} \,. \end{aligned}$$

These conditions are obviously equivalent to

$$\begin{aligned} 2Q\nu - q - Gu &= (u_{l+1}, \ldots, u_{l+k})^T + (u_{l+k+1} - u_{l+k+2})\mathbf{1} \,, \\ (g_i^T \nu - \gamma_i) u_i &= 0 \,, \quad i \in \{1, \ldots, l\} \,, \\ \nu_i u_{l+i} &= 0, \quad i \in \{1, \ldots, k\} \,, \\ \mathbf{1}^T \nu &= 1 \,, \\ \nu_i &\geq 0 \,, \quad i \in \{1, \ldots, k\} \,, \\ g_i^T \nu &\geq \gamma_i, \quad i \in \{1, \ldots, l\} \,, \\ u_i &\geq 0, \quad i \in \{1, \ldots, m\} \,. \end{aligned}$$

With the sets $K = \{i \in \{1, \ldots, k\} \mid \nu_i > 0\}$ and $L = \{i \in \{1, \ldots, l\} \mid g_i^T \nu = \gamma_i\}$ one obtains the result of Theorem 2. ∎

Proof of Corollary 2. Let $l = 2k$, $G = (g_1, \ldots, g_k, g_{k+1}, \ldots, g_l) = (I, -I) \in \mathbb{R}^{k \times l}$, $\gamma_i = \alpha_i$ and $\gamma_{k+i} = -\beta_i$ for $i \in \{1, \ldots, k\}$. Then the result of Corollary 2 follows from Theorem 2, if one observes that the subset K can be chosen as $\{1, \ldots, k\}$. ∎

References

[1] E. Blum and W. Oettli, *Mathematische Optimierung*, Springer, Berlin 1975.

[2] L. Chen, J. Eichenauer-Herrmann and J. Lehn, *Gamma-minimax estimators for the parameters of a multinomial distribution*, Zastos. Mat. 20 (1990), 561–564.

[3] S. Trybuła, *On the minimax estimation of the parameters in a multinomial distribution*, Selected Transl. Math. Statist. Probab. 3 (1962), 225–238.

[4] —, *Some investigations in minimax estimation theory*, Dissertationes Math. 240 (1985).

[5] A. Wald, *Statistical Decision Functions*, Wiley, New York 1950.
[6] M. Wilczyński, *Minimax estimation for the multinomial and multivariate hypergeometric distributions*, Sankhyā A47 (1985), 128–132.

MICHAEL BROCKMANN
UNIVERSITÄT HEIDELBERG
SFB 123
IM NEUENHEIMER FELD 294
D-W-6900 HEIDELBERG, FRG

JÜRGEN EICHENAUER-HERRMANN
TECHNISCHE HOCHSCHULE DARMSTADT
FACHBEREICH MATHEMATIK
SCHLOSSGARTENSTRASSE 7
D-W-6100 DARMSTADT, FRG

Received on 1.3.1991