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## A MIXED DUEL UNDER ARBITRARY MOVING

In the paper a duel is considered in which Player I has one silent bullet, Player II has  $n$  noisy bullets, the accuracy functions are arbitrary and the players can move as they like. It is assumed that the maximal speed of Player I is greater than that of Player II.

**1. Introduction.** Consider the game which will be called the *game*  $(1, n)$ . Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is  $v_1$ , the maximal speed of Player II is  $v_2$ , and it is assumed that  $v_1 > v_2 \geq 0$ . Player I has one bullet, Player II has  $n$  bullets and this fact is known to both players. Player I hears every shot of Player II, but Player II does not hear the shot of Player I (does not know whether or not his opponent has fired).

At the beginning of the duel the players are at distance 1 from each other. Let  $P_1(s)$  ( $P_2(s)$ ) be the probability of succeeding (destroying the opponent) by Player I (II) when the distance between them is  $1 - s$ . The functions  $P_1(s)$ ,  $P_2(s)$  will be called *accuracy functions*. It is assumed that they are increasing and continuous in  $[0, 1]$  and that, by definition,  $P_i(s) = 0$  for  $s \leq 0$ ,  $P_i(1) = 1$ ,  $i = 1, 2$ .

Player I gains 1 if only he succeeds, gains  $-1$  if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero-sum game.

Suppose that Player II has fired all his shots and missed. In this case the best what Player I can do if he has a bullet yet is to reach Player II in pursuit and succeed surely. This behaviour of Player I is assumed in the paper.

As will be seen from the sequel, without loss of generality we can suppose that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

For definitions and results in the theory of games of timing see [3]–[5], [7], [10], [11], [16], [18].

**2. Duel (1, 1).** Consider first the case when the players have one bullet each.

Denote by  $\xi_{11}^\varepsilon$ ,  $0 < \varepsilon < 1$ , the strategy of Player I defined as follows: Let  $a_{11}$  be the point for which

$$(1) \quad P_1(a_{11}) + 2P_2(a_{11}) = 1.$$

Then if Player II has not fired his shot yet, Player I moves back and forth between 0 and  $a_{11}$  and fires his (silent) shot only when he is at  $a_{11}$ , with probability  $\varepsilon(1 - \varepsilon)^{i-1}$  that he fires when reaching  $a_{11}$  for the  $i$ th time,  $i = 1, 2, \dots$  After firing his shot Player I escapes to 0 and never approaches Player II. If Player II has already fired his noisy shot and missed Player I stops his wandering, reaches Player II and succeeds surely.

The strategy  $\eta_{11}$  of Player II consists simply in that he fires when his opponent reaches the point  $a_{11}$  for the first time.

**THEOREM 1.** *The strategy  $\xi_{11}^\varepsilon$  is  $\varepsilon$ -maximin and the strategy  $\eta_{11}$  is minimax in the game (1, 1). The value of the game is  $v_{11} = P_1(a_{11})$ .*

**PROOF.** We say that Player II fires a shot at  $(k, a')$  if he fires it when Player I is at the point  $a'$  and if this happens during the  $k$ th approach of  $a_{11}$  by Player I or during his  $(k - 1)$ th escape from  $a_{11}$ ,  $k = 1, 2, \dots$  This strategy of Player II will also be denoted by  $(k, a')$ . Let  $K(\xi; \eta)$  be the payoff function, the expected gain of Player I if he applies the strategy  $\xi$  and Player II applies the strategy  $\eta$ . We have, if  $a' < a_{11}$ ,

$$\begin{aligned} K(\xi_{11}^\varepsilon; k, a') &= \sum_{i=1}^{k-1} \varepsilon(1 - \varepsilon)^{i-1} P_1(a_{11}) + \sum_{i=k}^{\infty} \varepsilon(1 - \varepsilon)^{i-1} (-P_2(a') + 1 - P_2(a')) \\ &\geq \sum_{i=1}^{k-1} \varepsilon(1 - \varepsilon)^{i-1} P_1(a_{11}) + \sum_{i=k}^{\infty} \varepsilon(1 - \varepsilon)^{i-1} (1 - 2P_2(a_{11})) \\ &\stackrel{(1)}{=} \sum_{i=1}^{\infty} \varepsilon(1 - \varepsilon)^{i-1} P_1(a_{11}) = P_1(a_{11}). \end{aligned}$$

If  $a' = a_{11}$  we obtain

$$\begin{aligned} K(\xi_{11}^\varepsilon; k, a') &= \sum_{i=1}^{k-1} \varepsilon(1 - \varepsilon)^{i-1} P_1(a_{11}) + \varepsilon(1 - \varepsilon)^{k-1} (P_1(a_{11}) - P_2(a_{11})) \\ &\quad + \sum_{i=k+1}^{\infty} \varepsilon(1 - \varepsilon)^{i-1} (1 - 2P_2(a_{11})) \end{aligned}$$

$$= P_1(a_{11}) - \varepsilon(1 - \varepsilon)^{k-1}P_2(a_{11}) \geq P_1(a_{11}) - \varepsilon.$$

Thus

$$(2) \quad K(\xi_{11}^\varepsilon; \eta) \geq P_1(a_{11}) - \varepsilon$$

for any strategy  $\eta$  of Player II.

Denote by  $(i_1, i_2, a')$ ,  $i_1, i_2 = 0$  or  $1$ , the strategy of Player I in which he fires when he is at  $a'$  (if Player II has not fired yet), and  $i_1 = 0$  if he never reaches  $a_{11}$  before the moment of the shot,  $i_1 = 1$  otherwise. The number  $i_2 = 0$  if he never reaches  $a_{11}$  later (after the shot),  $i_2 = 1$  otherwise. Under the assumed behaviour of Player I after the shot of his opponent we have, if  $a' < a_{11}$ ,

$$\begin{aligned} K(0, 0, a'; \eta_{11}) &= P_1(a') < P_1(a_{11}), \\ K(0, 1, a'; \eta_{11}) &= P_1(a') - (1 - P_1(a'))P_2(a_{11}) < P_1(a_{11}), \\ K(1, 0, a'; \eta_{11}) &= 1 - 2P_2(a_{11}) = P_1(a_{11}), \\ K(1, 1, a'; \eta_{11}) &= 1 - 2P_2(a_{11}) = P_1(a_{11}). \end{aligned}$$

If  $a' = a_{11}$  we have similarly

$$K(i_1, i_2, a'; \eta_{11}) \leq P_1(a_{11}) \quad \text{for } i_1, i_2 = 0 \text{ or } 1.$$

If  $a' > a_{11}$ ,

$$K(1, i_2, a'; \eta_{11}) = 1 - 2P_2(a_{11}) = P_1(a_{11}) \quad \text{for } i_2 = 0 \text{ or } 1.$$

Thus

$$(3) \quad K(\xi; \eta_{11}) \leq P_1(a_{11})$$

for any strategy  $\xi$  of Player I.

From (2) and (3) it follows that the strategy  $\xi_{11}^\varepsilon$  is  $\varepsilon$ -maximin, the strategy  $\eta_{11}$  is minimax and  $v_{11} = P_1(a_{11})$  is the value of the game (1, 1).

**3. Duel (1, 1),  $\langle a \rangle$ .** Denote by (1, 1),  $\langle a \rangle$  the duel in which at the beginning the distance between the players is  $1 - a$ ,  $0 \leq a \leq 1$ . All other assumptions made about the duel (1, 1) remain in force.

Denote by  $\xi_{11}^\varepsilon(a)$ ,  $a \leq a_{11}$ , the strategy of Player I in the duel (1, 1),  $\langle a \rangle$  defined similarly to the strategy  $\xi_{11}^\varepsilon$  with the only difference that the player starts his first approach to  $a_{11}$  from the point  $a$  instead of 0.

Denote by  $\eta_{11}(a)$ ,  $a \leq a_{11}$ , the strategy defined in the same way as the strategy  $\eta_{11}$  in the duel (1, 1).

It is obvious that if  $a \leq a_{11}$  the strategy  $\xi_{11}^\varepsilon(a)$  is  $\varepsilon$ -maximin and the strategy  $\eta_{11}(a)$  is minimax, and the value of the game is  $v_{11}^a = P_1(a_{11})$ .

Denote by  $\tilde{\xi}_{11}(a)$  the strategy of Player I consisting in that if Player II has not fired yet Player I moves from  $a$  to 0 back and forth and fires only when he is at  $a$ . Thus he can also fire at the beginning of the duel. The probability

that he fires when reaching  $a$  for the  $i$ th time is as before  $\varepsilon(1 - \varepsilon)^{i-1}$ ,  $i = 1, 2, \dots$ , and Player I behaves as before when his opponent fires his noisy shot.

Denote by  $\langle a \rangle$  the moment of the beginning of the duel. Denote by  $\hat{\eta}_{11}(a)$  the strategy of Player II in which he simply fires his shot at time  $\langle a \rangle$ .

Finally, denote by  $\hat{\xi}_{11}(a)$  the strategy of Player I in which he fires at time  $\langle a \rangle$  and escapes.

Let  $\hat{a}_{11}$  be the point for which  $P_2(\hat{a}_{11}) = 1/2$ . We have

**THEOREM 2.** *In the game  $(1, 1)$ ,  $\langle a \rangle$  the strategy  $\xi_{11}^\varepsilon(a)$  is  $\varepsilon$ -maximin and the strategy  $\eta_{11}(a)$  is minimax when  $0 \leq a \leq a_{11}$ , the strategy  $\hat{\xi}_{11}(a)$  is  $\varepsilon$ -maximin and  $\hat{\eta}_{11}(a)$  is minimax when  $a_{11} \leq a \leq \hat{a}_{11}$ , the strategy  $\hat{\xi}_{11}(a)$  is maximin and the strategy  $\hat{\eta}_{11}(a)$  is minimax when  $\hat{a}_{11} \leq a \leq 1$ . The value  $v_{11}^a$  of the game is*

$$v_{11}^a = \begin{cases} 1 - 2P_2(a_{11}) = P_1(a_{11}) & \text{if } 0 \leq a \leq a_{11}, \\ 1 - 2P_2(a) & \text{if } a_{11} \leq a \leq \hat{a}_{11}, \\ 0 & \text{if } \hat{a}_{11} \leq a \leq 1. \end{cases}$$

The proof is omitted.

When  $P_1(s) = P_2(s) \stackrel{\text{def}}{=} P(s)$  we have

$$v_{11}^a = \begin{cases} 1/3 & \text{if } P(a) \leq 1/3, \\ 1 - 2P(a) & \text{if } 1/3 \leq P(a) \leq 1/2, \\ 0 & \text{if } P(a) \geq 1/2. \end{cases}$$

**4. Duel  $(1, n)$ ,  $n \geq 2$ .** Suppose now that Player I has one silent bullet, Player II has  $n$  noisy bullets and that the game begins when the distance between the players is 1. Suppose that Player II cannot fire two or more bullets at the same time. We now define the strategy  $\xi_{1n}^\varepsilon$  of Player I.

Suppose that Player II has not fired yet. In this case Player I moves from 0 to  $a_{1n}$  back and forth and can fire his (silent) shot only when he is at  $a_{1n}$ . The number  $a_{1n}$  is determined recursively from equation (1) and the equation

$$(4) \quad P_1(a_{1n}) + (1 + P_1(a_{1,n-1}))P_2(a_{1n}) = P_1(a_{1,n-1}), \quad n = 2, 3, \dots$$

The other properties of the strategy  $\xi_{1n}^\varepsilon$  are the same as those of the strategy  $\xi_{11}^\varepsilon$  with the only difference that if Player II fires (when Player I is at  $a'$ ) and misses, Player I applies the strategy  $\xi_{1,n-1}^\varepsilon$  (starting at the beginning from  $a'$  instead of 0).

The strategy  $\eta_{1n}$  of Player II is defined as follows: he fires at the first moment when Player I reaches  $a_{1n}$ , and later applies the strategy  $\eta_{1,n-1}$ .

It is easy to see that  $a_{1n}$  always exists and that  $a_{1n} < a_{1,n-1}$ . Thus the strategies  $\xi_{1n}^\varepsilon$  and  $\eta_{1n}$  are well defined.

We prove

**THEOREM 3.** *The strategy  $\xi_{1n}^\varepsilon$  is  $\varepsilon$ -maximin and the strategy  $\eta_{1n}$  is minimax in the game  $(1, n)$ . The value of the game is  $v_{1n} = P_1(a_{1n})$ .*

The proof is by induction on  $n$ . For  $n = 1$  the theorem is already proved (see Theorem 1). Assume that it holds for  $n - 1$ . Let  $(k, a')$  be defined as in the proof of Theorem 1. For  $a' < a_{1n}$  we have

$$K(\xi_{1n}^\varepsilon; k, a') \geq \sum_{i=1}^{k-1} \varepsilon(1-\varepsilon)^{i-1} P_1(a_{1n}) + \sum_{i=k}^{\infty} \varepsilon(1-\varepsilon)^{i-1} (-P_2(a') + (1 - P_2(a'))v_{1,n-1}).$$

Since  $0 < v_{1,n-1} < 1$  we obtain

$$K(\xi_{1n}^\varepsilon; k, a') \geq \sum_{i=1}^{k-1} \varepsilon(1-\varepsilon)^{i-1} P_1(a_{1n}) + \sum_{i=k}^{\infty} \varepsilon(1-\varepsilon)^{i-1} (-P_2(a_{1n}) + (1 - P_2(a_{1n}))v_{1,n-1}).$$

But from the inductive assumption  $v_{1,n-1} = P_1(a_{1,n-1})$ , thus

$$\begin{aligned} & -P_2(a_{1n}) + (1 - P_2(a_{1n}))v_{1,n-1} \\ & = -P_2(a_{1n}) + (1 - P_2(a_{1n}))P_1(a_{1,n-1}) = P_1(a_{1n}), \end{aligned}$$

by (4), and we have

$$K(\xi_{1n}^\varepsilon; k, a') \geq \sum_{i=1}^{k-1} \varepsilon(1-\varepsilon)^{i-1} P_1(a_{1n}) + \sum_{i=k}^{\infty} \varepsilon(1-\varepsilon)^{i-1} P_1(a_{1n}) = P_1(a_{1n})$$

If  $a' = a_{1n}$  then

$$\begin{aligned} K(\xi_{1n}^\varepsilon; k, a') & \geq \sum_{i=1}^{k-1} \varepsilon(1-\varepsilon)^{i-1} P_1(a_{1n}) + \varepsilon(1-\varepsilon)^{k-1} (P_1(a_{1n})(1 - P_2(a_{1n})) \\ & \quad - P_2(a_{1n})(1 - P_1(a_{1n}))) \\ & \quad + \sum_{i=k+1}^{\infty} \varepsilon(1-\varepsilon)^{i-1} (-P_2(a_{1n}) + (1 - P_2(a_{1n}))v_{1,n-1}) \\ & = P_1(a_{1n}) - \varepsilon(1-\varepsilon)^{k-1} P_2(a_{1n}) > P_1(a_{1n}) - \varepsilon. \end{aligned}$$

On the other hand, if Player I fires when he is at  $a' < a_{1n}$  and never reaches  $a_{1n}$  (denote such a strategy simply by  $a'$ ) we obtain

$$K(a'; \eta_{1n}) = P_1(a') \leq P_1(a_{1n}).$$

Define  $Q_2(s) = 1 - P_2(s)$ . If Player I fires either when he is at  $a'$  with  $a_{1,k+1} < a' < a_{1k}$ , or at  $a' = a_{1,k+1}$  after the  $(n - k + 1)$ th shot of Player II, and in both cases never reaches the point  $a_{1k}$ , we have

$$\begin{aligned} K(a'; \eta_{1n}) &= -(1 - Q_2(a_{1n}) \dots Q_2(a_{1,k+1})) + Q_2(a_{1n}) \dots Q_2(a_{1,k+1})P_1(a') \\ &\leq -1 + Q_2(a_{1n}) \dots Q_2(a_{1,k+1})(1 + P_1(a_{1k})) = P_1(a_{1n}), \end{aligned}$$

since by (4)

$$(5) \quad Q_2(a_{1k}) = \frac{1 + P_1(a_{1k})}{1 + P_1(a_{1,k-1})}.$$

Let  $\langle a_{1k} \rangle$  denote the first moment when Player I reaches the point  $a_{1k}$ . If Player I fires at time  $\langle a_{1k} \rangle$  and never reaches the point  $a_{1,k-1}$ ,  $k = 2, \dots, n$ , or fires at time  $\langle a_{11} \rangle$  for  $k = 1$  we have

$$\begin{aligned} K(a'; \eta_{1n}) &= -1 + Q_2(a_{1n}) \dots Q_2(a_{1,k+1}) \\ &\quad + Q_2(a_{1n}) \dots Q_2(a_{1,k+1})(P_1(a_{1k}) - P_2(a_{1k})) \\ &\stackrel{(5)}{=} -1 + \frac{1 + P_1(a_{1n})}{1 + P_1(a_{1k})}(1 + P_1(a_{1k}) - P_2(a_{1k})) < P_1(a_{1n}). \end{aligned}$$

Suppose that Player I does not fire though he reaches  $a_{11}$ . For such a strategy, say  $\xi$ ,

$$\begin{aligned} K(\xi; \eta_{1n}) &= -1 + 2Q_2(a_{1n}) \dots Q_2(a_{11}) \\ &= -1 + 2 \frac{1 + P_1(a_{1n})}{1 + P_1(a_{11})} Q_2(a_{11}) = P_1(a_{1n}) \end{aligned}$$

by (1).

Since moving after his shot in the direction of Player II is for Player I no better than escape (Player I has no bullet) and since firing from the point  $a'$ ,  $a' > a''$ , reached by Player I before  $a''$  is for him not worse than firing from  $a''$ , if Player II applies his strategy  $\eta_{1n}$ , this ends the proof of the theorem.

**5. Final remarks.** In the definition of the strategy  $\xi_{1n}^\varepsilon$  instead of the geometrical distribution  $p_i = \varepsilon(1 - \varepsilon)^{i-1}$  many other distributions can be applied to obtain an  $\varepsilon$ -maximin strategy.

Notice that  $v_{1n} > 0$  for any  $P_1(s)$ ,  $P_2(s)$  and  $n!$  Thus the fact that Player I has greater speed and a silent bullet has substantial influence on the value of the game.

When  $P_1(s) = P_2(s) = P(s)$  we obtain from (1) and (4)

$$P(a_{1n}) = \frac{1}{2^{n+1} - 1}, \quad n = 1, 2, \dots$$

Duels under arbitrary moving, as far as the author knows, have never been considered before, except in the papers of the author (see [14], [15]).

For other results in the theory of games of timing see [1], [2], [6], [8], [9], [12], [13], [17].

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