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ASYMPTOTIC STABILITY OF TESTS BASED ON ORDER STATISTICS FOR THE SCALE PARAMETER

Abstract. Let $\mathcal{F} = \{\bigotimes_{i=1}^{\infty} F_\lambda : F_\lambda(x) = F(x/\lambda),\ \lambda > 0\}$ be a given parametric model. We consider the one sided scale testing problem in the class of consistent tests based on linear combinations of order statistics. It turns out that under the assumed violation of $\mathcal{F}$ the test based on the largest order statistic is asymptotically most power-stable and minimax for a given broad class of local alternatives.

1. Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables (r.v.'s). We assume that the distribution function (d.f.) belongs to the family $\{F_\lambda : F_\lambda(x) = F(x/\lambda),\ \lambda > 0\}$, where F is a fixed life d.f. ($F(0) = 0$) and the scale parameter is to be tested. Thus we consider the parametric model $\mathcal{F} = \{\bigotimes_{i=1}^{\infty} F_\lambda : \lambda > 0\}$. Consider the problem of testing the hypotheses:

$$H_0 : \lambda \le \lambda_0 \ (\lambda_0 > 0) \quad \text{versus} \quad H_1 : \lambda > \lambda_0\,. \tag{1}$$

Let α be an arbitrary number, $0 < \alpha < 1$. Let $C(\alpha)$ be a specified class of asymptotic tests for (1), all corresponding to the same asymptotic level of significance α. Thus for every sequence $\{\varphi_n\} \in C(\alpha)$,

$$\lim_{n\to\infty} \sup_{\lambda \le \lambda_0} E_{F_\lambda^n} \varphi_n = \alpha\,, \tag{2}$$

where $\varphi_n = \varphi_n(X_1, \ldots, X_n)$ and $E_{F_\lambda^n}\varphi_n$ denotes the expectation of φ_n if the sample $(X_1, \ldots, X_n)$ comes from F_λ.

Assume that all tests in $C(\alpha)$ are consistent. A well-known procedure of studying tests in $C(\alpha)$ (see e.g. [4]) is based on the method of local alternatives. It is connected with the power of tests evaluated under a class of sequences of alternatives approaching the hypothesis H_0. This procedure

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gives a possibility to provide numerical approximations to the true power functions. On the other hand, it reveals the general structure of power functions. Now, suppose that the model $\mathcal{F}$ is violated and the violation is described in some specified way. Then we can study the robustness of tests in $C(\alpha)$ by investigating their power on the class of sequences of violated alternatives.

2. Preliminaries. Let $\underline{X}_n$ be a sample of size n and let $\varphi_n = \varphi_n(\underline{X}_n)$ be a level-α test for (1). Given $\lambda > 0$, let $\pi_n(F_\lambda)$ denote a fixed set of d.f.'s that contains F_λ. Suppose that due to measurement errors the sample $\underline{X}_n$ has an unknown distribution $\boldsymbol{G} \in \boldsymbol{\pi}_n(F_\lambda)$, where $\boldsymbol{\pi}_n(F_\lambda) = \{\bigotimes_{i=1}^n G_i : G_i \in \pi_n(F_\lambda)\}$ and $\bigotimes_{i=1}^n G_i$ stands for the product of $G_1, \ldots, G_n$.

If $\boldsymbol{G}$ runs through the set $\boldsymbol{\pi}_n(F_\lambda), \lambda > \lambda_0$, then

$$r_\alpha(\varphi_n, \lambda) = \sup\{|E_{\boldsymbol{G}_1}\varphi_n - E_{\boldsymbol{G}_2}\varphi_n| : \boldsymbol{G}_1, \boldsymbol{G}_2 \in \boldsymbol{\pi}_n(F_\lambda)\} \tag{3}$$

is the oscillation of the power of φ_n over $\boldsymbol{\pi}_n(F_\lambda)$ and gives us a measure of stability (robustness) of the test φ_n with respect to its power, under the violation $\boldsymbol{\pi}_n$ (see [7]).

Let Γ denote a certain class of sequences $\{\lambda_n\}$ converging to λ_0 with $\lambda_n > \lambda_0$. Let Π be a specified class of sequences of neighbourhoods $\{\pi_n\}$ defined on $\{F_\lambda : \lambda > 0\}$. We assume that for every sequence of alternative hypotheses $\{F_{\lambda_n}\}$ the corresponding sequence of violations $\{\pi_n(F_{\lambda_n})\}$ does not contain any distribution from the hypothesis H_0 in the sense that for large n

$$\{F_\lambda : \lambda \le \lambda_0\} \cap \pi_n(F_{\lambda_n}) = \emptyset \quad \text{for every } \{(\lambda_n, \pi_n)\} \in \Gamma \times \Pi\,. \tag{4}$$

DEFINITION 1. We shall say that, relative to (Γ, Π), the test $\{\varphi_n\} \in C(\alpha)$ is *more stable* than the test $\{\psi_n\} \in C(\alpha)$ if, for any sequence $\{(\lambda_n, \pi_n)\} \in \Gamma \times \Pi$,

$$\limsup_{n\to\infty} r_\alpha(\varphi_n, \lambda_n) \le \liminf_{n\to\infty} r_\alpha(\psi_n, \lambda_n)\,,$$

and there exists $\{(\lambda_n^0, \pi_n^0)\} \in \Gamma \times \Pi$ for which the inequality is strict.

DEFINITION 2. We shall say that, relative to (Γ, Π), the test $\{\varphi_n\} \in C(\alpha)$ is *most stable* within the class $C(\alpha)$ if it is more stable than any other test in $C(\alpha)$.

We adopt the following notation. If $\varphi = \{\varphi_n\}$ and $\underline{X}_n$ has the distribution $\boldsymbol{G}_n$ then $E_{\boldsymbol{G}_n}\varphi_n$ is denoted by $\beta_\varphi(\boldsymbol{G}_n)$.

DEFINITION 3. We shall say that, relative to (Γ, Π) and $C(\alpha)$, the test $\varphi \in C(\alpha)$ is *asymptotically minimax* if for any other test $\psi \in C(\alpha)$,

$$\inf\{\liminf_{n\to\infty} \beta_\varphi(\boldsymbol{G}_n)\} > \inf\{\liminf_{n\to\infty} \beta_\psi(\boldsymbol{G}_n)\}\,,$$

where the infima are taken over all sequences $\{(\lambda_n, \pi_n)\} \in \Gamma \times \Pi$ with $G_n \in \pi_n(F_{\lambda_n})$.

3. Stability of asymptotic tests based on order statistics. In the sequel we assume that the d.f. F has a continuous positive density f such that

$$\text{(i)} \qquad \liminf_{x\to\infty} \frac{f(x)}{1-F(x)} > 0, \quad \limsup_{x\to\infty} \frac{f(x)}{1-F(x)} < \infty.$$

For the testing problem (1) we consider consistent tests based on order statistics $X_{1:n} \le \ldots \le X_{n:n}$ of the form

$$(5) \quad \varphi_n(aJ, \overline{a}p) = \begin{cases} 1 & \text{if } \frac{a}{n}\sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n} + \overline{a}X_{[np]:n} > c_n(aJ, \overline{a}p), \\ 0 & \text{otherwise,} \end{cases}$$

where $a = 0$ or 1, $\overline{a} = 1 - a$, $J \ne 0$ is an arbitrary nonnegative function with a continuous derivative on $[0,1]$, i.e. $J \in C_+^1[0,1]$, p is an arbitrary point in $(0,1]$ with $np \ge 1$ ($[t]$ denotes the integer part of t).

We assume that each sequence $\{\varphi_n\}$ satisfies the condition (2). It determines the asymptotic properties of the corresponding sequence $\{c_n\}$.

Let $\le_{st}$ denote the stochastic ordering. Let G, H be absolutely continuous life d.f.'s such that

(ii) $H \le_{st} F \le_{st} G$ and $F(x) \ne H(x)$ for every $x > 0$,

(iii) $\int_0^\infty x^3\, dG(x) < \infty$,

(iv) $\lim_{x\to\infty} \frac{1-G(x)}{\sqrt{1-F(x)}} = 0.$

Given n, we define a neighbourhood of the d.f. F_λ as follows:

$$(6) \qquad \pi_n(F_\lambda) = \{S \text{ d.f.'s} : (1-\delta_n)F_\lambda + \delta_n H_\lambda \le_{st} S \le_{st} (1-\varepsilon_n)F_\lambda + \varepsilon_n G_\lambda\},$$

where $\lambda > 0$, $\varepsilon_n, \delta_n \in [0,1]$, $G_\lambda(x) = G(x/\lambda)$, $H_\lambda(x) = H(x/\lambda)$. By (ii) we have $F_\lambda \in \pi_n(F_\lambda)$. It should be mentioned that for $\varepsilon_n = \delta_n, \lambda > 0$ we have $\pi_n(F_\lambda) = \{(1-\varepsilon_n)F_\lambda + \varepsilon_n S \text{ d.f.'s} : G_\lambda \le S \le H_\lambda\}$, i.e. a "neighbourhood of ε_n-contamination type with restrictions". Neighbourhoods generated by stochastic ordering have been considered in [1].

Let Γ be the class of all sequences $\{\lambda_n\}$ with $\lambda_n > \lambda_0$ and such that

$$0 < \limsup_{n\to\infty} \sqrt{n}(\lambda_n/\lambda_0 - 1) < \infty.$$

Let Π consist of all sequences $\{\pi_n\}$ of neighbourhoods of the form (6), where

$$\limsup_{n\to\infty} \sqrt{n}\varepsilon_n < \infty, \qquad \limsup_{n\to\infty} \sqrt{n}\delta_n < \infty.$$

From (6), (i), (ii) one can deduce that (Γ, Π) satisfies (4).

Let $C(\alpha)$ be the class of all tests $\{\varphi_n\}$ which are defined by (5) and (2).

Under the above definitions and assumptions, we prove the following theorem.

THEOREM. *Let $\alpha \in (0,1)$ and $\psi = \{\varphi_n(0,1)\} \in C(\alpha)$. Then relative to (Γ, Π) and $C(\alpha)$,*

(a) *ψ is most stable,*

(b) *ψ is asymptotically minimax.*

4. Proof. We first present some auxiliary lemmas. Let F satisfy the conditions of Section 3 and let S be an absolutely continuous life d.f. with the third moment finite. Let $R_S = -\ln(1-S)$ denote the hazard function of the d.f. S. We assume that $\lim_{n\to\infty}\sqrt{n}\varrho_n = \varrho$, $\lim_{n\to\infty}\sqrt{n}(\tau_n - 1) = \tau$, where $\{\varrho_n\} \subset [0,1]$, $\{\tau_n - 1\} \subset [0,\infty)$ and $\varrho, \tau \in [0,\infty)$. We define $S_n = (1-\varrho_n)F + \varrho_n S$.

Let Φ be the standard normal d.f. For simplicity, $E_{S_n}\varphi(\underline{X}_n)$ is denoted by $E_S\varphi(\underline{X}_n)$. Analogously, $\mathrm{Var}_S\varphi(\underline{X}_n)$ denotes the variance of $\varphi(\underline{X}_n)$.

LEMMA 1. *Let $p \in (0,1)$.*

(i) *If $\lim_{n\to\infty}\sqrt{n}(x_n - S_n^{-1}(p)) = \dfrac{\sqrt{p(1-p)}x}{f(F^{-1}(p))}$ then $\lim_{n\to\infty} P_{S_n}\{X_{[np]:n} \le x_n\} = \Phi(x)$.*

(ii) *$\lim_{n\to\infty}\sqrt{n}(F^{-1}(p) - \tau_n S_n^{-1}(p)) = \dfrac{\sqrt{p(1-p)}}{f(F^{-1}(p))}M_p(\varrho,\tau,S)$, where*

$$M_p(\varrho,\tau,S) = \left(\frac{\varrho(S(F^{-1}(p)) - p)}{f(F^{-1}(p))} - \tau F^{-1}(p)\right)\frac{f(F^{-1}(p))}{\sqrt{p(1-p)}}. \tag{7}$$

Proof. (i) follows from the Berry–Esseen theorem applied to the sum of independent exponentially distributed r.v.'s and is a simple modification of the proof of Corollary 1 of [2].

(ii) follows immediately from the fact that for $t = F^{-1}(p)$, $t_n = S_n^{-1}(p)$ we have

$$\sqrt{n}(t - t_n)\frac{F(t) - F(t_n)}{t - t_n} = \sqrt{n}\varrho_n(S(t_n) - F(t_n))$$

and $\sqrt{n}(t - \tau_n t_n) = \tau_n\sqrt{n}(t - t_n) - t\sqrt{n}(\tau_n - 1)$.

LEMMA 2. *Let $J \in C^1_+[0,1]$,*

$$L_n = \frac{1}{n}\sum_{i=1}^{n} J\left(\frac{i}{n+1}\right)X_{i:n},$$

$$\sigma_F^2(J) = \int_0^\infty\int_0^\infty J(F(x))J(F(y))(F(\min(x,y)) - F(x)F(y))\,dx\,dy.$$

Then

(i) $\lim_{n\to\infty} n \operatorname{Var}_{S_n} L_n = \sigma_F^2(J)$.

(ii) *If* $\lim_{n\to\infty} \frac{x_n - E_{S_n} L_n}{\sqrt{\operatorname{Var}_{S_n} L_n}} = x$ *then* $\lim_{n\to\infty} P_{S_n}\{L_n \le x_n\} = \Phi(x)$.

(iii) $\lim_{n\to\infty} \sqrt{n}(E_F L_n - \tau_n E_{S_n} L_n) = \sigma_F(J) N_J(\varrho, \tau, S)$, *where*

$$N_J(\varrho,\tau,S) = \int_0^\infty J(F(x))(\varrho(S(x) - F(x)) - \tau x f(x))\,dx/\sigma_F(J)\,. \tag{8}$$

Proof. (i) is a direct consequence of the proof of Theorem 1 of [5], and (ii) follows from Corollary 4.2 of [8].

(iii) Let

$$V_n = \sum_{i=1}^n \Big(\int_{(i-1)/n}^{i/n} J(x)\,dx\Big) X_{i:n}\,.$$

Note (see e.g. [3]) that

$$\sqrt{n}(E_F V_n - \tau_n E_{S_n} V_n) = \sqrt{n} E \int_0^\infty (\xi(\Gamma_n(F(x))) - \tau_n \xi(\Gamma_n(S_n(x))))\,dx \tag{9}$$
$$+ c\sqrt{n}(E_F X - \tau_n E_{S_n} X)\,,$$

where Γ_n denotes the empirical d.f. based on independent uniform $(0,1)$ r.v.'s, $c = \int_0^1 J(x)\,dx$, $\xi(u) = \int_u^1 J(x)\,dx - (1-u)c$, $u \in [0,1]$.

It is easily shown that $\sup\{|\xi''(u)| : u \in [0,1]\} = \sup\{|J'(u)| : u \in [0,1]\} = C_J < \infty$, $E\Gamma_n(x) = x$, $E(\Gamma_n(x) - x)^2 = x(1-x)/n$.

Consequently, using the Taylor expansion of order two and the Fubini theorem we find that for $P_n = F$ or S_n,

$$\begin{aligned} E\int_0^\infty \xi(\Gamma_n(P_n(x)))\,dx &= \int_0^\infty \xi(P_n(x))\,dx + o_n\,,\\ \int_0^\infty (\xi(S_n(x)) - \xi(F(x)))\,dx &= \int_0^\infty \xi'(F(x))(S_n(x) - F(x))\,dx + q_n\,, \end{aligned} \tag{10}$$

where

$$|o_n| \le \frac{C_J}{2n}\int_0^\infty x\,dP_n(x), \quad |q_n| \le \varrho_n^2 \frac{C_J}{2}\Big(\int_0^\infty x\,dF(x) + \int_0^\infty x\,dS(x)\Big).$$

Thus from (9), (10) it follows that

$$\lim_{n\to\infty} \sqrt{n}(E_F V_n - \tau_n E_{S_n} V_n) = \varrho \int_0^\infty \xi'(F(x))(F(x) - S(x))\,dx$$

$$-\tau \int_0^\infty \xi(F(x))\,dx + c(\varrho(E_F X - E_S X) - \tau E_F X).$$

In view of the definitions of c and ξ and the fact that

$$\int_0^\infty \int_{F(x)}^1 J(u)\,du\,dx = \int_0^\infty J(F(x)) x f(x)\,dx$$

the above equality proves the desired result for the statistic V_n. Since

$$\sqrt{n} E_{P_n} |L_n - V_n| \le \frac{C_J}{\sqrt{n}} E_{P_n} X$$

for $P_n = F$ or S_n, the proof is complete.

Under the assumptions (i), (ii), (iv) of Section 3 we have

LEMMA 3. *Let* $\lim_{n\to\infty} n(1 - F(y_n)) = \beta$, $\beta \in (0,\infty)$. *Then*

(i) $\lim\limits_{n\to\infty} n(1 - F(y_n/\tau_n)) = \beta$,

(ii) $\lim\limits_{n\to\infty} n\varrho_n (H(y_n/\tau_n) - G(y_n/\tau_n))/F(y_n/\tau_n) = 0$.

Proof. (i) In view of the obvious equality

$$n(1 - F(y_n/\tau_n)) = n(1 - F(y_n)) \exp(R_F(y_n) - R_F(y_n/\tau_n)), \quad \tau_n \ge 1,$$

it suffices to show that

$$\limsup_{n\to\infty} (R_F(y_n) - R_F(y_n/\tau_n)) = 0.$$

From the assumption it follows that $\lim_{n\to\infty}(\ln n - R_F(y_n)) = \ln\beta$ and consequently $\lim_{n\to\infty} R_F(y_n)/\sqrt{n} = 0$. Furthermore, the condition (i) of Section 3 states that

$$\underline{q} = \liminf_{x\to\infty} R_F'(x) > 0, \quad \overline{q} = \limsup_{x\to\infty} R_F'(x) < \infty.$$

Thus the obvious inequality

$$\limsup_{n\to\infty} (R_F(y_n) - R_F(y_n/\tau_n)) \le \frac{\overline{q}\tau}{\underline{q}} \limsup_{n\to\infty} R_F(y_n)/\sqrt{n}$$

completes the proof.

(ii) Note that

$$0 \le n\varrho_n \left(\frac{H(y_n/\tau_n)}{F(y_n/\tau_n)} - 1 \right) \le \varrho_n n(1 - F(y_n/\tau_n))/F(y_n/\tau_n),$$

$$n\varrho_n \left(1 - \frac{G(y_n/\tau_n)}{F(y_n/\tau_n)} \right) = \frac{n(1 - F(y_n/\tau_n))}{F(y_n/\tau_n)} \left(\frac{1 - G(y_n/\tau_n)}{\sqrt{1 - F(y_n/\tau_n)}} \frac{\sqrt{n}\varrho_n}{\sqrt{n(1 - F(y_n/\tau_n))}} - \varrho_n \right).$$

Consequently, by the condition (iv) of Section 3 and part (i) of the lemma the proof is complete.

Proof of Theorem. Let $\{(\lambda_n, \pi_n)\}$ be an arbitrary sequence in $\Gamma \times \Pi$, where, in view of (6), $\{\pi_n\}$ is defined by $\{\varepsilon_n\}$ and $\{\delta_n\}$. Let

$$\underset{\sim}{\varepsilon} = \liminf_{n\to\infty} \sqrt{n}\varepsilon_n, \quad \underset{\sim}{\delta} = \liminf_{n\to\infty} \sqrt{n}\delta_n,$$
$$\underset{\sim}{\tau} = \liminf_{n\to\infty} \sqrt{n}(\lambda_n/\lambda_0 - 1).$$

For the upper limits we use the notations $\widetilde{\varepsilon}, \widetilde{\delta}$ and $\widetilde{\tau}$ respectively.

Let $\{\varphi_n\} \in C(\alpha)$. It is easy to note that $\varphi_n(x_1, \ldots, x_n)$ is based on the statistic which is nondecreasing in each x_i, $i = 1, \ldots, n$. Consequently, we obtain (see e.g. [6])

$$r_\alpha(\varphi_n, \lambda_n) = E_{G_n}\varphi_n - E_{H_n}\varphi_n, \tag{11}$$

where $G_n = (1-\varepsilon_n)F_{\lambda_n} + \varepsilon_n G_{\lambda_n}$, $H_n = (1-\delta_n)F_{\lambda_n} + \delta_n H_{\lambda_n}$.

Let $p \in (0,1)$. Condition (2) implies that

$$\lim_{n\to\infty} \sqrt{n}(c_n - \lambda_0 F^{-1}(p)) = u_\alpha \lambda_0 \sqrt{p(1-p)}/f(F^{-1}(p)),$$

where $c_n = c_n(0,p)$ and $u_\alpha = \Phi^{-1}(1-\alpha)$.

Let P_n be equal to G_n or H_n. Since

$$\sqrt{n}(c_n - P_n^{-1}(p)) = \sqrt{n}(c_n - \lambda_0 F^{-1}(p)) + \lambda_0\sqrt{n}\left(F^{-1}(p) - \frac{P_n^{-1}(p)}{\lambda_0}\right),$$

(11) and Lemma 1 yield

$$\liminf_{n\to\infty} r_\alpha(\varphi_n, \lambda_n) \geq \inf_{\underset{\sim}{\tau}\leq\tau\leq\widetilde{\tau}} (\Phi(u_\alpha + M_p(\underset{\sim}{\delta}, \tau, H)) - \Phi(u_\alpha + M_p(\underset{\sim}{\varepsilon}, \tau, G))), \tag{12}$$

where $\varphi_n = \varphi_n(0,p)$ and M_p is defined by (7).

Analogously, from (11) and Lemma 2 we obtain

$$\liminf_{n\to\infty} r_\alpha(\varphi_n, \lambda_n) \geq \inf_{\underset{\sim}{\tau}\leq\tau\leq\widetilde{\tau}} (\Phi(u_\alpha + N_J(\underset{\sim}{\delta}, \tau, H)) - \Phi(u_\alpha + N_J(\underset{\sim}{\varepsilon}, \tau, G))), \tag{13}$$

where $\varphi_n = \varphi_n(J, 0)$ and N_J is defined by (8).

Consider the test $\{\varphi_n(0,1)\} \in C(\alpha)$. Let $y_n = c_n(0,1)/\lambda_0$, $\tau_n = \lambda_n/\lambda_0$. By (11) we get

$$\begin{aligned} r_\alpha(\varphi_n, \lambda_n) &= ((1-\delta_n)F(y_n/\tau_n) + \delta_n H(y_n/\tau_n))^n \\ &\quad - ((1-\varepsilon_n)F(y_n/\tau_n) + \varepsilon_n G(y_n/\tau_n))^n \\ &= F^n(y_n/\tau_n)((1+h_n/n)^n - (1-g_n/n)^n), \end{aligned}$$

where

$$g_n = n\varepsilon_n\left(1 - \frac{G(y_n/\tau_n)}{F(y_n/\tau_n)}\right), \quad h_n = n\delta_n\left(\frac{H(y_n/\tau_n)}{F(y_n/\tau_n)} - 1\right).$$

Obviously, condition (2) states that $\lim_{n\to\infty} n(1-F(y_n)) = -\ln(1-\alpha)$. Thus Lemma 3 implies that

$$\limsup_{n\to\infty} r(\varphi_n, \lambda_n) = 0\,, \quad \text{where } \varphi_n = \varphi_n(0,1)\,. \tag{14}$$

By the assumption (ii) of Section 3 we obtain

$$M_p(\underset{\sim}{\delta}, \tau, H) > M_p(\underset{\sim}{\varepsilon}, \tau, G)\,, \quad N_J(\underset{\sim}{\delta}, \tau, H) > N_J(\underset{\sim}{\varepsilon}, \tau, G)$$

for every $0 \le \tau < \infty$, $\underset{\sim}{\varepsilon} \ge 0$ and $\underset{\sim}{\delta} > 0$. Consequently, by (12)–(14) the proof of part (a) of the theorem is complete.

(b) The same argument as in (11) implies that

$$\inf\{\beta_\varphi(G_n) : G_n \in \pi_n(F_{\lambda_n})\} = E_{H_n}\varphi_n\,.$$

Therefore from the proof of (a) one concludes that for $\psi = \{\varphi_n(0,1)\} \in C(\alpha)$,

$$\inf\{\liminf_{n\to\infty} E_{H_n}\varphi_n : 0 < \widetilde{\tau} < \infty,\ 0 \le \widetilde{\delta} < \infty\} = \alpha\,.$$

Analogously, for any other test $\varphi \in C(\alpha)$ we have

$$\inf\{\liminf_{n\to\infty} E_{H_n}\varphi_n : 0 < \widetilde{\tau} < \infty,\ 0 \le \widetilde{\delta} < \infty\} = 0\,,$$

which completes the proof.

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