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# FULL-INFORMATION BEST CHOICE PROBLEMS WITH IMPERFECT OBSERVATION AND A RANDOM NUMBER OF OBSERVATIONS

**1. Introduction.** The following full-information best choice problem was studied by Gilbert and Mosteller [3]. A known number, $K$, of *iid* r.v.'s $X_1, \ldots, X_K$ from a known continuous distribution $F$ are observed sequentially with the object of choosing the largest. After $X_n$ is observed it must be chosen (and the observation is terminated) or rejected (and the observation is continued). Neither recall nor uncertainty of selection is allowed and one choice must be made.

For a finite number of observations this problem was solved basing on a heuristic argument by Gilbert and Mosteller [3]. The so-called *monotone case* was obtained. The optimal strategy is to accept (if possible) the first observation $X_n$ which is largest so far and exceeds $x_n$, where the sequence of the optimal decision labels $x_1, \ldots, x_K$ is non-increasing.

The full-information best choice problem when the number of observations $K$ is random was first posed by Sakaguchi [6] and Bojdecki [1] in some continuous time version. Porosiński [4] has characterized a class of distributions of $K$ for which the *monotone case* occurs and has given the solution for this case. The special examples when $K$ is geometric or uniform were considered in detail.

The problem with imperfect observation was first considered by Enns [2]. In his model the number of observations is known and the only available information is whether the observed r.v. is greater than, or less than some level specified by the observer (the exact value is not known). This model could describe e.g. the situation in destructive testing where each item is tested on whether it breaks or survives at some level of strength. This corresponds to the single level case of Gilbert and Mosteller [3]. Sakaguchi

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[6] generalized Enns' model to five natural kinds of objectives and two types of strategies.

In this paper the full-information best choice problem with imperfect observation is considered when the number $K$ of observations is allowed to be a r.v. with known distribution. Since $K$ is unknown the observer faces an additional risk. If he rejects any observation, he may then discover it was the last one, in which case he receives nothing at all.

We shall consider the following five cases where the objective is to choose:

(A) the largest r.v. by accepting one r.v.;
(B) the largest or the second largest by accepting exactly once;
(C) the largest by accepting twice;
(D) both the largest and the second largest by accepting twice;
(E) the largest or the second largest by accepting twice.

The optimal strategy for each of these problems when $K$ is geometric or uniform is obtained. In the uniform cases (A)–(E) and in case (E) when $K$ is concentrated at $N$ (this case was not considered by Sakaguchi [6]) simple asymptotically optimal strategies are found and the asymptotic probabilities of winning are obtained.

**2. Preliminaries.** Assume that $X_1, X_2, \ldots$ is a sequence of *iid* r.v.'s with a continuous distribution function $F$, defined on a probability space $(\Omega, \mathcal{F}, P)$, and the number of observations $K$ is a r.v. independent of the sequence $(X_n)_{n=1}^{\infty}$ with a known distribution

$$P(K = n) = p_n, \qquad n = 0, 1, 2, \ldots, \qquad \sum_{n=0}^{\infty} p_n = 1 .$$

Consider the five cases (A)–(E) of the full-information best choice problem defined in the introduction. The observer specifies a decision level $L$ and is only informed whether the observed r.v. is either greater or less than $L$. As long as the observation is less than or equal to $L$ it is rejected and as soon as it is greater it is accepted. If the r.v. $K$ is bounded, i.e. $P(K \le N) = 1$ and $p_N > 0$, then the observer slightly modifies his strategy. If no r.v. is accepted until $N-1$ for Cases (A) and (B) then the last observation $X_N$ is accepted (if it occurs) with respect to its value (i.e. the decision level for $N$ is 0). Analogously for (C), (D) and (E), if no r.v. is accepted until $N-2$ or one r.v. is accepted until $N-1$ then the last two r.v.'s are accepted or the last r.v. is accepted, respectively.

In each problem an event in which the objective is achieved is called a *win* and its probability is called the *probability of winning.* A strategy which maximizes the probability of winning is called *optimal.*

Since the distribution $F$ is known and continuous, without loss of generality it is assumed in the next sections that the observations have the uniform distribution on $[0,1]$.

**3. Probabilities of winning.** Let $N = \sup\{n : p_n > 0\}$ ($N = +\infty$ if $K$ is an unbounded r.v.).

Case (A). If either $1 \le n < N$ and $1 \le s \le n$, or $n = N$ and $1 \le s < N$, then

$$\begin{aligned} &P(\text{stop at } s \,\&\, \text{win} \mid K = n) \\ &\quad = P(X_1 \le L, \ldots, X_{s-1} \le L,\ X_s > L,\ X_{s+1} \le X_s, \ldots, X_n \le X_s) \\ &\quad = L^{s-1} \int_L^1 x_s^{n-s}\, dx_s = (L^{s-1} - L^n)/(n-s+1)\,. \end{aligned}$$

Moreover,

$$\begin{aligned} &P(\text{stop at } N \,\&\, \text{win} \mid K = N) \\ &\qquad = P(X_1 \le L, \ldots, X_{N-1} \le L, X_N > \max(X_1, \ldots, X_{N-1})) \\ &\qquad = L^{N-1}(1-L) + L^N/N\,. \end{aligned}$$

Thus

$$P(\text{win}) = \sum_{n=1}^{N} p_n L^n \sum_{r=1}^{n} (L^{-r} - 1)/r + p_N L^N/N\,.$$

The second term on the right side of the above equality is the additional probability which results from the modification of the strategy at the last step when the number of observations is a bounded r.v.

Case (B). If either $1 \le n < N$ and $1 \le s \le n$, or $n = N$ and $1 \le s < N$, then

$$\begin{aligned} &P(\text{stop at } s \,\&\, \text{win} \mid K = n) \\ &\quad = P(X_1 \le L, \ldots, X_{s-1} \le L,\ X_s > L,\ \max(X_{s+1}, \ldots, X_n) \le X_s \\ &\qquad\qquad \text{or exactly one of } X_{s+1}, \ldots, X_n \text{ is greater than } X_s) \\ &\quad = L^{s-1} \int_L^1 (x_s^{n-s} + (n-s)x_s^{n-s-1}(1-x_s))\, dx_s \\ &\quad = L^{s-1}(1 - L^{n-s} - (n-s-1)(1 - L^{n-s+1})/(n-s+1))\,. \end{aligned}$$

Moreover,

$$\begin{aligned} P(\text{stop at } N \,\&\, \text{win} \mid K = N) = P(X_1 \le L, \ldots, X_{N-1} \le L,& \\ X_N \text{ is greater than the second largest of } X_1, \ldots, X_{N-1})& \end{aligned}$$

$$= \int_0^L dy \int_0^y (N-1)(N-2)z^{N-3}(1-z)\,dz = L^{N-1}(1-L) + 2L^N/N\,.$$

Thus

$$P(\text{win}) = \sum_{n=1}^{N} p_n L^n \sum_{r=1}^{n} (2(L^{-r}-1)/r - L^{-1}(1-L)) + 2p_N L^N/N\,.$$

Case (C). In this case the observer can win if he stops once or twice up to moment $K$. If either $1 \le n < N$ and $1 \le s < t \le n$, or $n = N$ and $1 \le s < t < N$, then

$$\begin{aligned} &P(\text{stop at } s \,\&\, t \,\&\text{ win} \mid K = n) \\ &\quad = P(X_1 \le L, \dots, X_{s-1} \le L,\ X_s > L,\ X_{s+1} \le L, \dots, X_{t-1} \le L, \\ &\qquad\qquad X_t > L,\ X_{t+1} \le \max(X_s, X_t), \dots, X_n \le \max(X_s, X_t)) \\ &\quad = 2L^{t-2} \int_L^1 dx_s \int_{x_s}^1 x_t^{n-t}\,dx_t \\ &\quad = 2L^{t-2}(1 - L - (1 - L^{n-t+2})/(n-t+2))/(n-t+1)\,. \end{aligned}$$

If either $1 \le n < N-1$ and $1 \le s \le n$, or $n = N-1$ and $1 \le s \le N-2$, then

$$\begin{aligned} &P(\text{stop at } s \,\&\text{ win} \mid K = n) \\ &\qquad = P(X_1 \le L, \dots, X_{s-1} \le L,\ X_s > L,\ X_{s+1} \le L, \dots, X_n \le L) \\ &\qquad = L^{n-1}(1-L)\,. \end{aligned}$$

For $1 \le s \le N-2$,

$$P(\text{stop at } s \,\&\text{ no stop at } N-1 \,\&\text{ win} \mid K = N) = L^{N-2}(1-L)\,.$$

Moreover,

$$\begin{aligned} &P(\text{no stop in } \{1, \dots, N-2\} \,\&\text{ win} \mid K = N-1) \\ &\quad = P(\max(X_1, \dots, X_{N-2}) \le L,\ X_{N-1} > \max(X_1, \dots, X_{N-2})) \\ &\quad = \int_0^L (1-y)(N-2)y^{N-3}\,dy = L^{N-2}(1-L) + 2L^{N-1}/(N-1)\,, \end{aligned}$$

$$\begin{aligned} &P(\text{no stop in } \{1, \dots, N-2\} \,\&\text{ win} \mid K = N) \\ &\qquad = P(\max(X_1, \dots, X_{N-2}) \le L, \\ &\qquad\qquad\qquad \max(X_{N-1}, X_N) > \max(X_1, .., X_{N-2})) \\ &\qquad = \int_0^L (1-y^2)(N-2)y^{N-3}\,dy = L^{N-2}(1-L^2) + 2L^N/N\,. \end{aligned}$$

Thus

$$P(\text{win}) = \sum_{n=2}^{N} 2p_n L^n \sum_{r=2}^{n} (L^{-r} - 1)/r - \sum_{n=1}^{N} p_n (n-2) L^{n-1}(1-L) + p_{N-1} L^{N-1}/(N-1) + 2p_N L^N/N .$$

Case (D). If either $1 \le n < N$ and $1 \le s < t \le n$, or $n = N$ and $1 \le s < t \le N-1$, then

$$\begin{aligned} &P(\text{stop at } s \,\&\, t \,\&\, \text{win} \mid K = n) \\ &= P(X_1 \le L, \ldots, X_{s-1} \le L,\ X_s > L,\ X_{s+1} \le L, \ldots, X_{t-1} \le L, \\ &\qquad X_t > L,\ X_{t+1} \le \min(X_s, X_t), \ldots, X_n \le \min(X_s, X_t)) \\ &= 2L^{t-2} \int_L^1 dx_s \int_{x_s}^1 x_s^{n-t}\, dx_t \\ &= 2L^{t-2}(1 - (n-t+2)L^{n-t+1} \\ &\quad + (n-t+1)L^{n-t+2})/(n-t+1)(n-t+2) . \end{aligned}$$

For $1 \le s \le N-2$,

$$\begin{aligned} &P(\text{stop at } s \,\&\, \text{no stop at } N-1 \,\&\, \text{win} \mid K = N) \\ &\qquad = (1-L) \int_0^L (1-y)(N-2)y^{N-3}\, dy \\ &\qquad = (1-L)(L^{N-2}(1-L) + 2L^{N-1}/(N-1)) . \end{aligned}$$

Moreover,

$$\begin{aligned} &P(\text{no stop in } \{1, \ldots, N-2\} \,\&\, \text{win} \mid K = N) \\ &= P(\max(X_1, \ldots, X_{N-2}) \le L,\ \min(X_{N-1}, X_N) > \max(X_1, \ldots, X_{N-2})) \\ &= \int_0^L (1-y)^2 (N-2) y^{N-3}\, dy \\ &= (N-2)L^{N-2}(1/(N-2) - 2L/(N-1) + L^2/N) . \end{aligned}$$

Thus

$$P(\text{win}) = \sum_{n=2}^{N} 2p_n L^n \sum_{r=2}^{n} (n-r+1)((L^{-r}-1)/r - (L^{-1}-1))/(r-1) + p_N((1 + 1/(N-1))L^{N-1} - (1 - 1/(N-1) + 2/N)L^N) .$$

Case (E). If either $1 \le n < N$ and $1 \le s < t \le n$, or $n = N$ and

$1 \le s < t < N$, then

$$\begin{aligned}
&P(\text{stop at } s \,\&\, t \,\&\, \text{win} \mid K = n) \\
&\quad = P(X_1 \le L, \ldots, X_{s-1} \le L,\ X_s > L,\ X_{s+1} \le L, \ldots, X_{t-1} \le L, \\
&\qquad\quad X_t > L, \text{the second largest of } X_{t+1}, \ldots, X_n \text{ is } \le \max(X_s, X_t)) \\
&\quad = 2L^{t-2} \int_L^1 dx_s \int_{x_s}^1 ((n-t)(1-x_t)x_t^{n-t-1} + x_t^{n-t})\, dx_t \\
&\quad = 2L^{t-2}(2(1-L) - (1 - L^{n-t+1}) \\
&\qquad + (n-t-1)(1-L^{n-t+2})/(n-t+2))/(n-t+1)\,.
\end{aligned}$$

If either $1 \le n < N-1$ and $1 \le s \le n$, or $n = N-1$ and $1 \le s \le N-2$, then

$$\begin{aligned}
&P(\text{stop at } s \,\&\, \text{win} \mid K = n) \\
&\quad = P(X_1 \le L, \ldots, X_{s-1} \le L,\ X_s > L,\ X_{s+1} \le L, \ldots, X_n \le L) \\
&\quad = L^{n-1}(1-L)\,.
\end{aligned}$$

For $1 \le s \le N-2$,

$$P(\text{stop at } s \,\&\, \text{no stop at } N-1 \,\&\, \text{win} \mid K = N) = L^{N-2}(1-L)\,.$$

Moreover,

$$\begin{aligned}
&P(\text{no stop in } \{1, \ldots, N-2\} \,\&\, \text{win} \mid K = N-1) \\
&\quad = P(\max(X_1, \ldots, X_{N-2}) \le L, \\
&\qquad\qquad X_{N-1} > \text{second largest of } X_1, \ldots, X_{N-2}) \\
&\quad = \int_0^1 dz \int_0^z (1-y)(N-2)(N-3)y^{N-4}\, dy \\
&\quad = L^{N-2}(1-L) + 2L^{N-1}/(N-1)\,, \\
&P(\text{no stop in } \{1, \ldots, N-2\} \,\&\, \text{win} \mid K = N) \\
&\quad = \int_0^L dz \int_0^z (1-y^2)(N-2)(N-3)y^{N-4}\, dy \\
&\quad = L^{N-2} - (N-3)(N-2)L^N/(N-1)N\,.
\end{aligned}$$

Thus

$$\begin{aligned}
P(\text{win}) = \sum_{n=1}^{N} p_n n L^{n-1}(1-L) &+ 2p_{N-1}L^{N-1}/(N-1) \\
&+ p_N(4N-6)L^N/(N-1)N
\end{aligned}$$

$$+\sum_{n=2}^{N} 2p_n L^n \sum_{r=2}^{n}(n-r+1)((2r-3)L^{-r}/r$$
$$-2L^{-r+1}+L^{-1}-(r-3)/r)/(r-1).$$

It is impossible to check open forms of optimal strategies when the distribution of the number of observations is not specified. In the next sections optimal strategies and probabilities of winning are given when $K$ has geometric or uniform distribution.

**4. Optimal strategies for $K$ geometric.** Let $K$ be a geometrically distributed r.v., i.e. $p_n = pq^n$ for $n = 0,1,2,\ldots, 0 < p < 1, p+q = 1$. In this case our results are summarized in the following theorem.

THEOREM. *Under the assumptions of Section 2 for $K$ geometric the solutions are:*

*Case* (A): *If $p < e^{-1}$ then the optimal decision level is $L^* = (1-ep)/q$ and the probability of winning is $P(\text{win}) = e^{-1} \cong 0.36789$. If $p \geq e^{-1}$ then $L^* = 0$ and $P(\text{win}) = -p\ln p$.*

*Case* (B): *If $p < \alpha$, where $\alpha \cong 0.30171$ is a root of the equation $2x - 2\ln x - 3 = 0$ in $(0,1)$ (or equivalently if $2p - 2\ln p - 3 > 0$) then $L^* = (1-\alpha^{-1}p)/q$ and $P(\text{win}) = \alpha(2-\alpha) \cong 0.51239$. If $p \geq \alpha$ then $L^* = 0$ and $P(\text{win}) = p(p-1-2\ln p)$.*

*Case* (C): *The solution is identical with Case* (B).

*Case* (D): *If $p < \alpha$, where $\alpha \cong 0.28466$ is a root of the equation $2x\ln x - x + 1 = 0$ in $(0,1)$ (or equivalently if $2p\ln p - p + 1 > 0$) then $L^* = (1-\alpha^{-1}p)/q$ and $P(\text{win}) = \alpha(1-\alpha) \cong 0.20363$. If $p \geq \alpha$ then $L^* = 0$ and $P(\text{win}) = 2p(p\ln p - p + 1)$.*

*Case* (E): *If $p < \alpha$, where $\alpha \cong 0.23978$ is a root of the equation $8x - 4(x+1)\ln x - 9 = 0$ in $(0,1)$ (or equivalently if $8p - 4(p+1)\ln p - 9 > 0$) then $L^* = (1-\alpha^{-1}p)/q$ and $P(\text{win}) = \alpha(5\alpha - 5 - 2(\alpha+2)\ln\alpha) \cong 0.62244$. If $p \geq \alpha$ then $L^* = 0$ and $P(\text{win}) = p(5p - 2(p+2)\ln p - 5)$.*

Proof. In Case (B) we obtain

$$P(\text{win}) = \sum_{n=1}^{\infty} pq^n L^n \sum_{r=1}^{n}(2(L^{-r}-1)/r - L^{-1}(1-L))$$
$$= \sum_{r=1}^{\infty} p(2(L^{-r}-1)/r - L^{-1}(1-L)) \sum_{n=r}^{\infty}(qL)^n$$
$$= p\Big(2\sum_{r=1}^{\infty} q^r/r - 2\sum_{r=1}^{\infty}(qL)^r/r - L^{-1}(1-L)\sum_{r=1}^{\infty}(qL)^r\Big)/(1-qL\Big)$$

$$= (p/(1-qL))(p/(1-qL) - 2\ln(p/(1-qL)) - 1).$$

Let $p/(1-qL) = x$. Since $0 \le L \le 1$ we have $p \le x \le 1$. The function $f(x) = x(x - 2\ln x - 1)$ in $(0,1)$, connected with $P(\text{win})$, has a unique local extremum (maximum) in $(0,1)$ at the point $\alpha$ for which the derivative $f'(x) = 2x - 2\ln x - 3$ is equal to zero. So, since $x \in [p,1]$ the function $f(x)$ attains its maximum at $\alpha$ if $p < \alpha$ (or equivalently if $f'(p) > 0$), and at $p$ if $p \ge \alpha$. Thus $L^* = (1 - \alpha^{-1}p)/q$ and $P(\text{win}) = f(\alpha) = \alpha(2-\alpha)$ if $p < \alpha$, and $L^* = 0$ and $P(\text{win}) = f(p)$ if $p \ge \alpha$. The Theorem is thus proved for problem (B). The proof for the remaining cases is analogous. We obtain the functions connected with $P(\text{win})$:

$$f(x) = \begin{cases} x\ln x & \text{for (A)}, \\ x(x - 1 - 2\ln x) & \text{for (C)}, \\ 2x(1 - x + x\ln x) & \text{for (D)}, \\ x(5x - 5 - 2(x+2)\ln x) & \text{for (E)}, \end{cases}$$

where $x = p/(1-qL)$, and find their maxima in $[p,1]$ as previously. This is easy, hence full particulars are omitted.

**5. Optimal strategies for $K$ uniform.** Let $K$ be uniformly distributed on $\{1,\ldots,n\}$, i.e. $p_k = 1/n$ for $k = 1,\ldots,n$. In this case denote the probability of winning at level $L$ by $P(L,n)$, the optimal decision level by $L(n)$ and set $P(L(n),n) = P(n)$. From the formulae for $P(\text{win})$ it is easy to get, by changing the order of sums and after some simplifications, the following:

Case (A):

$$P(L,n) = (n(1-L))^{-1}\sum_{r=1}^{n}(1-L^r)(1-L^{n-r+1})/r + L^n/n^2,$$

Case (B):

$$P(L,n) = (n(1-L))^{-1}\Big(L^n - 1 + 2\sum_{r=1}^{n}(1-L^r)(1-L^{n-r+1})/r\Big) + (1+2/n^2)L^n,$$

Case (C):

$$P(L,n) = (n(1-L))^{-1}\Big(L^{n-2} - L^2 + 2\sum_{r=2}^{n}(1-L^r)(1-L^{n-r+1})/r\Big)$$
$$+ (1 - L - (n-2)L^{n-1}/(n-1) - L^{n-2} + (n-2+2/n)L^n)/n,$$

Case (D):

$$P(L,n) = 2n^{-1}(1-L)^{-2}\sum_{r=2}^{n}((1-L^r) + L^{n+1}(1-L^{-r}))/r(r-1)$$

$$+ 2(n(1-L))^{-1}L^{n+1}\sum_{r=2}^{n}(n-r+1)(1-L^{-r})/r(r-1)$$
$$- 2(n(1-L))^{-1}\sum_{r=2}^{n}(L^{r-1}-L^n-(n-r+1)L^n(1-L))/(r-1)$$
$$+ ((1+1/(n-1))L^{n-1}-(1-1/(n-1)+2/n)L^n)/n,$$

Case (E):

$$P(L,n) = 2(L^{n-1}+(2n-3)L^n/n)/n(n-1) - L^n$$
$$+ 2(n(1-L))^{-1}\sum_{r=2}^{n}(2(1-L^{n-r+1})+(L^{r-1}-L^n))/(r-1)$$
$$- 6n^{-1}(1-L)^{-2}\sum_{r=2}^{n}(1-L^r)(1-L^{n-r+1})/r(r-1)$$
$$- 2n^{-1}L^n\sum_{r=2}^{n}(n-r+1)(2L^{-r+1}+1)/(r-1)$$
$$- 6(n(1-L))^{-1}L^{n+1}\sum_{r=2}^{n}(n-r+1)(L^{-r}-1)/r(r-1).$$

Numerical results on optimal decision levels $L(n)$ for which the probabilities of winning attain their maxima $P(n)$ and the values of $P(n)$ are given in Table 1.

In each case as $n$ grows large $L(n)$ tends to 1 but in such a manner that it is approximately linear in $1/n$:

$$L(n) = 1 - \alpha/n + o(1/n),$$

or in other words,

$$n(1-L(n)) \to \alpha \quad \text{or} \quad L^n \to e^{-\alpha}.$$

Using the above property in the formulae for $P(L,n)$ we find by passing to the limit that $P(n) \to P(\alpha)$, where

Case (A):

$$P(\alpha) = (-I_1(-\alpha) - e^{-\alpha}I_1(\alpha))/\alpha,$$

Cases (B) and (C):

$$P(\alpha) = e^{-\alpha} - (1-e^{-\alpha})/\alpha + 2(-I_1(-\alpha) - e^{-\alpha}I_1(\alpha))/\alpha,$$

Case (D):

$$P(\alpha) = -2e^{-\alpha}(1+I_2(\alpha))$$
$$+ 2(e^{-\alpha}(I_1(\alpha)-I_2(\alpha)) - (I_1(-\alpha)-I_2(-\alpha)))/\alpha,$$

TABLE 1

The optimal decision levels $L(n)$ and the optimal probabilities of winning $P(n)$ for $K$ uniformly distributed on $\{1,\ldots,n\}$

Case (A): largest when accepting once

| $n$ | $L(n)$ | $P(n)$ | $n$ | $L(n)$ | $P(n)$ |
|---|---|---|---|---|---|
| 2 | 0 | 0.75 | 11 | 0.7544 | 0.4791 |
| 3 | 0.2403 | 0.6346 | 13 | 0.7904 | 0.4708 |
| 4 | 0.3934 | 0.5791 | 15 | 0.8173 | 0.4647 |
| 5 | 0.4970 | 0.5463 | 20 | 0.8617 | 0.4549 |
| 6 | 0.5711 | 0.5257 | 30 | 0.9069 | 0.4453 |
| 7 | 0.6265 | 0.5108 | 50 | 0.9438 | 0.4376 |
| 8 | 0.6694 | 0.4998 | 70 | 0.9597 | 0.4344 |
| 9 | 0.7035 | 0.4913 | 100 | 0.9717 | 0.4319 |
| 10 | 0.7313 | 0.4846 | $\infty$ | | 0.4263 |

$L(n) = 1-2.8397/n+1.516/n^2$, $P(n) = 0.42632+0.561/n+0.22/n^2$ with error less than 0.0001 for $n \geq 10$

Case (B): largest or second largest when accepting once

| $n$ | $L(n)$ | $P(n)$ | $n$ | $L(n)$ | $P(n)$ |
|---|---|---|---|---|---|
| 2 | 0 | 1 | 11 | 0.6728 | 0.6694 |
| 3 | 0 | 0.8889 | 13 | 0.7207 | 0.6577 |
| 4 | 0.1914 | 0.8098 | 15 | 0.7565 | 0.6492 |
| 5 | 0.3295 | 0.7642 | 20 | 0.8156 | 0.6355 |
| 6 | 0.4285 | 0.7346 | 30 | 0.8759 | 0.6219 |
| 7 | 0.5024 | 0.7138 | 50 | 0.9250 | 0.6112 |
| 8 | 0.5595 | 0.6984 | 70 | 0.9463 | 0.6066 |
| 9 | 0.6050 | 0.6865 | 100 | 0.9623 | 0.6032 |
| 10 | 0.6420 | 0.6771 | $\infty$ | | 0.5953 |

$L(n) = 1-3.7900/n+2.091/n^2$, $P(n) = 0.59531+0.789/n+0.29/n^2$ with error less than 0.0001 for $n \geq 10$

Case (C): largest when accepting twice

| $n$ | $L(n)$ | $P(n)$ | $n$ | $L(n)$ | $P(n)$ |
|---|---|---|---|---|---|
| 3 | 0 | 0.8889 | 13 | 0.7211 | 0.6578 |
| 4 | 0.1994 | 0.8105 | 14 | 0.7401 | 0.6532 |
| 5 | 0.3345 | 0.7648 | 15 | 0.7567 | 0.6493 |
| 6 | 0.4316 | 0.7351 | 20 | 0.8157 | 0.6355 |
| 7 | 0.5044 | 0.7142 | 30 | 0.8759 | 0.6220 |
| 8 | 0.5609 | 0.6987 | 50 | 0.9250 | 0.6112 |
| 9 | 0.6060 | 0.6867 | 70 | 0.9463 | 0.6066 |
| 10 | 0.6428 | 0.6773 | 100 | 0.9623 | 0.6032 |
| 11 | 0.6733 | 0.6696 | $\infty$ | | 0.5953 |
| 12 | 0.6991 | 0.6632 | | | |

$L(n) = 1 - 3.7900/n + 2.10/n^2$ for $n \geq 15$ and
$P(n) = 0.59531 + 0.789/n + 0.31/n^2$ for $n \geq 10$
with error less than 0.0001

Case (D): largest or second largest when accepting twice

| $n$ | $L(n)$ | $P(n)$ | $n$ | $L(n)$ | $P(n)$ |
|---|---|---|---|---|---|
| 3 | 0 | 0.4444 | 13 | 0.6984 | 0.2961 |
| 4 | 0.1689 | 0.3911 | 14 | 0.7188 | 0.2935 |
| 5 | 0.2979 | 0.3613 | 15 | 0.7367 | 0.2912 |
| 6 | 0.3951 | 0.3425 | 20 | 0.8001 | 0.2834 |
| 7 | 0.4699 | 0.3295 | 30 | 0.8653 | 0.2759 |
| 8 | 0.5287 | 0.3201 | 50 | 0.9185 | 0.2700 |
| 9 | 0.5761 | 0.3130 | 70 | 0.9416 | 0.2675 |
| 10 | 0.6150 | 0.3074 | 100 | 0.9590 | 0.2657 |
| 11 | 0.6474 | 0.3029 | $\infty$ | | 0.2614 |
| 12 | 0.6749 | 0.2992 | | | |

$L(n) = 1 - 4.1255/n + 2.60/n^2$ for $n \geq 15$ and
$P(n) = 0.26140 + 0.422/n + 0.38/n^2$ for $n \geq 10$
with error less than 0.0001

Case (E): largest or second largest when accepting twice

| $n$ | $L(n)$ | $P(n)$ | $n$ | $L(n)$ | $P(n)$ |
|---|---|---|---|---|---|
| 3 | 0 | 1 | 13 | 0.6202 | 0.7868 |
| 4 | 0 | 0.9583 | 14 | 0.6462 | 0.7815 |
| 5 | 0.0855 | 0.9096 | 15 | 0.6689 | 0.7770 |
| 6 | 0.2206 | 0.8755 | 20 | 0.7493 | 0.7611 |
| 7 | 0.3220 | 0.8515 | 30 | 0.8313 | 0.7453 |
| 8 | 0.4002 | 0.8337 | 50 | 0.8980 | 0.7327 |
| 9 | 0.4624 | 0.8201 | 70 | 0.9269 | 0.7276 |
| 10 | 0.5130 | 0.8092 | 100 | 0.9487 | 0.7238 |
| 11 | 0.5549 | 0.8003 | $\infty$ | | 0.7143 |
| 12 | 0.5901 | 0.7930 | | | |

$L(n) = 1 - 5.15283/n + 2.79/n^2$,
$P(n) = 0.71427 + 0.930/n - 0.14/n^2$
with error less than 0.0001 for $n \geq 15$

Case (E):

$$P(\alpha) = 5(1-e^{-\alpha})/\alpha + e^{-\alpha}(1 + 6I_2(\alpha) - 4I_1(\alpha)) + 2(e^{-\alpha}(3I_2(\alpha) - 5I_1(\alpha)) - 3I_2(-\alpha) + I_1(-\alpha))/\alpha\,,$$

where

$$I_1(\alpha) = \int_0^\alpha x^{-1}(e^x - 1)\,dx, \quad I_2(\alpha) = \int_0^\alpha x^{-2}(e^x - x - 1)\,dx\,.$$

For example in Case (C) as $n$ tends to infinity we have

$$P(\alpha) = \alpha^{-1}\Big(e^{-\alpha} - 1 + 2\int_0^1 x^{-1}(1-e^{-\alpha x})(1-e^{-\alpha+\alpha x})\,dx\Big) + e^{-a}$$

$$= e^{-\alpha} - (1 - e^{-\alpha})/\alpha + 2\Big( \int_{-\alpha}^{0} x^{-1}(e^x - 1)\,dx - e^{-\alpha} \int_{0}^{\alpha} x^{-1}(e^x - 1)\,dx\Big)/\alpha .$$

From the necessary condition that the derivative of $P(\alpha)$ is zero we find the unique point $\alpha$ at which the maximum of $P(\alpha)$ is attained. E.g. in Case (C), $\alpha$ is a solution of the equation

$$1 - e^{-\alpha}(\alpha^2 + \alpha + 1) + 2(\alpha + 1)e^{-\alpha} I_1(\alpha) + I_1(-\alpha) + 4(e^{-\alpha} - 1) = 0 .$$

Numerical approximations of $\alpha$ and the asymptotic probabilities of winning $P(\alpha)$ are presented in Table 1.

**6. Optimal strategy for fixed $K$ in problem (E).** Now suppose $P(K = n) = 1$ for some $n > 2$. Then the probability of winning at level $L$ is

$$\begin{aligned} P(\text{win}) &= nL^{n-1}(1 - L) + (4n - 6)L^n/(n - 1)n \\ &\quad + 2L^n \sum_{r=2}^{n} (n - r + 1) \\ &\quad \times ((2r - 3)L^{-r}/r - 2L^{-r+1} + L^{-1} - (r - 3)/r)/(r - 1) . \end{aligned}$$

Numerical results on the optimal decision levels $L(n)$ for which $P(\text{win})$ attains its maximum $P(n)$ and the values of $P(n)$ are given in Table 2.

TABLE 2

Case (E): largest or second largest when accepting twice

| $n$ | $L(n)$ | $P(n)$ | $n$ | $L(n)$ | $P(n)$ |
|---|---|---|---|---|---|
| 3 | 0 | 1 | 13 | 0.8123 | 0.9035 |
| 4 | 0.5 | 0.9792 | 14 | 0.8245 | 0.9006 |
| 5 | 0.5773 | 0.9614 | 15 | 0.8353 | 0.8979 |
| 6 | 0.6342 | 0.9475 | 20 | 0.8739 | 0.8886 |
| 7 | 0.6777 | 0.9368 | 30 | 0.9142 | 0.8790 |
| 8 | 0.7121 | 0.9283 | 50 | 0.9476 | 0.8712 |
| 9 | 0.7398 | 0.9214 | 70 | 0.9623 | 0.9678 |
| 10 | 0.7627 | 0.9157 | 100 | 0.9735 | 0.8652 |
| 11 | 0.7819 | 0.9110 | $\infty$ | | 0.8591 |
| 12 | 0.7982 | 0.9070 | | | |

$L(n) = 1 - 2.6861/n + 3.25/n^2$ for $n \geq 15$ and
$P(n) = 0.85912 + 0.612/n - 0.45/n^2$ for $n \geq 10$
with error less than 0.0001

The dependence of $L(n)$ on $1/n$ is approximately linear:

$$L(n) = 1 - \alpha/n + o(1/n) .$$

Using the relations $n(1-L(n)) \to \alpha$ and $L^n \to e^{-\alpha}$ following from this property we find that if $n$ tends to infinity $P(n)$ tends to $P(\alpha)$ where

$$P(\alpha) = (4-\alpha)e^{-\alpha} - 4 - 6\alpha e^{-\alpha} I_2(\alpha) + 2e^{-\alpha}(2\alpha+3)I_1(\alpha),$$

and $I_1(\alpha)$, $I_2(\alpha)$ are defined in Section 5. $\alpha = 2.68614$ is a numerical value of $\alpha$ at which $P(\alpha)$ attains its maximum approximately equal to 0.85912.

## 7. Remarks

1. The full-information best choice problems when the observation is perfect have been solved, as far as the author knows, in Case (A) (Sakaguchi [5] and Bojdecki [1]) and (C) (Tamaki [7]) when the number $K$ of observations is known, and only in Case (A) (Porosiński [4]) when $K$ is random.

2. In Cases (C)–(E) for $K$ geometric, when two choices are allowed, one might suppose that the optimal strategy ought to depend on two decision levels, but consideration of the probability of winning as a function of two levels $0 \le L_1 \le L_2 \le 1$ shows that its maximum is attained for $L_1 = L_2$. The probabilities of winning are connected with the following functions of two variables:

$$f(x,y) = \begin{cases} x(-1-x+2y) - (x+y)\ln y & \text{for (C)}, \\ x(2-2y-\ln x+\ln y+y(\ln x+\ln y)) & \text{for (D)}, \\ -4x-x^2+5xy-y+y^2-2(x+xy+y)\ln y & \text{for (E)}, \end{cases}$$

where $p \le x = p/(1-qL_1) \le y = p/(1-qL_2) \le 1$. These functions have no local extrema in the interior of the triangle $p \le x \le y \le 1$ and the maxima are attained on the interval $x = y$. This is the reason why in this paper in all cases only the single level strategy is considered.

3. In the geometric case it is interesting and quite unexpected that the probabilities of winning in all *natural* situations (i.e. when $p$ is *small*) are constants independent of $p$ (see the Theorem). It is also interesting that in this case the more complex models have solutions which turn out to be simpler. (Compare these with solutions of (A)–(D) when $K$ is nonrandom and known in Sakaguchi [6].)

4. In Case (A) for $K$ geometric the optimal decision level does not depend on the number of preceding observations even if the class of admissible strategies is not bounded (Porosiński [4]). This interesting property of the optimal strategy is presumably a consequence of the *memoryless* property of the geometric distribution. It seems that optimal strategies for (B)–(E) when $K$ is geometric ought to obey this principle. Then our results for (B)–(E) would also be optimal when the observation is perfect.

5. In Case (A) for $K$ uniform and perfect observation the probability of winning is estimated by Porosiński [4] as 0.4352. It is surprising that the optimal strategy when the observation is imperfect has the probability of winning and its asymptotic value (0.4263) only a little less, in spite of its very simple structure (cf. Table 2 in [4]). It seems that also in Cases (B)–(E) when $K$ is uniform our single level strategy may be only a little worse than the optimal strategy for perfect observation.

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