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SYSTEMATIC ESTIMATION AND PREDICTION FOR PROCESSES WITH BOUNDED SUM

A minimax n-estimator (see Section 1) of the parameter $m = (m_1, \ldots, m_r)$, $m_i = E(X_i)$, is determined for the loss function (2) in the case when the random variables $X_1, \ldots, X_r$ satisfy the conditions

$$X_1 \geq 0, \ldots, X_r \geq 0, \quad X_1 + \ldots + X_r \leq s\,.$$

The problem of minimax n-prediction for such a process is also solved.

1. Let $X = (X_1, \ldots, X_r)$ be a random variable satisfying the conditions

$$(1) \quad X_1 \geq 0, \ldots, X_r \geq 0, \quad X_1 + \ldots + X_r = s, \quad s > 0,\ r \in \{2, 3, \ldots\}.$$

Let $X^{(1)}, \ldots, X^{(n)}$, $X^{(j)} = (X_1^{(j)}, \ldots, X_r^{(j)})$, $j = 1, \ldots, n$, be independent random variables having the same distribution as X. Let $\widehat{X}^{(k)} = (X^{(1)}, \ldots, X^{(k)})$, $k = 1, \ldots, n$, $m_i = E(X_i)$, $i = 1, \ldots, r$. We consider the situation when the statistician estimates systematically the parameter $m = (m_1, \ldots, m_r)$ in steps $1, \ldots, n$ on the basis of $\widehat{X}^{(1)}, \ldots, \widehat{X}^{(n)}$, respectively, and when the loss function is the sum of the losses at particular steps. The sequence

$$d(\widehat{X}) = \{d^{(1)}(\widehat{X}^{(1)}), \ldots, d^{(n)}(\widehat{X}^{(n)})\}, \quad \widehat{X} = \widehat{X}^{(n)}\,,$$

where

$$d^{(k)}(\widehat{X}^{(k)}) = (d_1^{(k)}(\widehat{X}^{(k)}), \ldots, d_r^{(k)}(\widehat{X}^{(k)}))\,,$$

is called an *n-estimator*.

Let the loss function be

$$(2) \qquad L(m, d) = \sum_{k=1}^{n} c_k \sum_{i,j=1}^{r} c_{ij}(d_i^{(k)}(\widehat{X}^{(k)}) - p_i)(d_j^{(k)}(\widehat{X}^{(k)}) - p_j)\,,$$

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where $c_k \geq 0$, $k = 1, \ldots, n$, at least one $c_k > 0$, and the matrix $C = \|c_{ij}\|_1^r$ is symmetric and nonnegative definite.

The problem is to determine a minimax n-estimator for the above loss function.

Let

$$Y_i^{(k)} = \sum_{t=1}^{k} X_i^{(t)}, \quad k = 1, \ldots, n, \quad Y^{(k)} = (Y_1^{(k)}, \ldots, Y_r^{(k)}).$$

Consider the n-estimator $d(\widehat{X})$ for which

$$d_i^{(k)}(\widehat{X}^{(k)}) = \frac{Y_i^{(k)} + \beta_i \alpha}{k + \alpha}, \quad i = 1, \ldots, r,\ k = 1, \ldots, n, \tag{3}$$

where $\beta_i \geq 0, \sum_{i=1}^{r} \beta_i = s$, $\alpha > 0$. For this n-estimator the risk function is

$$\begin{aligned} R(m,d) &= E[L(m, d(X))] \\ &= \sum_{k=1}^{n} c_k \sum_{i,j=1}^{r} c_{ij} E\left[\left(\frac{Y_i^{(k)} + \beta_i \alpha}{k+\alpha} - m_i\right)\left(\frac{Y_j^{(k)} + \beta_j \alpha}{k+\alpha} - m_j\right)\right] \\ &= \sum_{k=1}^{n} \frac{c_k}{(k+\alpha)^2} \\ &\quad \times \sum_{i,j=1}^{r} c_{ij}\{E[(Y_i^{(k)} - m_i)(Y_j^{(k)} - m_j)] + \alpha^2(\beta_i - m_i)(\beta_j - m_j)\} \\ &= \sum_{k=1}^{n} \frac{c_k}{(k+\alpha)^2} \sum_{i,j=1}^{r} c_{ij}[kE(X_i X_j) - k m_i m_j + \alpha^2(\beta_i - m_i)(\beta_j - m_j)]. \end{aligned}$$

But

$$\begin{aligned} &\sum_{i,j=1}^{r} c_{ij} X_i X_j - s \sum_{i=1}^{r} c_{ii} X_i \\ &\qquad = \sum_{i,j=1}^{r} c_{ij} X_i X_j - \frac{1}{2} \sum_{i,j=1}^{r} c_{ii} X_i X_j - \frac{1}{2} \sum_{i,j=1}^{r} c_{jj} X_i X_j \\ &\qquad = -\frac{1}{2} \sum_{i,j=1}^{r} (c_{ii} + c_{jj} - 2c_{ij}) X_i X_j \leq 0, \end{aligned} \tag{4}$$

since the matrix C is nonnegative definite and $X_i \geq 0$. Hence

$$R(m,d) \leq \sum_{k=1}^{n} \frac{c_k}{(k+\alpha)^2} \tag{5}$$

$$\times \Big\{ \sum_{i,j=1}^{r} c_{ij}[-km_i m_j + \alpha^2(\beta_i - m_i)(\beta_j - m_j)] + ks \sum_{i=1}^{r} c_{ii} m_i \Big\}.$$

Suppose that

$$\sum_{k=1}^{n} \frac{c_k}{(k+\alpha)^2}(\alpha^2 - k) = 0. \tag{6}$$

For the parameter α determined in this way we have from (5)

$$\begin{aligned} R(m,d) &\le \sum_{k=1}^{r} \frac{c_k}{(k+\alpha)^2}\Big[\sum_{i,j=1}^{r} c_{ij}(\beta_i\beta_j - 2\beta_j m_i)\alpha^2 + k \sum_{i,j=1}^{r} c_{ii}\beta_j m_i\Big] \\ &= \sum_{k=1}^{r} \frac{c_k\alpha^2}{(k+\alpha)^2}\Big[\sum_{i,j=1}^{r} c_{ij}\beta_i\beta_j + \sum_{i,j=1}^{r}(c_{ii} - 2c_{ij})\beta_j m_i\Big]. \end{aligned} \tag{7}$$

Let

$$P(X = e_i) = m_i/s \overset{\text{df}}{=} p_i, \tag{8}$$

where $e_1 = (s, 0, \dots, 0)$, $\dots$, $e_r = (0, 0, \dots, s)$. Then

$$E(X_i) = m_i, \quad E(X_i X_j) = 0 \quad \text{for } i \neq j, \quad E(X_i^2) = sm_i$$

and

$$R(m,d) = \sum_{k=1}^{r} \frac{c_k\alpha^2}{(k+\alpha)^2}\Big\{\sum_{i,j=1}^{r} c_{ij}\beta_i\beta_j + \sum_{i,j=1}^{r}(c_{ii} - 2c_{ij})\beta_j m_i\Big\}. \tag{9}$$

Suppose that there exist a set $A \subset R = \{1, \dots, r\}$, $|A| \ge 2$, and constants $\beta_1, \dots, \beta_r, v$ such that

$$\sum_{j\in A}(c_{ii} - 2c_{ij})\beta_j = v \quad \text{if } i \in A, \tag{10}$$

$$\sum_{j\in A}(c_{ii} - 2c_{ij})\beta_j \le v \quad \text{if } i \in R - A, \tag{11}$$

$\beta_i > 0$ for $i \in A$, $\beta_i = 0$ for $i \in R - A$, $\sum_{i=1}^{r} \beta_i = s$. From [5] it follows that either such a set A and constants $\beta_1, \dots, \beta_r$ exist, or $c_{ij} = \text{const}$.

We prove that *for $\beta_1, \dots, \beta_r$ chosen in this way and α determined from* (6) *the n-estimator* (3) *is minimax.*

Let $\beta_1, \dots, \beta_r, v$ be chosen according to (10) and (11). For α being the solution of (6) we find from (7) and (9) that

$$R(m,d) = \sum_{k=1}^{r} \frac{c_k\alpha^2}{(k+\alpha)^2}\Big(\sum_{i,j=1}^{r} c_{ij}\beta_i\beta_j + vs\Big) \overset{\text{df}}{=} c$$

if X is distributed according to (8) and

$$R(m,d) \le c$$

for any distribution of X.

One can view the problem of determining a minimax n-estimator of the parameter $m = (m_1, \ldots, m_r)$ as the problem of finding a minimax strategy in a game against nature: the nature chooses a distribution of the random variable X, the statistician chooses an n-estimator of $m = E(X)$ and the payoff is a risk function $R(m, d)$. Choose a mixed strategy τ for the nature in the following way:

First choose the parameter $p = (p_1, \ldots, p_r)$ according to the density

$$(12) \quad g(p_1, \ldots, p_r) = \begin{cases} \dfrac{\Gamma(\sum_{j=1}^{q} \alpha_{i_j})}{\Gamma(\alpha_{i_1}) \ldots \Gamma(\alpha_{i_q})} p_{i_1}^{\alpha_{i_1}-1} \ldots p_{i_q}^{\alpha_{i_q}-1} & \\ & \text{if } p_{i_k} > 0,\ \sum_{k=1}^{q} p_{i_k} = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$\{i_1, \ldots, i_q\} = A$, $\alpha_i = \beta_i \alpha / s$ (α and β_i determined in (6), (10) and (11)), and later choose the distribution P of the random variable X according to (8).

It can be verified that the n-estimator d defined by (3), (6), (10) and (11) is Bayes with respect to such a mixed strategy of nature and thus it is minimax.

2. Let the random variable $X = (X_1, \ldots, X_r)$ satisfy the conditions

$$X_1 \geq 0, \ldots, X_r \geq 0, \quad X_1 + \ldots + X_r \leq s, \quad s > 0,\ r \in \{1, 2, \ldots\},$$

and let the loss function be given by (2). Define $X_{r+1} = s - \sum_{i=1}^{r} X_i$ and $c_{i,r+1} = 0$ for $i = 1, \ldots, r+1$. Then we are in the situation considered in the previous section and a minimax n-estimator may be determined using the formulae (3), (6), (10) and (11) for $R = \{1, \ldots, r+1\}$.

3. Let $X = (X_1, \ldots, X_r)$ be a random variable satisfying the conditions (1) and let $X^{(1)}, \ldots, X^{(n+1)}$, $X^{(j)} = (X_1^{(j)}, \ldots, X_r^{(j)})$, $j = 1, \ldots, n+1$, be independent random variables having the same distribution as X. Define

$$\widehat{X}^{(k)} = (X^{(1)}, \ldots, X^{(k)}), \quad k = 1, \ldots, n,$$

$$Y_i^{(k)} = \sum_{t=1}^{k} X_i^{(t)}, \quad Y_i^k = \sum_{t=k+1}^{n+1} X_i^{(t)}, \quad i = 1, \ldots, r,$$

$$Y^k = (Y_1^k, \ldots, Y_r^k), \quad \widehat{Y} = (Y^1, \ldots, Y^n), \quad \widehat{X} = \widehat{X}^{(n)}.$$

At the kth step we predict the random variable Y^0 using $\widehat{X}^{(k)}$. Since the random variables $X^{(t)}$ in Y^0 are known for $t \leq k$ it is sufficient to predict at this step Y^k using $\widehat{X}^{(k)}$. Then let

$$d(\widehat{X}) = \{d^{(1)}(\widehat{X}^{(1)}), \ldots, d^{(n)}(\widehat{X}^{(n)})\}, \quad \text{where}$$

$$d^{(k)}(\widehat{X}^{(k)}) = (d_1^{(k)}(\widehat{X}^{(k)}), \ldots, d_r^{(k)}(\widehat{X}^{(k)})),$$

be an n-predictor of Y. Let the loss function be

$$L(\widehat{Y}, d) = \sum_{k=1}^{n} c_k \sum_{i,j=1}^{r} c_{ij}(d_i^{(k)}(\widehat{X}^{(k)}) - Y_i^k)(d_j^{(k)}(\widehat{X}^{(k)}) - Y_j^k),$$

where c_k and c_{ij} satisfy the same conditions as in Section 1. For this loss function the risk function can be represented in the form

$$\begin{aligned} R(m,d) &= E[L(\widehat{Y}, d(\widehat{X}))] \\ &= \sum_{k=1}^{n} c_k \sum_{i,j=1}^{r} c_{ij}\{E[(d_i^{(k)}(\widehat{X}^{(k)}) - (n-k+1)m_i) \\ &\quad \times (d_j^{(k)}(\widehat{X}^{(k)}) - (n-k+1)m_j)] \\ &\quad + E[(Y_i^k - (n-k+1)m_i)(Y_j^k - (n-k+1)m_j)]\}. \end{aligned}$$

Notice that the second term does not depend on the n-predictor d.

Let us study the n-predictor for which

$$(13) \quad d_i^{(k)}(\widehat{X}^{(k)}) = (n-k+1)\frac{Y_i^{(k)} + \beta_i\alpha}{k+\alpha}, \quad i = 1, \ldots, r,\ k = 1, \ldots, n.$$

For this n-predictor the risk function is

$$\begin{aligned} (14) \quad & R(m,d) \\ &= \sum_{k=1}^{n} c_k \Bigg\{ \sum_{i,j=1}^{r} c_{ij} \Bigg\{ \Bigg[\left(\frac{n-k+1}{k+\alpha}\right)^2 k + n - k + 1\Bigg] E(X_iX_j) \\ &\quad + \Bigg[\left(\frac{n-k+1}{k+\alpha}\right)^2 (\alpha^2 - k) - (n-k+1)\Bigg] m_i m_j \\ &\quad + \left(\frac{n-k+1}{k+\alpha}\right)^2 \alpha^2(\beta_i\beta_j - 2m_i\beta_j)\Bigg\}\Bigg\} \\ &\le \sum_{k=1}^{n} c_k \Bigg\{ \sum_{i,j=1}^{r} c_{ij} \Bigg\{ \Bigg[\left(\frac{n-k+1}{k+\alpha}\right)^2 (\alpha^2 - k) - (n-k+1)\Bigg] m_i m_j \\ &\quad + \alpha^2 \left(\frac{n-k+1}{k+\alpha}\right)^2 (\beta_i\beta_j - 2m_i\beta_j)\Bigg\} \\ &\quad + \sum_{i,j=1}^{r} c_{ii} \Bigg[\left(\frac{n-k+1}{k+\alpha}\right)^2 k + n - k + 1\Bigg] m_i\beta_j \Bigg\} \end{aligned}$$

by (4).

Let $\alpha > 0$ be a solution of the equation

$$\varphi(\alpha) = \sum_{k=1}^{r} c_k \left[\left(\frac{n-k+1}{k+\alpha} \right)^2 (\alpha^2 - k) - (n-k+1) \right] = 0 . \tag{15}$$

This solution always exists except for the case when

$$c_1 = \ldots = c_{n-1}, \quad c_n > 0 . \tag{16}$$

Under the condition given in (15) the inequality (14) takes the form

$$R(m,d) \le \sum_{k=1}^{r} c_k \alpha^2 \left(\frac{n-k+1}{k+\alpha} \right)^2 \Big[\sum_{i,j=1}^{r} c_{ij} \beta_i \beta_j + \sum_{i,j=1}^{r} (c_{ii} - 2c_{ij}) m_i \beta_j \Big] .$$

Notice that the expression in square brackets is the same as that in (7). Moreover, the predictor d given by (13) is Bayes with respect to the strategy of nature τ defined in (12) (with α obtained from (15)). Then in the same way as in Section 1 one can prove that *the n-predictor defined by* (10), (11), (13) *and* (15) *is minimax.*

When conditions (16) hold there does not exist a solution $\alpha > 0$ of the equation $\varphi(\alpha) = 0$ but still $\varphi(\alpha) \to 0$ as $\alpha \to \infty$. In this case one can prove that the n-predictor d for which $d_i^{(n)}(\widehat{X}^{(n)}) = \beta_i$, β_i determined by (10) and (11), is a Bayes predictor of $\widehat{Y}$ with respect to the strategy τ_0 for which the condition (8) holds and

$$P(m_i = \beta_i \text{ for } i \in A, m_i = 0 \text{ for } i \notin A) = 1 .$$

Then *this n-predictor is minimax.*

4. Similarly to Section 3 one can solve the problem of minimax n-prediction when

$$X_1 \ge 0, \ldots, X_r \ge 0, \quad X_1 + \ldots + X_r \le s, \quad r \in \{1, 2, \ldots\}, \ s > 0 .$$

For related problems see [1]–[5].

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