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MULTIVARIATE TAIL EQUIVALENCE OF DISTRIBUTIONS IN EXTREME VALUE THEORY

We consider some relations between multivariate tail equivalence of distribution functions and the weak convergence of extremes for independent identically distributed random vectors. S. I. Resnick in [5] has considered the same problem for the case of random variables.

1. Introduction. Let F and G be m-dimensional continuous distribution functions. Introduce the following notation:

$$\underline{x}_F^0 = (x_{F_1}^0, \ldots, x_{F_m}^0),$$

where for $i = 1, \ldots, m$

$$x_{F_i}^0 = \sup\{x \in \mathbf{R} : F_i(x) < 1\}$$

and F_i is the ith marginal of F.

DEFINITION 1. Two distribution functions F and G are *tail equivalent* if $\underline{x}_F^0 = \underline{x}_G^0 = \underline{x}^0$ and

$$\lim_{x_1 \to x_1^0-, \ldots, x_m \to x_m^0-} \frac{1 - F(x_1, \ldots, x_m)}{1 - G(x_1, \ldots, x_m)} = \gamma$$

for some constant $0 < \gamma < \infty$. We denote this property by $F \overset{\gamma}{\sim} G$.

Let $\underline{X}_1, \underline{X}_2, \ldots$ be a sequence of independent identically distributed m-dimensional random vectors with continuous distribution function F. Then

$$P\{\max_{1 \le k \le n} (X_k^i) \le x_i : i = 1, \ldots, m\} = F^n(x_1, \ldots, x_m),$$

where $\underline{X}_n = (X_n^1, \ldots, X_n^m)$.

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DEFINITION 2. A distribution function F belongs to the *domain of attraction* of a distribution function H ($F \in D(H)$) if for some m-dimensional sequences of constants $\underline{a}_n > \underline{0}$, $\underline{b}_n$

$$\lim_{n\to\infty} F^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{x}), \tag{1}$$

where

$$\underline{a}_n\underline{x} + \underline{b}_n = (a_n^1 x_1 + b_n^1, \ldots, a_n^m x_m + b_n^m).$$

The convergence is understood to occur at continuity points of H. A distribution function H for which (1) holds is called *extremal*.

We assume in the paper that the extremal distribution H has nondegenerate marginals H_i for $i = 1, \ldots, m$. It is easy to notice that (1) holds if and only if

$$\lim_{n\to\infty} P\{\max_{1\le k\le n}(X_k^i) \le a_n^i x_i + b_n^i : i = 1, \ldots, m\} = H(\underline{x})$$

for a sequence of independent identically distributed random vectors with distribution F.

It is shown in [2] that every extremal distribution function is continuous and its marginals are of extremal types:

$$\begin{aligned} \Phi_\alpha(x) &= \exp(-x^{-\alpha}), && x > 0,\ \alpha > 0, \\ \Psi_\alpha(x) &= \exp(-(-x)^\alpha), && x \le 0,\ \alpha > 0, \\ \Lambda(x) &= \exp(-e^{-x}), && -\infty < x < \infty. \end{aligned}$$

An important role in multivariate extreme value theory is played by the notion of depending function.

DEFINITION 3. For a distribution function F, a function D_F for which

$$F(x_1, \ldots, x_m) = D_F(F_1(x_1), F_2(x_2), \ldots, F_m(x_m))$$

is called a *depending function* for F. Some properties of depending functions are given in [2].

2. Main results. Now we obtain some relations between tail equivalence of F and G and the asymptotic properties of $F^n(\underline{a}_n\underline{x} + \underline{b}_n)$ and $G^n(\underline{a}_n\underline{x} + \underline{b}_n)$. Before presenting the main theorems, we give a useful lemma whose proof is the same as in the one-dimensional case and is therefore omitted.

LEMMA 1. *For a distribution F and an extremal distribution H we have*

$$\lim_{n\to\infty} F^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{x})$$

if and only if

$$\lim_{n\to\infty} n[1 - F(\underline{a}_n\underline{x} + \underline{b}_n)] = -\log H(\underline{x})$$

for all $\underline{x} \in \mathbf{R}^m$ *such that* $H(\underline{x}) > 0$.

Using this lemma we prove the following theorem:

THEOREM 1. *Let F and G be distributions and let H be an extremal distribution. If*

$$\lim_{n\to\infty} F^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{X}) \quad \text{and} \quad F \overset{\gamma}{\sim} G$$

then

$$\lim_{n\to\infty} G^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{A}\,\underline{x} + \underline{B})$$

for some $\underline{A} > \underline{0}$, $\underline{B} \in \mathbf{R}^m$

Proof. From Lemma 1 we have

$$\lim_{n\to\infty} n[1 - F(\underline{a}_n\underline{x} + \underline{b}_n)] = -\log H(\underline{x}),$$

for $\underline{x} \in \mathbf{R}^m$ such that $H(\underline{x}) > 0$. Notice that this convergence implies $\underline{a}_n\underline{x} + \underline{b}_n \to \underline{x}^0$. Hence

$$\lim_{n\to\infty} n[1 - G(\underline{a}_n\underline{x} + \underline{b}_n)] = \lim_{n\to\infty} n[1 - F(\underline{a}_n\underline{x} + \underline{b}_n)]\frac{1 - G(\underline{a}_n\underline{x} + \underline{b}_n)}{1 - F(\underline{a}_n\underline{x} + \underline{b}_n)}$$
$$= -\log H^{\gamma^{-1}}(\underline{x}).$$

Applying Lemma 1 once again we have

$$\lim_{n\to\infty} G^n(\underline{a}_n\underline{x} + \underline{b}_n) = H^{\gamma^{-1}}(\underline{x}).$$

If we show that $H^{\gamma^{-1}}(\underline{x}) = H(\underline{A}\,\underline{x} + \underline{B})$ the proof will be complete.

Define

$$\mathrm{I}(H) = \{1 \le i \le m : H_i = \Phi_{\alpha_i}\},$$
$$\mathrm{II}(H) = \{1 \le i \le m : H_i = \Psi_{\beta_i}\},$$
$$\mathrm{III}(H) = \{1 \le i \le m : H_i = \Lambda\}.$$

It may occur that some of these sets (not all) are empty. Without loss of generality we may assume that there exist $1 \le i_1 < i_2 \le m$ such that $\mathrm{I}(H) = \{1, \ldots, i_1\}$, $\mathrm{II}(H) = \{i_1 + 1, \ldots, i_2\}$ and $\mathrm{III}(H) = \{i_2 + 1, \ldots, m\}$.

Since H is extremal we have (see [2])

$$H(x_1, \ldots, x_m) = D_H^{\gamma}(H_1^{\gamma^{-1}}(x_1), \ldots, H_m^{\gamma^{-1}}(x_m)),$$

that is,

$$H^{\gamma^{-1}}(x_1, \ldots, x_m) = D_H(H_1^{\gamma^{-1}}(x_1), \ldots, H_m^{\gamma^{-1}}(x_m)). \tag{2}$$

For $i \in \mathrm{I}(H)$ there exist $A_i > 0$ such that

$$\gamma = A_i^{\alpha_i} \quad \text{and} \quad \Phi_{\alpha_i}^{\gamma^{-1}}(x_i) = \Phi_{\alpha_i}(A_i x_i).$$

For $i \in \mathrm{II}(H)$ we may find A'_i such that

$$\gamma^{-1} = A_i'^{\beta i} \quad \text{and} \quad \Psi_{\beta_i}^{\gamma^{-1}}(x_i) = \Psi_{\beta_i}(A'_i x_i),$$

and similarly for $i \in \mathrm{III}(H)$ we have

$$\gamma^{-1} = e^{-B_i} \quad \text{and} \quad \Lambda^{\gamma^{-1}}(x_i) = \Lambda(x_i + b_i) \quad \text{for some } B_i .$$

Now if

$$\underline{A} = (A_1, \ldots, A_{i_1}, A'_{i_1+1}, \ldots, A'_{i_2}, 1, \ldots, 1),$$
$$\underline{B} = (0, \ldots, 0, B_{i_2+1}, \ldots B_m),$$

then from (2) we obtain $H^{\gamma^{-1}}(\underline{x}) = H(\underline{A}\,\underline{x} + \underline{B})$.

As an easy consequence of Theorem 1 we have the following corollary.

COROLLARY 1. *Let* $\lim_{n\to\infty} F^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{x})$ *and* $F \overset{\gamma}{\sim} G$, *where* $H_i = L$ *for* $i = 1, \ldots, m$. *Then*

(i) *if* $L = \Phi_\alpha$ *then* $\lim_{n\to\infty} G^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\gamma^{1/\alpha} \cdot \mathbf{1} \cdot \underline{x})$,
(ii) *if* $L = \Psi_\alpha$ *then* $\lim_{n\to\infty} G^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\gamma^{-1/\alpha} \cdot \mathbf{1} \cdot \underline{x})$,
(iii) *if* $L = \Lambda$ *then* $\lim_{n\to\infty} G^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{x} + \log\gamma \cdot \mathbf{1})$,

where $\mathbf{1} = (1, \ldots, 1) \in \mathbf{R}^m$.

Now we prove two technical lemmas needed in the sequel. For this purpose we introduce the following notations for a given distribution function G:

$$A_n = \bigtimes_{i=1}^{m} [\mu_n^{G_i}, \mu_{n+1}^{G_i}], \qquad n \ge 2,$$

where for $i = 1, \ldots, m$

$$\mu_n^{G_i} = G_i^{-1}(1 - 1/n), \qquad G_i^{-1}(y) = \inf\{x \in \mathbf{R} : G(x) > y\}.$$

It can be shown (see [5]) that $\mu_n^{G_i} < \mu_{n+1}^{G_i} \nearrow x_{G_i}^0$ as $n \to \infty$.

Let $\underline{\beta}(t) = (\beta_1(t), \ldots, \beta_m(t))$, $t \in \mathbf{R}$, be a continuous curve for which

(3) $$\beta_i(t) \nearrow x_{G_i}^0 \quad \text{as } t \to \infty \text{ for } i = 1, \ldots, m,$$

(4) $$\exists\, t_{\underline{\beta}} \in \mathbf{R}\ \forall t > t_{\underline{\beta}}\ \exists n \ge 2 : \quad \underline{\beta}(t) \in A_n .$$

Now we can formulate the above-mentioned lemmas:

LEMMA 2. *If*

$$\lim_{n\to\infty} F^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{x}), \qquad \lim_{n\to\infty} G^n(\underline{a}_n\underline{x} + \underline{b}_n) = H(\underline{A}\,\underline{x} + \underline{B})$$

where $H_i = \Psi_\alpha$ *for* $i = 1, \ldots, m$, *and* $\underline{A} = A \cdot \mathbf{1}$ *for some* $A > 0$, *then* $\underline{B} = \underline{0}$, $\underline{x}_F^0 = \underline{x}_G^0$ *and*

$$\lim_{t\to\infty} \frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} = A^{-\alpha}$$

for any curve $\underline{\beta}$ *such that* (3) *and* (4) *hold.*

Proof. From the assumptions we have for $i = 1, \ldots, m$

$$\lim_{n\to\infty} F_i^n(a_n^i x_i + b_n^i) = \Psi_\alpha(x_i),$$
$$\lim_{n\to\infty} G_i^n(a_n^i x_i + b_n^i) = \Psi_\alpha(Ax_i + B_i). \tag{5}$$

We can take (see [3]) $a_n^i = x_{F_i}^0 - \mu_n^{F_i}$ and $b_n^i = x_{F_i}^0$, where $x_{F_i}^0 < \infty$. Hence and from (5) we obtain

$$\lim_{n\to\infty} G_i^n((x_{F_i}^0 - \mu_n^{F_i})A^{-1}x_i + x_{F_i}^0 - (x_{F_i}^0 - \mu_n^{F_i})A^{-1}B_i) = \Psi_\alpha(x_i),$$

that is, $G_i \in D(\Psi_\alpha)$, and therefore, as above, $x_{G_i}^0 < \infty$ and

$$\lim_{n\to\infty} G_i^n((x_{G_i}^0 - \mu_n^{G_i})x_i + x_{G_i}^0) = \Psi_\alpha(x_i).$$

From Khinchin's theorem (see [5], p. 138)

$$\frac{x_{F_i}^0 - \mu_n^{F_i}}{x_{G_i}^0 - \mu_n^{G_i}} \to A \quad \text{and} \tag{6}$$

$$\frac{x_{G_i}^0 - (x_{F_i}^0 - (x_{F_i}^0 - \mu_n^{F_i})A^{-1}B_i)}{x_{G_i}^0 - \mu_n^{G_i}} \to 0 \qquad \text{as } n \to \infty.$$

Hence we obtain

$$\frac{x_{G_i}^0 - x_{F_i}^0}{x_{G_i}^0 - \mu_n^{G_i}} + B_i \to 0 \qquad \text{as } n \to \infty. \tag{7}$$

Since $\mu_n^{G_i} \nearrow x_{G_i}^0$ we see from (7) that $x_{G_i}^0 = x_{F_i}^0$ $(= x_i^0$, say) and $B_i = 0$ for $i = 1, \ldots, m$.

Now we prove the second part of the lemma. Let $\underline{\beta}$ be a curve for which (3) and (4) hold. Let $\varepsilon > 0$ be given. For sufficiently large t there exists $n = n(t)$ such that $\underline{\beta}(t) \in A_n$ and by (6) we have

$$x_i^0 - (A^{-1} + \varepsilon)(x_i^0 - \mu_n^{F_i}) < \mu_n^{G_i} < x_i^0 - (A^{-1} - \varepsilon)(x_i^0 - \mu_n^{F_i})$$

for $i = 1, \ldots, m$. Hence

$$\frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} \le \frac{1 - F(\underline{\mu}_n^G)}{1 - G(\underline{\mu}_{n+1}^G)} \le \frac{n[1 - F(\underline{a}_n(-(A^{-1} + \varepsilon)) + \underline{b}_n)]}{n[1 - G(\underline{a}_{n+1}(-(A^{-1} - \varepsilon)) + \underline{b}_{n+1})]}.$$

We have $n = n(t) \to \infty$ as $t \to \infty$, and therefore, by Lemma 1,

$$\limsup_{t\to\infty} \frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} \le \frac{\log H(-(A^{-1} + \varepsilon)\mathbf{1})}{\log H(-A(A^{-1} - \varepsilon)\mathbf{1})}. \tag{6'}$$

It is easy to see that for sufficiently small $\varepsilon > 0$

$$0 < H(-(A^{-1} + \varepsilon)\mathbf{1}) < 1, \quad 0 < H(-A(A^{-1} - \varepsilon)\mathbf{1}) < 1,$$

and hence

$$-\log H(-A(A^{-1} - \varepsilon)\mathbf{1}) > 0:$$

Notice that $0 < H(-1) < 1$, that is, $\log H(-1)$ makes sense. We also have $H(-A^{-1}\cdot\mathbf{1}) = H^{A^{-\alpha}}(-1)$. From (6′) and the above it follows that

$$\limsup_{t\to\infty}\frac{1-F\circ\underline{\beta}(t)}{1-G\circ\underline{\beta}(t)}\le A^{-\alpha}$$

if we let $\varepsilon \searrow 0$. Similarly it may be shown that

$$\liminf_{t\to\infty}\frac{1-F\circ\underline{\beta}(t)}{1-G\circ\underline{\beta}(t)}\ge A^{-\alpha}.$$

Thus the proof is complete.

LEMMA 3. *If*

$$\lim_{n\to\infty}F^n(\underline{a}_n\underline{x}+\underline{b}_n)=H(\underline{x}),\qquad \lim_{n\to\infty}G^n(\underline{a}_n\underline{x}+\underline{b}_n)=H(\underline{A}\,\underline{x}+\underline{B}),$$

where $H_i = \Lambda$ *for* $i=1,\ldots,m$ *and* $\underline{B} = B\cdot\mathbf{1}$ *for some* $B\in\mathbf{R}$, *then* $\underline{A}=\mathbf{1}$, $\underline{x}_F^0 = \underline{x}_G^0$ *and*

$$\lim_{t\to\infty}\frac{1-F\circ\underline{\beta}(t)}{1-G\circ\underline{\beta}(t)}=e^B$$

for any curve $\underline{\beta}$ *such that* (3) *and* (4) *hold.*

Proof. From the assumptions and Lemma 2.5 of [5] we have $A_i = 1$ and $x^0_{F_i} = x^0_{G_i}$ $(= x^0_i$, say) for $i=1,\ldots,m$. Hence

(8)
$$\lim_{n\to\infty}G_i^n(a_n^ix_i+b_n^i)=\Lambda(x_i+B),$$
$$\lim_{n\to\infty}F_i^n(a_n^ix_i+b_n^i)=\Lambda(x_i),\qquad i=1,\ldots,m.$$

Similarly to Lemma 2 we may set (see [3])

$$b_n^i=\mu_n^{F_i},\qquad a_n^i=F_i^{-1}(1-(ne)^{-1})-\mu_n^{F_i}.$$

From (8) we have

$$\lim_{n\to\infty}G_i^n(a_n^ix_i+\mu_n^{F_i}-a_n^iB)=\Lambda(x_i),$$

that is, $G_i\in D(\Lambda)$ and as above we have

$$\lim_{n\to\infty}G_i^n([G_i^{-1}(1-(ne)^{-1})-\mu_n^{G_i}]x_i+\mu_n^{G_i})=\Lambda(x_i).$$

By Khinchin's theorem

$$(\mu_n^{F_i}-a_n^iB-\mu_n^{G_i})/a_n^i\to0\quad\text{as } n\to\infty,$$

and this means that

$$(\mu_n^{F_i}-\mu_n^{G_i})/a_n^i\to B\quad\text{as } n\to\infty,\ i=1,\ldots,m.$$

Let $\varepsilon>0$ be given. For sufficiently large n and for $i=1,\ldots,m$

(9)
$$\mu_n^{F_i}-(B+\varepsilon)a_n^i<\mu_n^{G_i}<\mu_n^{F_i}-(B-\varepsilon)a_n^i.$$

If $\underline{\beta}$ is a curve for which both (3) and (4) hold then for sufficiently large t there exists $n = n(t)$ such $\underline{\beta}(t) \in A_n$ and (9) holds. Hence

$$(10) \qquad \frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} \le \frac{1 - F(\underline{\mu}_n^G)}{1 - G(\underline{\mu}_{n+1}^G)} \le \frac{n[1 - F(-(B+\varepsilon)\underline{a}_n + \underline{\mu}_n^F)]}{n[1 - G(-(B-\varepsilon)\underline{a}_{n+1} + \underline{\mu}_{n+1}^F)]}.$$

Since (see [4])

$$H(\underline{x}) = \exp\Big\{ - \int_{\mathcal{H}} \max_{1 \le i \le m} (u_i e^{-x_i})\, S(d\underline{u}) \Big\},$$

where $\mathcal{H} = \{\underline{u} \in [0,\infty]^m \setminus \{0\} : \|\underline{u}\| = 1\}$ and S is some finite measure on $\mathcal{H}$ such that $\int_{\mathcal{H}} u_i\, S(d\underline{u}) = 1$ for $i = 1,\dots,m$, we obtain

$$-\log H(\underline{0}) > 0, \qquad -\log H(-B \cdot \mathbf{1}) = e^B(-\log H(\underline{0})).$$

Since $n = n(t) \to \infty$ as $t \to \infty$, applying Lemma 1 to (10) we have

$$\limsup_{t\to\infty} \frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} \le e^B.$$

In a similar way it may be shown that

$$\liminf_{t\to\infty} \frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} \ge e^B.$$

This completes the proof.

Applying Lemma 2 we show the following theorem:

THEOREM 2. *Let* $\lim_{n\to\infty} F^n(\underline{a}_n \underline{x} + \underline{b}_n) = H(\underline{x})$, *where* $H_i = \Psi_\alpha$ *and* $x^0_{F_i} \ne 0$ *for* $i = 1,\dots,m$. *Then*

$$\lim_{n\to\infty} G^n(\underline{a}_n \underline{x} + \underline{b}_n) = H(A \cdot \mathbf{1} \cdot \underline{x} + \underline{B}), \qquad A > 0,$$

if and only if $\underline{B} = \underline{0}$ *and* $F \overset{A^{-\alpha}}{\sim} G$.

Proof. If $\underline{B} = \underline{0}$ and $F \overset{A^{-\alpha}}{\sim} G$ the proof is an immediate consequence of Corollary 1, so let us consider the converse implication. From Lemma 2 we have $\underline{x}^0_F = \underline{x}^0_G = \underline{x}^0$ and

$$\lim_{t\to\infty} \frac{1 - F \circ \underline{\beta}(t)}{1 - G \circ \underline{\beta}(t)} = A^{-\alpha},$$

where $\underline{\beta}$ is a continuous curve for which both (3) and (4) hold. Let $\underline{x}_n = (x_n^1,\dots,x_n^m)$, $n \ge 1$, be any sequence for which $x_n^i \nearrow x_i^0$ as $n \to \infty$, for $i = 1,\dots,m$. Let $\underline{z}_n = (z_n^1,\dots,z_n^m), n \ge 1$, be defined by $z_n^i = \beta_i(t_n)$, where $t_n \nearrow \infty$ as $n \to \infty$. Since $F_i \in D(\Psi_\alpha)$ we have $x_i^0 < \infty$ for $i = 1,\dots,m$ (see [3]). Hence

$$\lim_{n\to\infty} x_n^i / z_n^i = x_i^0 / x_i^0 = 1, \qquad i = 1,\dots,m.$$

Since F and G are continuous (as in the Introduction) we have

$$\lim_{n\to\infty}\frac{1-F(\underline{x}_n)}{1-G(\underline{x}_n)}=\lim_{n\to\infty}\frac{1-F\left(\frac{x_n^1}{z_n^1}z_n^1,\dots,\frac{x_n^m}{z_n^m}z_n^m\right)}{1-G\left(\frac{x_n^1}{z_n^1}z_n^1,\dots,\frac{x_n^m}{z_n^m}z_n^m\right)}$$
$$=\lim_{n\to\infty}\frac{1-F\circ\underline{\beta}(t_n)}{1-G\circ\underline{\beta}(t_n)}=A^{-\alpha}.$$

Since $\underline{x}_n$ is arbitrary, the proof is complete.

If Lemma 3 is used in an analogous way the following theorem may be shown:

THEOREM 3. *Let* $\lim_{n\to\infty}F^n(\underline{a}_n\underline{x}+\underline{b}_n)=H(\underline{x})$, *where* $H_i=\Lambda$ *and* $0\neq x^0_{F_i}<\infty$ *for* $i=1,\dots,m$. *Then*

$$\lim_{n\to\infty}G^n(\underline{a}_n\underline{x}+\underline{b}_n)=H(\underline{A}\,\underline{x}+B\cdot\mathbf{1})$$

if and only if $\underline{A}=\mathbf{1}$, $\underline{x}^0_F=\underline{x}^0_G$ *and* $F\overset{e^B}{\sim}G$.

3. Asymptotic independence. The problem of asymptotic independence of the random vector

$$(\max_{1\le k\le n}(X_k^1),\max_{1\le k\le n}(X_k^2),\dots,\max_{1\le k\le n}(X_k^m))\qquad\text{as } n\to\infty$$

for a sequence $\underline{X}_1,\underline{X}_2,\dots$ of independent identically distributed random vectors was considered in [1], [4]. In [1] the following theorem was proved:

THEOREM 4 (S. M. Berman). *For independent identically distributed random vectors* $\underline{X}_1,\underline{X}_2,\dots$ *with distribution function* F, *suppose that* $F\in D(H)$ *and*

$$\lim_{x_i\nearrow x^0_{F_i},\,x_j\nearrow x^0_{F_j}}\frac{1-F_i(x_i)-F_j(x_j)+F_{ij}(x_i,x_j)}{1-F_{ij}(x_i,x_j)}=0\tag{11}$$

for $1\le i\neq j\le m$. *Then*

$$H(\underline{x})=\prod_{i=1}^{m}H_i(x_i).$$

As a consequence, we can prove the following result.

THEOREM 5. *Suppose that* $F\in D(H^{(1)})$, $G\in D(H^{(2)})$ *and*

(i) $F_{ij}\overset{\gamma}{\sim}G_{ij}$ *for* $1\le i\neq j\le m$,
(ii) (11) *holds for at least one of* F *and* G.

Then

$$H^{(1)}(\underline{x}) = \prod_{i=1}^{m} H_i^{(1)}(x_i), \qquad H^{(2)}(\underline{x}) = \prod_{i=1}^{m} H_i^{(2)}(x_i),$$

$$H^{(1)}(\underline{x}) = H^{(2)}(\underline{A}\,\underline{x} + \underline{B}) \quad \textit{for some } \underline{A} > \underline{0},\ \underline{B} \in \mathbf{R}^m .$$

Proof. Assume that (11) holds for F. Then, by Theorem 4, $H^{(1)}(\underline{x}) = \prod_{i=1}^{m} H_i^{(1)}(x_i)$. Define

$$u_{ij}^F(x_i, x_j) = \frac{1 - F_i(x_i) + 1 - F_j(x_j)}{1 - F_{ij}(x_i, x_j)}$$

and u_{ij}^G similarly.

From (i) we have $x_{F_i}^0 = x_{G_i}^0$ ($= x_i^0$, say) for $i = 1, \ldots, m$. It is easy to notice that (11) is equivalent to

(12) $$\lim_{x_i \nearrow x_i^0,\, x_j \nearrow x_j^0} u_{ij}^F(x_i, x_j) = 1 .$$

Since $F_{ij} \overset{\mathcal{T}}{\sim} G_{ij}$ we also have $F_i \overset{\mathcal{T}}{\sim} G_i$. Hence for any $\varepsilon > 0$ we have

$$\gamma^{-1} - \varepsilon \le \frac{1 - G_i(x_i)}{1 - F_i(x_i)} \le \gamma^{-1} + \varepsilon, \quad i = 1, \ldots, m,$$

and

$$\gamma - \varepsilon \le \frac{1 - F_{ij}(x_i, x_j)}{1 - G_{ij}(x_i, x_j)} \le \gamma + \varepsilon, \quad 1 \le i \ne j \le m,$$

for x_i, x_j sufficiently close to x_i^0, x_j^0. Hence

$$\begin{aligned}(\gamma^{-1} - \varepsilon)(\gamma - \varepsilon) u_{ij}^F(x_i, x_j) &\le u_{ij}^G(x_i, x_j) \\ &\le (\gamma^{-1} + \varepsilon)(\gamma + \varepsilon) u_{ij}^F(x_i, x_j)\end{aligned}$$

and since (12) holds we obtain

$$\begin{aligned}(\gamma^{-1} - \varepsilon)(\gamma - \varepsilon) &\le \liminf_{x_i \nearrow x_i^0,\, x_j \nearrow x_j^0} u_{ij}^G(x_i, x_j) \\ &\le \limsup_{x_i \nearrow x_i^0,\, x_j \nearrow x_j^0} u_{ij}^G(x_i, x_j) \le (\gamma^{-1} + \varepsilon)(\gamma + \varepsilon) .\end{aligned}$$

Since ε is arbitrary we have

$$\lim_{x_i \nearrow x_i^0,\, x_j \nearrow x_j^0} u_{ij}^G(x_i, x_j) = 1$$

and so, by Theorem 4, $H^{(2)}(\underline{x}) = \prod_{i=1}^{m} H_i^{(2)}(x_i)$. Hence and from (i) it is easy to see that $H^{(2)}(\underline{x}) = H^{(1)}(\underline{A}\,\underline{x} + \underline{B})$ for some $\underline{A} > \underline{0}$ and $\underline{B} \in \mathbf{R}^m$. Thus the theorem is proved.

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