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B. KOPOCIŃSKI (Wrocław)

THE EXTREME GAP IN THE MULTIVARIATE POISSON PROCESS

Abstract. Let N be a Poisson process and let G be a bounded set in $\mathbf{R}^d$. Define D to be the radius of the largest sphere in $\mathbf{R}^d$ with centre in G which contains no signals of N. We study the limiting distribution of the suitably standardized random variable D when the norm in $\mathbf{R}^d$ is $|x| = \max_{1\le j\le d}|x_j|$, $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, and G is a straight line segment.

1. Problem. Consider a Poisson process N in Euclidean space $\mathbf{R}^d$. Let $|\cdot|$ be any arbitrary norm in $\mathbf{R}^d$ and let $G \subset \mathbf{R}^d$ be a bounded set. Suppose that the signals of the process N are arranged in a sequence $X_1, X_2, \ldots$ (for example in the order of increasing distances from the origin). Define the random variable

$$D = \sup_{Y\in G} \min_{1\le i<\infty} |X_i - Y|,$$

which is the radius of the largest sphere with centre in G which contains no signals of the process. The problem is to study the limiting distribution of the suitably standardized random variable D as the parameter λ of the process N tends to infinity.

This problem arises in the analysis of morbidity in a population localized in some region and in testing the importance of regions without illnesses.

Note that Deheuvels [1] defined the maximal spacing for points uniformly distributed in the d-cube $[0,1]^d$ as the size of the largest cubical gap parallel to the unit cube. As an extension Janson [4] considered the volume of the maximal gap of the shape and orientation of a convex set. In those papers the condition that the centre of the gap lies in a given set G was not assumed. Dette and Henze [2] introduced the largest nearest-neighbour link

$$D = \max_{1\le i\le n} \min_{j\ne i} |X_i - X_j|$$

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where X_i, $i = 1, \ldots, n$, is a sequence of independent random points each uniformly distributed within the unit d-cube. Our problem was to find strong bounds or limiting distribution for D.

2. Main result. Consider the norm $|x| = \max_{1 \le j \le d} |x_j|$, $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, so $w_d(\xi) = (2\xi)^d$ is the volume of the d-sphere $S = \{x : |x| < \xi\}$, and let $G = \{x : 0 \le x_1 \le t,\ x_2 = \ldots = x_d = 0\}$, $t \ge 0$, be a straight line segment in $\mathbf{R}^d$. Then $D = D(t)$ is the radius of the largest ball with centre in G which contains no signal of the process N.

In what follows, the symbol $\xrightarrow{d}$ means convergence in distribution, Λ is the extreme-value distribution: $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbf{R}$. The main result concerns the limiting behaviour of $D(t)$ as $t \to \infty$. It is worthwhile to notice the dimensional effect with standardizing functions when $n \ge 2$. A similar effect was observed by Dette and Henze [2] in their version of the problem when $d \ge 3$.

THEOREM. *As $t \to \infty$, $\xi \in \mathbf{R}$, we have*

$$(1) \qquad P(\lambda(2D(t))^d - \log t - \log((\lambda \log^{d-1} t)^{1/d}) \le \xi) \xrightarrow{d} \Lambda(\xi).$$

LEMMA. *The function $R(t,\xi) = P(D(t) > \xi)$ satisfies the recursion relation*

$$(2) \qquad R(t,\xi) = \exp(-\lambda w_d(\xi)) + \int_0^{\min(2\xi,t)} R(t-u,\xi)\exp(-\lambda w_{d-1}(\xi)u)\lambda w_{d-1}(\xi)\,du$$

whose solution is

$$(3) \qquad R(t,\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}(-\lambda w_{d-1}(\xi))\exp(-\lambda w_d(\xi))^k \times [\exp(-\lambda w_d(\xi))(t-2k\xi)_+^k - (1_{k>0})(t-2(k-1)\xi)_+^k],$$

where $a_+ = \max(0,a)$.

Proof of Lemma. The equation (2) may be obtained from the total probability formula under the condition of occurrence of the signals of the process N in the ball S. The equation is of renewal type and it may be solved using the Laplace transform:

$$R^*(s,\xi) = \int_0^{\infty} e^{-st} R(t,\xi)\,dt = \frac{1}{s}\exp(-\lambda w_d(\xi)) + R^*(s,\xi)\frac{\lambda w_{d-1}(\xi)}{s+\lambda w_{d-1}(\xi)}(1-\exp(-\lambda w_d(\xi) - 2\xi s)).$$

It follows that

$$R^*(s,\xi) = \frac{s + w_{d-1}(\xi)\lambda}{s} \frac{\exp(-\lambda w_d(\xi))}{s + \lambda w_{d-1}(\xi)\exp(-\lambda w_d(\xi) - 2\xi s)}$$
$$= \frac{s + \lambda w_{d-1}(\xi)}{s^2} \sum_{k=0}^{\infty} \left(-\frac{\lambda}{s} w_{d-1}(\xi)\right)^k$$
$$\times \exp(-\lambda(k+1) w_d(\xi)) \exp(-2k\xi s),$$

and hence

$$R(t,\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}(-\lambda w_{d-1}(\xi))^k \exp(-\lambda w_d(\xi)(k+1))(t - 2k\xi)_+^k$$
$$- \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(-\lambda w_{d-1}(\xi))^{k+1}$$
$$\times \exp(-\lambda w_d(\xi)(k+1))(t - 2k\xi)_+^{k+1},$$

which implies (3).

Proof of Theorem. Define

$$a(t) = (\lambda \log^{d-1} t)^{1/d},$$
$$\xi(t) = \frac{1}{2}\left(\frac{1}{\lambda}(\xi + \log t + \log a(t))\right)^{1/d}.$$

Then

$$\exp(-\lambda w_d(\xi(t))) = \exp(-\xi)/(ta(t)),$$
$$\xi(t)/t \to 0, \quad \lambda w_{d-1}(\xi(t))/a(t) \to 1, \quad t \to \infty.$$

An easy computation shows that

$$P(\lambda(2D(t))^d - \log t - \log a(t) > \xi) = R(t, \xi(t))$$
$$= \frac{1}{ta(t)} e^{-\xi} + \sum_{k=1}^{\infty} \frac{1}{k!}(-\lambda w_{d-1}(\xi(t)) e^{-\xi}/(ta(t)))^k$$
$$\times \left[\frac{1}{ta(t)} e^{-\xi}(t - 2k\xi(t))_+^k - (t - 2(k-1)\xi(t))_+^k\right]$$
$$= -\sum_{k=1}^{\infty} \frac{1}{k!}(-1)^k e^{-k\xi} + o(1) = 1 - \Lambda(\xi) + o(1), \quad t \to \infty.$$

3. Two related cases. For one dimension two related problems were formulated for the largest nearest link among random points which may be recognized as modifications of our problem. Now, in both cases we prove the limiting theorems on the base of the exact probability distribution function.

3.1. *The extreme space in a uniformly distributed sample.* Let $X_1, \ldots, X_n$ be independent random variables each uniformly distributed on $[0,t]$, let $X_{1,n} \le \ldots \le X_{n,n}$ be the order statistics and let $X_{0,n} = 0$, $X_{n+1,n} = t$. Write

$$D_n(t) = \max_{0\le i\le n}(X_{i+1,n} - X_{i,n}).$$

It is easy to show that

$$P\left(\frac{n}{t}D_n(t) - \log n \le \xi\right) \xrightarrow{d} \Lambda(\xi), \quad n \to \infty.$$

Note that the function $R_n(t,\xi) = P(D_n(t) > \xi)$ satisfies the recursion relation.

$$R_n(t,\xi) = \left(1 - \frac{\xi}{t}\right)_+^n + \int_0^{\min(t,\xi)} R_{n-1}(t-u,\xi)\left(1-\frac{u}{t}\right)^{n-1} n\,du$$

whose solution is (see Feller [3], I, (9.9))

$$R_n(t,\xi) = 1 - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\left(1-\frac{k\xi}{t}\right)_+^n.$$

Since

$$n^k\left(1 - \frac{k}{n}(\xi + \log n)\right)^n \to e^{-\xi k}, \quad n \to \infty,$$

we deduce that

$$P\left(\frac{n}{t}D_n(t) - \log n > \xi\right) = R_n\left(t, \frac{t}{n}(\xi + \log n)\right) \xrightarrow{d} 1 - \Lambda(\xi).$$

3.2. *The extreme gap in a univariate Poisson process.* Consider the Poisson process on $\mathbf{R}$ generated by independent random variables $\ldots, U_{-1}, U_1, U_2, \ldots$ each exponentially distributed with parameter λ. Set first $S_0 = -U_{-1}$, $S_1 = U_1$, $S_{n+1} = S_n + U_{n+1}$, $n = 1,2,\ldots$, and

$$N(t) = \sum_{n=1}^{\infty} 1_{S_n \le t},$$

$$\gamma_1(t) = t - S_{N(t)}, \quad \gamma_2(t) = S_{N(t)+1} - t.$$

Then

$$D(t) = \sup_{0\le u\le t}\min(\gamma_1(u), \gamma_2(u))$$

defines the extreme gap in the Poisson process. The distribution function of $D(t)$ is given by the Lemma and the Theorem for $d = 1$.

Now let $\widetilde{S}_0 = 0$, $\widetilde{S}_{n+1} = \widetilde{S}_n + U_{n+1}$, $n = 0,1,\ldots$, and

$$\gamma(t) = t - \widetilde{S}_{N(t)}.$$

Then

$$\widetilde{D}(t) = \sup_{0\le u\le t} \gamma(u)$$

defines the extreme gap in Feller's sense. We prove that $\widetilde{D}(t)$ and $D(t)$ are asymptotically equidistributed and the relation (1) holds for $d = 1$. To prove

$$P(\lambda\widetilde{D}(t) - \log\lambda t \le \xi) \xrightarrow{d} \Lambda(\xi)$$

recall from [5] (also [3], XIV, (2.6) after a suitable transformation) that

$$\begin{aligned}\widetilde{R}(t,\xi) &= P(\widetilde{D}(t) > \xi)\\ &= e^{-\lambda\xi}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}(-\lambda(t-k\xi)_+e^{-\lambda\xi})^{k-1}\left(1+\frac{\lambda}{k}(t-k\xi)_+\right).\end{aligned}$$

Now, some algebra gives

$$P(\lambda\widetilde{D}(t) - \log t > \xi) = \widetilde{R}\left(t, \frac{1}{\lambda}(\xi + \log t)\right) \xrightarrow{d} 1 - \Lambda(\xi).$$

References

[1] P. Deheuvels, *Strong bounds for miltidimensional spacings*, Z. Wahrsch. Verw. Gebiete 64 (1983), 411–424.

[2] H. Dette and N. Henze, *The limit distribution of the largest nearest-neighbour link in the unit d-cube*, J. Appl. Probab. 26 (1989), 67–80.

[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, Wiley, New York 1966.

[4] S. Janson, *Maximal spacings in several dimensions*, Ann. Probab. 15 (1987), 274–280.

[5] I. Kopocińska and B. Kopociński, *On the extreme gap in the renewal process*, this issue, 127–136.

BOLESLAW KOPOCIŃSKI
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCLAW
PL. GRUNWALDZKI 2/4
50-384 WROCLAW, POLAND

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