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R. H. WARREN (King of Prussia, PA)

POLYNOMIALLY SOLVABLE TRAVELING SALESMAN PROBLEMS

Abstract. We introduce a class of symmetric traveling salesman problems which can be solved in polynomial time. Roughly, it can be decided in polynomial time if there is an ordering of the cities such that there is a band of small, equal elements about the diagonal of the distance matrix. In this case, an optimal tour is known. An open question about polynomial time special cases is included.

Introduction. The special case of the traveling salesman problem (TSP) in Lemma 1 of [2] has a specific tour of minimum length. Then it is shown in [2] that there is a polynomial time recognition algorithm for TSPs which can be transformed by renumbering cities so that a simpler case of Lemma 1 holds. The results for the TSP in this paper have the same general pattern.

In Lemma 1 we state a special case of the TSP which has a specific tour of minimum length. The proof of Theorem 1 contains a polynomial time algorithm for recognizing TSPs which can be transformed into a simpler case of Lemma 1 by renumbering cities. We conclude with a general open question about special cases of the TSP.

A TSP is characterized by a square matrix whose i,j entry ($i \neq j$) is the distance from city i to city j. A *tour* for the salesman is a cyclic permutation of the cities. A solution for a TSP (called an *optimal tour*) is a tour which has minimum length, i.e., the sum of the entries from the matrix for the tour is a minimum over the sums for all tours.

Some of the properties which characterize TSPs are found in [1], [3], and [4].

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Special case. We assume all TSPs are symmetric.

In Lemma 1 a specific tour $\widehat{t}$ is shown to be optimal for all TSPs with a band characteristic. An induction argument lifts the optimal tour $\widehat{t}$ on $n-1$ cities to the optimal tour $\widehat{t}$ on n cities when the band characteristic is satisfied.

LEMMA 1. *Let $A = [a_{ij}]$ be an n by n TSP where $n \geq 5$. Let i, j, p, q be four distinct integers. Let m be an integer such that $m = q$ or $m \neq i, j, p, q$. Let*

$$a_{ij} + a_{jm} + a_{pq} \geq a_{im} + a_{jp} + a_{jq}$$

whenever $|i-j| \geq 3$, $|p-q| \leq 3$, $|j-p| < 3$, and $|j-q| < 3$. Let

$$a_{15} + a_{43} + a_{32} \geq a_{13} + a_{35} + a_{42}\,.$$

Then $\widehat{t} = (135\ldots642)$ is an optimal tour for A.

Proof. Induction on n. We denote the length of a tour t by $f(t)$. For $n = 5$ there are 11 tours t_i to be considered. We group them according to the pattern of $f(t_i) - f(\widehat{t})$ where $\widehat{t} = (13542)$.

Group I	Group II	Group III	Group IV
t_1=(15324)	t_2=(14532)	t_3=(15234)	t_4=(15432)
t_5=(14352)	t_6=(15342)		
t_7=(15243)	t_8=(13452)		
t_9=(14523)	t_{10}=(14253)		
	t_{11}=(15423)		

Group I is the case when $m \neq i, j, p, q$. Group II is the case $m = q$, and Group III uses both $m \neq i, j, p, q$ and $m_1 = q_1$. We will show one calculation from Groups I and II. The others are similar.

$$\begin{aligned} f(t_1) - f(\widehat{t}) &= a_{51} + a_{14} + a_{23} - (a_{54} + a_{12} + a_{13}) \geq 0\,, \\ f(t_2) - f(\widehat{t}) &= a_{14} + a_{32} - (a_{13} + a_{42}) \\ &= a_{14} + a_{43} + a_{23} - (a_{13} + a_{42} + a_{43}) \geq 0\,. \end{aligned}$$

For Group III,

$$\begin{aligned} f(t_3) - f(\widehat{t}) = {} & a_{14} + a_{43} + a_{52} - (a_{13} + a_{45} + a_{42}) \\ & + a_{51} + a_{13} + a_{23} - (a_{53} + a_{12} + a_{13}) \geq 0\,. \end{aligned}$$

For Group IV, $f(t_4) - f(\widehat{t}) \geq 0$ is a direct result of the hypothesis.

Let t be a tour on $n-1$ cities. If the salesman proceeds directly from city i to city j on tour t, then by t_j^i we denote the tour on n cities that is obtained when n is inserted in t between i and j. By the induction assumption and symmetry (which accounts for the odd and even cases of $\widehat{t}$), it follows that

$$\begin{aligned} f(t^i_j) - f(\widehat{t}) &\geq a_{in} + a_{nj} - a_{ij} - (a_{n-1,n} + a_{n,n-2} - a_{n-1,n-2}) \\ &= a_{in} + a_{nj} + a_{n-1,n-2} - (a_{ij} + a_{n,n-1} + a_{n,n-2}). \end{aligned}$$

To verify that $f(t^i_j) - f(\widehat{t}) \geq 0$, the following four cases need to be checked:

$$\begin{aligned} &\{i,j\} \cap \{n-1, n-2\} = \{n-1, n-2\}, \\ &\{i,j\} \cap \{n-1, n-2\} = \{n-1\}, \\ &\{i,j\} \cap \{n-1, n-2\} = \{n-2\}, \\ &\{i,j\} \cap \{n-1, n-2\} = \emptyset. \ \blacksquare \end{aligned}$$

We consider a simpler case of Lemma 1 when there is a band of equal elements about the main diagonal which are smaller than the elements outside the band. For z in $\{2, 3, \ldots, n-1\}$, an n by n TSP $A = [a_{ij}]$ is said to have a *z-band* if $0 < |k-m| \leq z$ and $0 < |p-q| \leq z < |r-s|$ implies that $a_{km} = a_{pq} < a_{rs}$. Note that if a TSP has a z-band, then $\widehat{t} = (135\ldots642)$ is an optimal tour.

EXAMPLE 1. For $n = 5$ we will illustrate the 0, 1 symmetric TSPs with various z-bands.

$$z = 2 \quad \begin{bmatrix} - & 0 & 0 & 1 & 1 \\ 0 & - & 0 & 0 & 1 \\ 0 & 0 & - & 0 & 0 \\ 1 & 0 & 0 & - & 0 \\ 1 & 1 & 0 & 0 & - \end{bmatrix} \qquad z = 3 \quad \begin{bmatrix} - & 0 & 0 & 0 & 1 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 1 & 0 & 0 & 0 & - \end{bmatrix}$$

$$z = 4 \quad \begin{bmatrix} - & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & 0 & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} - & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 \\ 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & - & 1 \\ 1 & 1 & 1 & 1 & - \end{bmatrix}$$

THEOREM 1. *Let A be an n by n TSP and $n \geq 4$. Then it can be determined in polynomial time if there is a renumbering of the cities of A such that the resulting TSP has a z-band.*

Proof. Let $A = [a_{ij}]$. The number of elements a_{pq} such that $0 < |p-q| \leq z$ is $z(2n-z-1)$. The a_{ij} $(i \neq j)$ are ordered to determine if for some z in $\{2, 3, \ldots, n-1\}$ there are $z(2n-z-1)$ equal elements which are smallest. If so, this set is called B. Then we look for a row of A, say k, which has exactly z members of B. If so, we interchange cities 1 and k, and renumber the cities so that the z members of B in row 1 are in positions

$(1,2),(1,3),\ldots,(1,z+1)$. The entry in each of the positions

$$\begin{aligned}(2,3),(2,4),\ldots,(2,z+1)&,\\ (3,4),\ldots,(3,z+1)&,\\ &\vdots\\ (z,z+1)&\end{aligned}$$

is checked for membership in B. If $z < n-1$, then for $r = 2,\ldots,n-z$, the entries in row r in positions $(r,r+z),\ldots,(r,n)$ are checked for exactly one member of B. If this member of B is not in position $(r,r+z)$, it is moved to position $(r,r+z)$ by interchanging two cities. Finally, for $z < n-1$, the entry in column $r+z$ in each of the positions $(r+1,r+z),\ldots,(r+z-1,r+z)$ is checked for membership in B. The result is an $O(n^3)$ algorithm. ■

EXAMPLE 2. We will illustrate the proof of Theorem 1 for $n = 5$ and

$$A = \begin{bmatrix} - & 1 & 0 & 0 & 0 \\ 1 & - & 1 & 0 & 0 \\ 0 & 1 & - & 0 & 1 \\ 0 & 0 & 0 & - & 0 \\ 0 & 0 & 1 & 0 & - \end{bmatrix}.$$

Ordering the elements of A indicates that for $z = 2$ there are $z(2n-z-1) =$ 14 equal elements which are smallest. So $B = \{a_{ij} : a_{ij} = 0\}$ contains 14 members. Since row 1 of A has more than 2 members of B and row 2 of A has exactly 2 members of B, we exchange cities 1 and 2. Then we move the two zeros in row 1 to positions $(1,2)$ and $(1,3)$ by exchanging cities 2 and 5, and cities 3 and 4. The result is

$$\begin{bmatrix} - & 0 & 0 & 1 & 1 \\ 0 & - & 0 & 1 & 0 \\ 0 & 0 & - & 0 & 0 \\ 1 & 1 & 0 & - & 0 \\ 1 & 0 & 0 & 0 & - \end{bmatrix}.$$

Then for $r = 2$, position $(2,4)$ does not have a member of B. Exchanging cities 4 and 5 shifts a member of B from position $(2,5)$ to position $(2,4)$. The result is the first matrix in Example 2.

Open question. There appears to be more than 20 special cases of the TSP in the literature which are solvable in polynomial time. Ten or so of them are in [1]. It seems that it would be useful to formulate a general theory which encompasses special cases and shows their relationships. This may reveal some of the structure of the TSP.

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RICHARD H. WARREN
VALLEY FORGE SPACE CENTER
GENERAL ELECTRIC COMPANY
KING OF PRUSSIA, PENNSYLVANIA 19406
U.S.A.

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