ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
21,2 (1991), pp. 265–272

A. KRZYWICKI and T. NADZIEJA (Wrocław)

SOME RESULTS CONCERNING THE POISSON–BOLTZMANN EQUATION

If a gas filling up some domain Ω is in thermodynamical equilibrium and the only acting forces are those of gravity then their potential V satisfies the Poisson equation

$$\Delta V = 4\pi\varrho$$

with the density ϱ of the gas given by the Boltzmann formula

$$\varrho = M\mu\exp(-V/kT),$$

where M is the total mass of the gas, k the Boltzmann constant, T the absolute temperature of the gas, constant in Ω by assumption, and

$$\mu = \Big(\int_\Omega \exp(-V/kT)\Big)^{-1}.$$

Putting $u = -V/kT$, we obtain

$$\Delta u + \sigma\mu\exp u = 0, \tag{1}$$

where $\sigma = 4\pi(kT)^{-1}M$. One of the possible boundary conditions imposed upon u is

$$u|_{\partial\Omega} = 0. \tag{2}$$

Problems of the form (1), (2) arise in the theory of gravitational equilibrium of polytropic stars [5] and in thermal ignition [6] (cf. also [7]).

I. If Ω is the unit ball in $\mathbf{R}^n$, $n = 2,3$, then u is radially symmetric (for $n = 3$ the additional assumption $u \in C^2(\overline{\Omega})$ is needed, cf. [2], [8]). In this particular case the equation (1) takes the form

$$(r^n u')' + \sigma\mu r^n \exp u = 0 \tag{3}$$

1991 *Mathematics Subject Classification*: Primary 35J60.
Key words and phrases: Poisson–Boltzmann equation, radially symmetric solution.

with $n = 1, 2$, resp. To visualize some differences between the two- and three-dimensional cases we consider the family of equations

$$(4)\qquad (r^\beta u')' + \sigma\mu r^\beta \exp u = 0\,, \quad \beta \in [1,2]\,,$$

$$(5)\qquad \mu = \Big(\int_0^1 r^\beta \exp u\, dr\Big)^{-1}$$

(we omit in μ the coefficient containing π) with u subject to the boundary conditions

$$(6)\qquad u'(0) = u(1) = 0\,.$$

The equation (4) is invariant under the translation $u \to u+$ const., therefore we can replace (6) by the initial conditions

$$(7)\qquad u'(0) = u(0) = 0\,.$$

To get the original solution one has only to subtract from the modified solution u its value at $r = 1$.

THEOREM 1. *There exists $\sigma^* > 0$ such that for $\sigma \le \sigma^*$ there is a solution of* (4), (6) *and for $\sigma > \sigma^*$ there is no solution. Moreover, for sufficiently small σ the solution is unique. For $\beta = 1$, $\sigma^* = 4$ and the solution is unique for all $\sigma \in [0, 4]$.*

Proof. The case $\beta = 1$ is integrable [2] and the unique solution u, μ of (4), (7) is

$$u(r) = -2\ln(1 + \sigma\mu r^2/8)\,, \quad \mu = 8/(4-\sigma)\,.$$

Passing to the case $\beta > 1$ we begin by showing that the initial value problem

$$(8)\qquad (r^\beta \phi')' + r^\beta \exp\phi = 0\,,$$

$$(9)\qquad \phi(0) = \phi'(0) = 0$$

has a unique solution ϕ defined for all $r > 0$. In the proof Schauder's theorem is used. On the space X of continuous functions over $[0, R]$, R any fixed positive constant, equipped with the supremum norm $\|\ \|$, we define the operator T by

$$(Tw)(r) = -\int_0^r t^{-\beta}\, dt \int_0^t s^\beta \exp w(s)\, ds\,.$$

T is continuous and compact. Moreover, it maps $\{w : w \le 0, \|w\| \le R^2(\beta+1)^{-1}/2\}$, which is a closed and convex subset of X, into itself. Hence T has a fixed point which is a solution of (8), (9).

If w_1, w_2 are solutions of (8), (9) then

$$|w_2(r) - w_1(r)| \le C \int_0^r t^{-\beta}\, dt \int_0^t s^{\beta} |w_2(s) - w_1(s)|\, ds$$
$$\le \frac{C}{\beta - 1} \int_0^r s |w_2(s) - w_1(s)|\, ds$$

by changing the order of integration. The last inequality allows us to apply Gronwall's lemma, from which the desired unicity of solution results.

Now integrating (4) over $[0,1]$ and using (5) we get

$$u'(1) + \sigma = 0\,. \tag{10}$$

Hence our problem (4), (6) is equivalent to the following: find u and μ such that (4), (10) and $u(0) = 0$ are satisfied. To do this we put

$$u(r) = \phi(Ar), \quad A > 0\,,$$

and by (10) the problem of existence of the solution of (4)–(6) reduces to the existence of A satisfying

$$A\phi'(A) = -\sigma\,.$$

The key point is the introduction of the function $\chi(x) = -x\phi'(x)$ whose behaviour is as in Fig. 1.

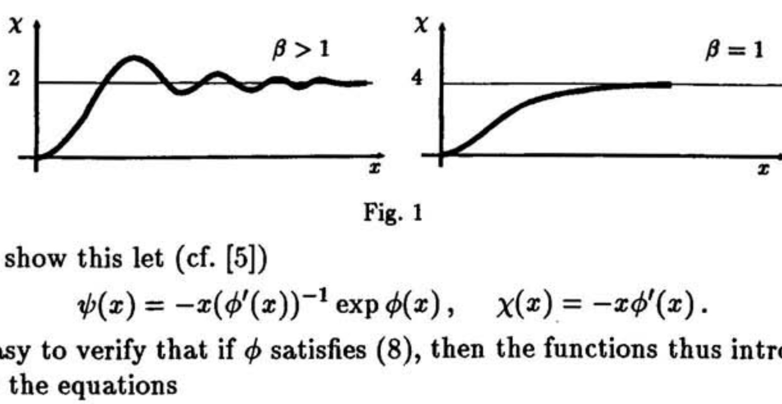

Fig. 1

To show this let (cf. [5])

$$\psi(x) = -x(\phi'(x))^{-1} \exp \phi(x)\,, \quad \chi(x) = -x\phi'(x)\,.$$

It is easy to verify that if ϕ satisfies (8), then the functions thus introduced satisfy the equations

$$\psi' = \frac{\psi}{x}(\beta + 1 - \chi - \psi)\,, \quad \chi' = \frac{\chi}{x}(1 - \beta + \psi)\,.$$

In the new independent variable $s = \ln x$ the last equations may be rewritten in the form

$$\psi' = \psi(\beta + 1 - \chi - \psi)\,, \quad \chi' = \chi(1 - \beta + \psi)\,. \tag{11}$$

If $\beta > 1$ then the equations (11) have singular points $(0,0)$, $(\beta + 1, 0)$, $(\beta - 1, 2)$; the first two are saddles, the third is a sink. The curve corresponding to the solution of (8), (9) is a separatrice $\gamma = (\psi(t), \chi(t))$, $\psi(t) > 0$, starting at the saddle $(\beta + 1, 0)$. It is easy to see that γ is contained in a bounded subset of $\mathbf{R}^2$, hence its ω-limit set is either a periodic orbit or the singular point $(\beta - 1, 2)$. Theorem 31 in [1] (p. 226 and Ex. 7, p. 234) excludes the first possibility, therefore γ looks like the curve presented in Fig. 2. To make it clearer we have only drawn the curve corresponding to the desired solution.

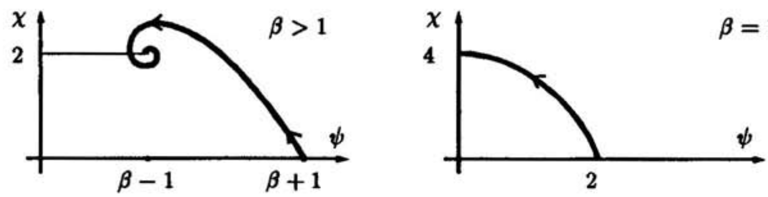

Fig. 2

Looking at the figure we conclude from the graph of χ that the dependence of the solutions of (4), (7) upon the parameter σ may be represented by the diagram of Fig. 3 corresponding to the case $\beta > 1$.

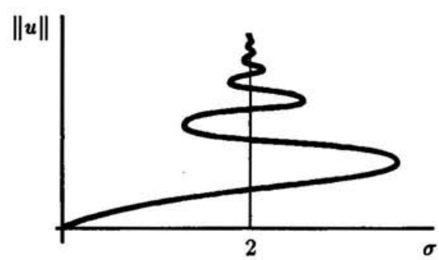

Fig. 3

In the case $\beta = 2$, σ^* may be estimated from above, $\sigma^* < 6$. To see this note that for $r \geq 0$

$$r^2\phi'(r) = -\int_0^r t^2 \exp\phi(t)\,dt$$

$$= -(r^3/3)\exp\phi(r) + \int_0^r (t^3/3)\phi'(t)\exp\phi(t)\,dt\,.$$

Now since $\phi' \leq 0$ we have $r^2\phi'(r) \leq -(r^3/3)\exp\phi(r)$. Integrating and using the preceding equality we get

$$r^2\phi'(r) \geq -6\int_0^r \frac{t^2}{t^2+6}\,dt > -6r\,,$$

from which $\sigma^* < 6$ follows.

II. As in the preceding part, we here also consider radially symmetric solutions of (4), restricting ourselves to $\beta = 2$ (for other values of β the situation is completely similar). However, we modify the domain of definition of the unknown function u to the annulus $a < r < 1$ with a positive and less than one and consider the boundary conditions of the following two forms:

$$u'(a) = k/a^2\,, \quad u(1) = 0\,, \tag{12}$$

$$u(a) = u(1) = 0\,. \tag{13}$$

We show that in contrast to the case considered in part I the equation

$$(r^2u')' + \sigma\mu r^2 \exp u = 0\,, \quad \mu = \Big(\int_a^1 r^2 \exp u \, dr\Big)^{-1}\,, \tag{14}$$

with either of the boundary conditions (12), (13) has a solution for any value of $\sigma > 0$.

Consider the case (12). Integrating (14) we get

$$u'(r) = k/r^2 - (\sigma\mu/r^2) \int_a^r s^2 \exp u \, ds\,, \quad a \le r \le 1\,.$$

We have $\mu \int_a^r s^2 \exp u\, ds \le 1$, hence $(k-\sigma)/r^2 \le u'(r) \le k/r^2$ and

$$|u(r)| \le (|k| + \sigma)/a\,. \tag{15}$$

The problem (14), (12) is equivalent to the equation $u(r) = Tu(r)$, where

$$Tu(r) = k(1 - 1/r) + \sigma\mu \int_r^1 t^{-2}\, dt \int_a^t s^2 \exp u \, ds\,.$$

The operator T considered on the class of continuous functions defined on $[a, 1]$ with supremum norm is continuous and compact and the a priori estimate (15) holds true for all solutions of the family of equations

$$u = \lambda T u\,, \qquad 0 \le \lambda \le 1\,;$$

therefore the theorem of Leray–Schauder may be applied to show the existence of a solution of the problem under consideration.

In a completely similar way the problem with the boundary condition (13) may be treated. This time, using the corresponding Green function we transform the problem to the form

$$u(r) = \frac{\sigma\mu}{1-a}\Big[(1 - a/r) \int_r^1 (1/s - 1)s^2 \exp u\, ds + (1/r - 1) \int_a^r (1 - a/s)s^2 \exp u \, ds\Big]$$

and this is the starting point for a procedure parallel to the preceding one.

It seems that in spite of their simplicity, the last two examples indicate a nontrivial fact of influence of the topology of Ω on the existence of a solution of (1); for other problems that phenomenon was noted earlier (cf. [4]).

III. Consider now the general case of the problem (1), (2) with u defined on a bounded domain Ω in $\mathbf{R}^3$, $\mu = (\int_\Omega \exp u)^{-1}$. We assume the boundary $\partial\Omega$ to be regular enough to guarantee the existence of a Green function G (see below). We prove the following local existence theorem:

THEOREM 2. *There exists a positive constant σ_0 such that the problem* (1), (2) *has a solution for any σ, $0 \le \sigma \le \sigma_0$.*

Proof. The proof is a slight modification of the reasoning given in [3]. Let G be the Green function of Δ for the domain Ω with zero Dirichlet data. Then (1), (2) may be replaced by the equivalent equation

$$u(x) = \sigma\mu(G\exp u)(x)\,, \tag{16}$$

where $(G\exp u)(x) = \int_\Omega G(x,y)\exp u(y)\,dy$.

Consider the space $X = C^0(\Omega)\times\mathbf{R}$, the norm of its elements (v,t) being given by $\|v\| + |t|$, $\|v\| = \sup|v(x)|$. Then X is a Banach space and $B = X_M \times [0,L]$, where $X_M = \{v \in C^0(\Omega) : 0 \le v \le M\}$ and L, M are positive constants, is a closed, bounded, convex subset of X.

Consider on B the continuous transformation $\mathcal{G}$ defined by

$$\mathcal{G}(u,t) = \left(t\frac{G\exp u}{\|G\exp u\|}, \sigma\frac{\|G\exp u\|}{\int_\Omega \exp u}\right).$$

Moreover, by the obvious inequality $\int_\Omega \exp u \ge |\Omega|$ valid for $u \ge 0$, where $|\Omega|$ is the volume of Ω, we have

$$\sigma\frac{\|G\exp u\|}{\int_\Omega \exp u} \le C\sigma|\Omega|^{-1}\exp M\,,$$

where $C = \sup_{x\in\Omega}\int_\Omega G(x,y)\,dy < \infty$. Therefore, if σ, L, M are chosen so that the right hand side of the last inequality does not exceed $L \le M$ then $\mathcal{G} : B \to B$ and the Schauder theorem may be applied to show the existence of a fixed point (u,t) of $\mathcal{G}$, i.e.

$$u = t\frac{G\exp u}{\|G\exp u\|}, \quad t = \sigma\frac{\|G\exp u\|}{\int_\Omega \exp u},$$

which is equivalent to (16) and the proof is complete.

The questions of nonexistence of solutions of (1), (2) for large σ is partly answered by the following theorem.

THEOREM 3. *Suppose that $\partial\Omega$ is of class C^1 and satisfies the following strong starlikeness type property with respect to the point 0 lying inside Ω:*

$$\int_{\partial\Omega} \frac{dS}{\langle x, n\rangle} = A < \infty ,$$

where n is the exterior unit normal to $\partial\Omega$. Then the problem (1), (2) has no solution in the class $C^2(\Omega) \cap C^1(\overline{\Omega})$ for $\sigma > \overline{\sigma} = 3/A$.

Proof. We make use of the Pokhozhaev identity, which for the general equation $-\Delta u = g(u)$ in Ω, $u = 0$ on $\partial\Omega$, has the form (cf. [4])

$$(17) \qquad (1 - 3/2) \int_{\Omega} ug(u) + 3 \int_{\Omega} G(u) = \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 \langle x, n\rangle \, dS ,$$

where $G(u) = \int_0^u g(t)\, dt$. In our case $g(u) = \sigma\mu \exp u$, therefore the left hand side of (17) is

$$-\frac{1}{2}\sigma\mu \int_{\Omega} u \exp u + 3\sigma\mu \int_{\Omega} (\exp u - 1) < 3\sigma ,$$

since $u \geq 0$. Applying now the inequality

$$\sigma^2 = \left(\int_{\partial\Omega} \frac{\partial u}{\partial n} \, dS \right)^2 \leq \int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 \langle x, n\rangle \int_{\partial\Omega} \frac{dS}{\langle x, n\rangle}$$

we get $\sigma < 3/A$. In dimension two the number 3 in (17) should be replaced by 2 and the conclusion is similar.

References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon and A. G. Maĭer, *Qualitative Theory of Second Order Dynamical Systems*, Nauka, Moscow 1966 (in Russian).

[2] G. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, New York 1980.

[3] M. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York 1977.

[4] H. Brezis, *Elliptic equations with limiting Sobolev exponents—the impact of topology*, Comm. Pure Appl. Math. 39 (1986), 17–39.

[5] S. Chandrasekhar, *An Introduction to the Study of Stellar Structures*, Dover Publ., New York 1957.

[6] I. M. Gel'fand, *Some problems in the theory of quasilinear equations*, Uspekhi Mat. Nauk 14 (2) (1959), 87–158 (in Russian).

[7] B. Gidas, Wei-Ming Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68 (1979), 209–243.
[8] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. 49 (1973), 241–269.

ANDRZEJ KRZYWICKI
TADEUSZ NADZIEJA
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCLAW
PL. GRUNWALDZKI 2/4
50-384 WROCLAW, POLAND

Received on 8.6.1989