A REMARK ON CYCLIC TRIDIAGONAL MATRICES

Abstract. A cyclic tridiagonal matrix $A$ is seen as a perturbed tridiagonal matrix $B$. We present processes to find the inverse of $A$ and to solve $Ax = f$ using the inverse of $B$ and the solution of $Bx = f$. These processes are equivalent to those obtained by using the Woodbury formula.

Introduction. A real $(n, n)$-matrix $A$ is said to be a cyclic tridiagonal matrix [3] if

$$A = \begin{bmatrix}
    b_1 & c_1 & & & a_1 \\
    a_2 & b_2 & c_2 & & \\
    & \ddots & \ddots & \ddots & \\
    & & a_{n-1} & b_{n-1} & c_{n-1} \\
    c_n & & & a_n & b_n
\end{bmatrix}.$$ 

Let $B$ be the $(n, n)$-tridiagonal matrix obtained from $A$ when we replace $a_1$ and $c_n$ by 0,

$$B = \begin{bmatrix}
    b_1 & c_1 & & & \\
    a_2 & b_2 & c_2 & & \\
    & \ddots & \ddots & \ddots & \\
    & & a_{n-1} & b_{n-1} & c_{n-1} \\
    & & & a_n & b_n
\end{bmatrix}.$$ 

It follows that $A$ can be obtained from $B$ by using two rank-one modifications

$$(1) \quad A = B + c_n e_n e_n^T + a_1 e_1 e_1^T$$

where $e_j$ is the $(n, 1)$-column matrix having 0 components except the $j$th component which is 1. We also remark that (1) is equivalent to a rank-two

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modification

\[ A = B + RST \]

where \( R = [c_1 e_n, a_1 e_1] \) and \( S = [e_1, e_n] \).

In this paper we will assume that \( \delta_k \neq 0 \), \( k = 1, \ldots, n \), where \( \delta_k \) denotes the \( k \)th leading minor of \( B \). Note that \( \delta_n = \det B \).

The purpose of this paper is to extend the results presented in [3] and [4] using the form (1) and the well known results about \( B \). In particular, we will see how to update the solution of \( Bx = f \) to find the solution of \( Ax = f \), and how to update \( B^{-1} \) to obtain \( A^{-1} \). We begin with a brief review of useful results about the tridiagonal matrix \( B \).

2. Tridiagonal matrix. Under the assumption that \( \delta_k \neq 0 \) for \( k = 1, \ldots, n \) it is well known that it is possible to write \( B = LU \) with

\[
L = \begin{bmatrix}
  r_1 & & & \\
  a_2 & r_2 & & \\
  & \ddots & \ddots & \\
  & & a_n & r_n
\end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix}
  1 & z_1 & & \\
  & 1 & z_2 & \\
  & & \ddots & \ddots \\
  & & & 1 & z_{n-1}
\end{bmatrix}
\]

where \( r_k = \delta_k / \delta_{k-1} \) and \( z_k = c_k / r_k \). The inverse of \( B \) is \( B^{-1} = U^{-1} L^{-1} \),

\[
L^{-1} = (p_{ij})_{n \times n}, \quad p_{ij} = \begin{cases} 0 & \text{if } i < j, \\
(-1)^{i+j} \frac{\delta_{j-1}}{\delta_i} \prod_{k=j+1}^{i} a_k & \text{if } i \geq j;
\end{cases}
\]

\[
U^{-1} = (q_{ij})_{n \times n}, \quad q_{ij} = \begin{cases} 0 & \text{if } i > j, \\
(-1)^{i+j} \frac{\delta_{i-1}}{\delta_j} \prod_{k=i}^{j-1} c_k & \text{if } i \leq j;
\end{cases}
\]

where we have used the notation

\[
\prod_{k=i}^{j} \varepsilon_k = \begin{cases} 1 & \text{if } j < i, \\
\varepsilon_i \varepsilon_{i+1} \ldots \varepsilon_j & \text{if } j \geq i.
\end{cases}
\]

From these expressions we can find the following explicit formulas for the elements of \( B^{-1} = (\beta_{ij})_{n \times n} \):

\[
\beta_{ij} = (-1)^{i+j} \delta_{j-1} \delta_{i-1} \sum_{k=\max(i,j)}^{n} \frac{1}{\delta_{k-1} \delta_k} \left[ \prod_{\ell=i}^{k-1} c_{\ell} \right] \left[ \prod_{\ell=j+1}^{k} a_{\ell} \right]
\]

(see also [4]).

3. Cyclic tridiagonal matrix as perturbed tridiagonal matrix. From the assumption that \( \det B = \delta \neq 0 \), using (1) and \( B^{-1} = [\beta_1, \ldots, \beta_n] \) we obtain

\[
B^{-1} A = I + c_n \beta_n e_1^T + a_1 \beta_1 e_n^T
\]
where \( \beta_j \) denotes the \( j \)th column of \( B^{-1} \).

Using (4) for solving \( Ax = f \) we obtain

\[
x + c_n \beta_n x_1 + a_1 \beta_{11} x_n = g
\]

where \( g = B^{-1} f \). This expression suggests that we first solve for \( x_1 \) and \( x_n \) using the first and the last equations in (5), and obtain

\[
\begin{bmatrix}
  x_1 \\
x_n
\end{bmatrix} = G^{-1} \begin{bmatrix}
g_1 \\
g_n
\end{bmatrix}
\]

where

\[
G = \begin{bmatrix}
  1 + c_n \beta_{1n} & a_1 \beta_{11} \\
c_n \beta_{nn} & 1 + a_1 \beta_{n1}
\end{bmatrix}.
\]

Then we determine \( x_k \) for \( k = 2, \ldots, n-1 \) using the remaining \( n-2 \) equations in (5),

\[
x_k = g_k - c_n \beta_{kn} x_1 - a_1 \beta_{k1} x_n.
\]

The matrix \( G \) is invertible because from (4) we find directly that \( \det A = \det G \det B \). Using (3) we also have

\[
\det A = \det B + (-1)^{n+1} \left\{ \prod_{i=1}^{n} a_i + \prod_{i=1}^{n} c_i \right\} - a_1 c_n \delta_{n-1} \sum_{k=1}^{n-1} \frac{1}{\delta_{k-1} \delta_k} \left( \prod_{i=2}^{k} a_i \right) \left( \prod_{i=1}^{k-1} c_i \right)
\]

(compare with equation (9) in [3], which is incorrect).

Hence, we can solve \( Ax = f \) in four steps:

1. Factor \( B = LU \).
2. Solve \( Bg = f \) for the unknown \( g \) \((g = B^{-1} f)\),
   \[Bu = a_1 e_1\] for the unknown \( u \) \((u = a_1 \beta_{11})\),
   \[Bv = c_n e_n\] for the unknown \( v \) \((v = c_n \beta_{nn})\).
3. Compute \( G^{-1} \) and
   \[
   \begin{bmatrix}
   x_1 \\
x_n
   \end{bmatrix} = G^{-1} \begin{bmatrix}
g_1 \\
g_n
   \end{bmatrix}
   \]
   where
   \[
   G = \begin{bmatrix}
   1 + v_1 & u_1 \\
v_n & 1 + u_n
   \end{bmatrix}.
   \]
4. Compute \( x_k = g_k - x_1 v_k - x_n u_k \) for \( k = 2, \ldots, n-1 \).

This requires \( O(3n) \) divisions, \( O(8n) \) multiplications and \( O(7n) \) additions/subtractions (the expression \( O(h(n)) \) means that \( O(h(n)) = h(n) + o(n) \) where \( \lim_{n \to \infty} o(n)/h(n) = 0 \)).

We can also apply this process to update \( B^{-1} \) and obtain \( A^{-1} = (\alpha_{ij})_{n \times n} \). We solve \( Ax = e_j \) for \( j = 1, \ldots, n \). The inverse is obtained in four steps:

1. Factor \( B = LU \).
2. Find \( B^{-1} \) using the \( LU \) factorization.
3. Form
   \[
   G = \begin{bmatrix}
   1 + c_n \beta_{1n} & a_1 \beta_{11} \\
c_n \beta_{nn} & 1 + a_1 \beta_{n1}
   \end{bmatrix}
   \]
   and compute \( G^{-1} \).
4. For each column $j$: compute

$$
\begin{bmatrix}
\alpha_{1j} \\
\alpha_{nj}
\end{bmatrix} = G^{-1}
\begin{bmatrix}
\beta_{1j} \\
\beta_{nj}
\end{bmatrix}
$$

and

$$
\alpha_{kj} = \beta_{kj} - \alpha_{1j}c_n\beta_{kn} - \alpha_{nj}a_1\beta_{k1}.
$$

This requires $O(3n)$ divisions, $O(3/2n^2)$ multiplications and $O(1/2n^2)$ additions/subtractions for steps 1 and 2, and 4 divisions, $O(2n^2)$ multiplications and $O(2n^2)$ additions/subtractions for steps 3 and 4.

It is interesting to point out that these processes are those obtained using the Woodbury formula [2] for the inverse of $A$. When we consider (2), the Woodbury formula states that

$$
A^{-1} = B^{-1} - B^{-1}R(I + S^TB^{-1}R)^{-1}S^TB^{-1}
$$

where

$$
I + S^TB^{-1}R = G.
$$

References


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