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A REMARK ON CYCLIC TRIDIAGONAL MATRICES

Abstract. A cyclic tridiagonal matrix A is seen as a perturbed tridiagonal matrix B. We present processes to find the inverse of A and to solve $Ax = f$ using the inverse of B and the solution of $Bx = f$. These processes are equivalent to those obtained by using the Woodbury formula.

Introduction. A real (n,n)-matrix A is said to be a *cyclic tridiagonal matrix* [3] if

$$A = \begin{bmatrix} b_1 & c_1 & & & a_1 \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ c_n & & & a_n & b_n \end{bmatrix}.$$

Let B be the (n,n)-tridiagonal matrix obtained from A when we replace a_1 and c_n by 0,

$$B = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix}.$$

It follows that A can be obtained from B by using two rank-one modifications

$$A = B + c_n e_n e_1^T + a_1 e_1 e_n^T \tag{1}$$

where e_j is the $(n,1)$-column matrix having 0 components except the jth component which is 1. We also remark that (1) is equivalent to a rank-two

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modification

$$A = B + RS^T \tag{2}$$

where $R = [c_n e_n, a_1 e_1]$ and $S = [e_1, e_n]$.

In this paper we will assume that $\delta_k \neq 0$, $k = 1, \dots, n$, where δ_k denotes the kth leading minor of B. Note that $\delta_n = \det B$.

The purpose of this paper is to extend the results presented in [3] and [4] using the form (1) and the well known results about B. In particular, we will see how to update the solution of $Bx = f$ to find the solution of $Ax = f$, and how to update B^{-1} to obtain A^{-1}. We begin with a brief review of useful results about the tridiagonal matrix B.

2. Tridiagonal matrix. Under the assumption that $\delta_k \neq 0$ for $k = 1, \dots, n$ it is well known that it is possible to write $B = LU$ with

$$L = \begin{bmatrix} r_1 & & & \\ a_2 & r_2 & & \\ & \ddots & \ddots & \\ & & a_n & r_n \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & z_1 & & & \\ & 1 & z_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & z_{n-1} \\ & & & & 1 \end{bmatrix}$$

where $r_k = \delta_k / \delta_{k-1}$ and $z_k = c_k / r_k$. The inverse of B is $B^{-1} = U^{-1} L^{-1}$,

$$L^{-1} = (p_{ij})_{n\times n}, \quad p_{ij} = \begin{cases} 0 & \text{if } i < j, \\ (-1)^{i+j} \frac{\delta_{j-1}}{\delta_i} \prod_{k=j+1}^{i} a_k & \text{if } i \geq j; \end{cases}$$

$$U^{-1} = (q_{ij})_{n\times n}, \quad q_{ij} = \begin{cases} 0 & \text{if } i > j, \\ (-1)^{i+j} \frac{\delta_{i-1}}{\delta_{j-1}} \prod_{k=i}^{j-1} c_k & \text{if } i \leq j; \end{cases}$$

where we have used the notation

$$\prod_{k=i}^{j} \varepsilon_k = \begin{cases} 1 & \text{if } j < i, \\ \varepsilon_i \varepsilon_{i+1} \dots \varepsilon_j & \text{if } j \geq i. \end{cases}$$

From these expressions we can find the following explicit formulas for the elements of $B^{-1} = (\beta_{ij})_{n\times n}$:

$$\beta_{ij} = (-1)^{i+j} \delta_{i-1} \delta_{j-1} \sum_{k=\max\{i,j\}}^{n} \frac{1}{\delta_{k-1}\delta_k} \Big[\prod_{\ell=i}^{k-1} c_\ell \Big] \Big[\prod_{\ell=j+1}^{k} a_\ell \Big] \tag{3}$$

(see also [4]).

3. Cyclic tridiagonal matrix as perturbed tridiagonal matrix. From the assumption that $\det B = \delta \neq 0$, using (1) and $B^{-1} = [\beta_{\cdot 1}, \dots, \beta_{\cdot n}]$ we obtain

$$B^{-1} A = I + c_n \beta_{\cdot n} e_1^T + a_1 \beta_{\cdot 1} e_n^T \tag{4}$$

where $\beta_{\cdot j}$ denotes the jth column of B^{-1}.

Using (4) for solving $Ax = f$ we obtain

$$x + c_n\beta_{\cdot n}x_1 + a_1\beta_{\cdot 1}x_n = g \tag{5}$$

where $g = B^{-1}f$. This expression suggests that we first solve for x_1 and x_n using the first and the last equations in (5), and obtain

$$\begin{bmatrix} x_1 \\ x_n \end{bmatrix} = G^{-1}\begin{bmatrix} g_1 \\ g_n \end{bmatrix} \quad \text{where} \quad G = \begin{bmatrix} 1 + c_n\beta_{1n} & a_1\beta_{11} \\ c_n\beta_{nn} & 1 + a_1\beta_{n1} \end{bmatrix}.$$

Then we determine x_k for $k = 2, \ldots, n-1$ using the remaining $n-2$ equations in (5),

$$x_k = g_k - c_n\beta_{kn}x_1 - a_1\beta_{k1}x_n.$$

The matrix G is invertible because from (4) we find directly that $\det A = \det G \det B$. Using (3) we also have

$$\begin{aligned} \det A = {} & \det B + (-1)^{n+1}\Big\{\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} c_i\Big\} \\ & - a_1c_n\delta_{n-1}\sum_{k=1}^{n-1}\frac{1}{\delta_{k-1}\delta_k}\Big(\prod_{i=2}^{k} a_i\Big)\Big(\prod_{i=1}^{k-1} c_i\Big) \end{aligned}$$

(compare with equation (9) in [3], which is incorrect).

Hence, we can solve $Ax = f$ in four steps:

1. Factor $B = LU$.
2. Solve $Bg = f$ for the unknown g $(g = B^{-1}f)$,
   $Bu = a_1e_1$ for the unknown u $(u = a_1\beta_{\cdot 1})$,
   $Bv = c_ne_n$ for the unknown v $(v = c_n\beta_{\cdot n})$.
3. Compute G^{-1} and $\begin{bmatrix} x_1 \\ x_n \end{bmatrix} = G^{-1}\begin{bmatrix} g_1 \\ g_n \end{bmatrix}$ where $G = \begin{bmatrix} 1 + v_1 & u_1 \\ v_n & 1 + u_n \end{bmatrix}$.
4. Compute $x_k = g_k - x_1v_k - x_nu_k$ for $k = 2, \ldots, n-1$.

This requires $\mathcal{O}(3n)$ divisions, $\mathcal{O}(8n)$ multiplications and $\mathcal{O}(7n)$ additions/subtractions (the expression $\mathcal{O}(h(n))$ means that $\mathcal{O}(h(n)) = h(n) + o(n)$ where $\lim_{n\to\infty} o(n)/h(n) = 0$).

We can also apply this process to update B^{-1} and obtain $A^{-1} = (\alpha_{ij})_{n\times n}$. We solve $Ax = e_j$ for $j = 1, \ldots, n$. The inverse is obtained in four steps:

1. Factor $B = LU$.
2. Find B^{-1} using the LU factorization.
3. Form $G = \begin{bmatrix} 1 + c_n\beta_{1n} & a_1\beta_{11} \\ c_n\beta_{nn} & 1 + a_1\beta_{n1} \end{bmatrix}$ and compute G^{-1}.

4. For each column j: compute $\begin{bmatrix} \alpha_{1j} \\ \alpha_{nj} \end{bmatrix} = G^{-1} \begin{bmatrix} \beta_{1j} \\ \beta_{nj} \end{bmatrix}$

 and $\alpha_{kj} = \beta_{kj} - \alpha_{1j} c_n \beta_{kn} - \alpha_{nj} a_1 \beta_{k1}$.

This requires $\mathcal{O}(3n)$ divisions, $\mathcal{O}(\frac{3}{2}n^2)$ multiplications and $\mathcal{O}(\frac{1}{2}n^2)$ additions/subtractions for steps 1 and 2, and 4 divisions, $\mathcal{O}(2n^2)$ multiplications and $\mathcal{O}(2n^2)$ additions/subtractions for steps 3 and 4.

It is interesting to point out that these processes are those obtained using the Woodbury formula [2] for the inverse of A. When we consider (2), the Woodbury formula states that

$$A^{-1} = B^{-1} - B^{-1}R(I + S^T B^{-1} R)^{-1} S^T B^{-1}$$

where

$$I + S^T B^{-1} R = G\,.$$

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