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MINIMAX CONTROL OF A SYSTEM WITH ACTUATION ERRORS

0. Introduction. The minimax approach to the problem of control of stochastic systems has been studied in various aspects (see [2], [5]–[9], [11]–[13]). The variety of the problems is caused by the variety of assumptions concerning the uncertainty about the examined system. The description of the uncertainty in our paper is similar to the one considered in [13]. We are looking for strategies which are minimax in the sense determined in that paper.

In this paper a discrete-time linear stochastic system is considered. We assume that the system is observed via random actions which influence its plant and which have an unknown mean. It is also assumed that the parameters of the system are random matrices of known Gaussian distributions. In view of the plant equation (1) these assumptions imply that the states of the system are unknown and it turns out that we deal with the state estimation problem as well. For references connected with the state estimation problems see [6] or [9]; both estimation and control problems are considered in [5].

1. Preliminary remarks and notations. Throughout the paper, Greek and upper-case bold letters except **T**–**Z** indicate matrices, upper-case bold letters **T**–**Z** and lower-case bold letters indicate vectors. Scalars are denoted by italics.

In the sequel we will use the following notations:

$\mathbf{o}_l, \mathbf{e}_l =$ the l-dimensional vectors all of whose components are, respectively, zero and unity,

$\mathbf{I}_l =$ the $l \times l$ identity matrix,

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$\lceil \mathbf{A}_1 \ldots \mathbf{A}_n \rfloor =$ the square matrix in the block-diagonal form with the blocks $\mathbf{A}_i$ on the principal diagonal,

$\|\mathbf{a}\|^2 =$ $\mathbf{a}^T\mathbf{a}$ for every vector $\mathbf{a}$,

$\operatorname{tr} \mathbf{A} =$ the trace of the matrix $\mathbf{A}$,

$\mathbf{A}^+ =$ the Moore–Penrose pseudoinverse matrix to $\mathbf{A}$,

$\mathbf{A} \otimes \mathbf{B} =$ the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$,

$\mathbf{A} * \mathbf{B} =$ the Hadamard product of $\mathbf{A}$ and $\mathbf{B}$.

For definitions and properties of the pseudoinverse matrix and the above matrix products see e.g. [10].

Let $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ be square matrices having the same dimensions. We denote the product $\mathbf{A}_n\mathbf{A}_{n-1} \ldots \mathbf{A}_k$ by $\mathbf{A}_{n,k}$. If $k > n$ then $\mathbf{A}_{n,k}$ is the identity matrix of the appropriate dimension.

All random vectors which can be observed up to the moment n will be denoted by $\mathbf{W}_n$ (they will be specified in the sequel).

In our model we assume that random vectors $\mathbf{V}_n$ which influence the controlled system have distribution depending on an unknown parameter V. The parameter is assumed to be a random variable. The following notations connected with this parameter will be very useful:

$P_{\mathbf{X}|\mathbf{Y}}, f_{\mathbf{X}|\mathbf{Y}}(\mathbf{x} \mid \mathbf{y}) =$ the conditional distribution of a random vector $\mathbf{X}$ given a random vector $\mathbf{Y}$, and its density function, respectively,

$P^v_{\mathbf{X}|\mathbf{Y}}, f_{\mathbf{X}|\mathbf{Y},v}(\mathbf{x} \mid \mathbf{y}, v) =$ the conditional distribution of a random vector $\mathbf{X}$ given a random vector $\mathbf{Y}$ and $V = v$, and its density function, respectively.

Let $\mathbf{X}_i$ be a random vector. Then

$\widehat{\mathbf{X}}_{i|j} =$ the conditional expectation of $\mathbf{X}_i$ given $\mathbf{W}_j$,

$\operatorname{Cov}(\mathbf{X}_i \mid j) =$ the covariance matrix of the conditional distribution of $\mathbf{X}_i$ given $\mathbf{W}_j$,

$\widehat{\mathbf{X}}^v_{i|j} =$ the conditional expectation of $\mathbf{X}_i$ given $\mathbf{W}_n$ and $V{=}v$,

$\operatorname{Cov}(\mathbf{X}^v_i \mid j) =$ the covariance matrix of the conditional distribution of $\mathbf{X}_i$ given $\mathbf{W}_j$ and $V = v$.

2. Model description. Consider the l-dimensional, discrete time linear stochastic system given by the following plant equation:

$$\mathbf{X}_{n+1} = \alpha_n\mathbf{X}_n + \beta_n\mathbf{u}_n + \gamma_n\mathbf{V}_n, \quad n = 0, 1, \ldots, M, \tag{1}$$

where $\mathbf{X}_n$, $\mathbf{u}_n$, $\mathbf{V}_n$ are the state, control and disturbance, respectively.

The initial state $\mathbf{X}_0$ is assumed to be a random vector with the known distribution $N(\mathbf{m}, \Theta)$. It is also assumed that for each n the matrix β_n is random; it represents a disturbance which is usually called a *random actuation error* (for more details see e.g. [4]). It is introduced into the system by exercising control over it and its magnitude depends on the magnitude of the control. It is assumed to have the Gaussian distribution $N(\mathbf{B}_n, \Lambda_n \otimes \Delta_n)$ (see [3]), where $\mathbf{B}_n$ is a known mean matrix and Λ_n, Δ_n are given, nonsingular, $l \times l$ matrices.

We assume that for each n the random vector $\mathbf{V}_n$ has the distribution

$$N(qV\mathbf{e}_l, q\mathbf{I}_l) \tag{2}$$

where q is a given positive constant and V is an unknown real parameter.

Let N denote a horizon of control. The horizon is assumed to be a random variable with the distribution

$$P(N = k) = p_k, \quad k = 0, 1, \ldots, M, \quad p_M > 0, \ \sum_{k=0}^{M} p_k = 1 .$$

In our model the observation equation has the form

$$\mathbf{Y}_n = \varepsilon_n \mathbf{V}_n + \mathbf{Z}_n, \quad n = 0, 1, \ldots, M , \tag{3}$$

where $\mathbf{Y}_n$, $\mathbf{Z}_n$ are p-dimensional vectors, ε_n is a given $p \times l$ matrix and $\mathbf{Z}_n$ has a known distribution $N(\mathbf{o}_p, \Sigma_n)$.

The following data are available at the moment n:

$$\mathbf{Y}^{n-1} = (\mathbf{Y}_0^T, \ldots, \mathbf{Y}_{n-1}^T)^T, \quad \mathbf{U}^{n-1} = (\mathbf{u}_0^T, \ldots, \mathbf{u}_{n-1}^T)^T .$$

As we have already mentioned, $\mathbf{W}_n^T$ denotes the vector $((\mathbf{Y}^{n-1})^T, (\mathbf{U}^{n-1})^T)$. The control vector $\mathbf{u}_n$ is a Borel function of $\mathbf{W}_n$.

The random elements $N, \mathbf{X}_0, \mathbf{V}_n, \mathbf{Z}_n, \beta_n$, $n = 0, 1, \ldots$, are independent.

3. Game-theoretic aspects of the control problem. Let $\mathbf{u}_0, \ldots, \mathbf{u}_M$ be controls. The system of vectors $\mathbf{U} = (\mathbf{u}_0, \ldots, \mathbf{u}_M)$ is called a *control strategy*. For the given control strategy $\mathbf{U}$ we define the *risk function* by

$$R(v, \mathbf{U}) \stackrel{\mathrm{df}}{=} E_N E_v \sum_{i=0}^{N} L_i(\mathbf{X}_i, \mathbf{u}_i, v)$$

where $E_N(\,)$ denotes the expectation with respect to (w.r.t.) the distribution of N, and $E_v(\,)$ denotes the expectation w.r.t. the distributions of $\mathbf{X}_0$, V_n, β_n, $n = 0, 1, \ldots$, given $V = v$. The functions $L_i(\cdot, \cdot, \cdot)$ are given by

$$L_i(\mathbf{X}_i, \mathbf{u}_i, v) = (1, v, \mathbf{X}_i^T)\mathbf{R}_i(1, v, \mathbf{X}_i^T)^T + \mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i$$

with $\mathbf{R}_i$ and $\mathbf{K}_i$ being nonnegative definite matrices of appropriate dimensions. In order to simplify our further formulae we write $\mathbf{R}_n$ in the block form

$$\mathbf{R}_n = \begin{bmatrix} \mathbf{R}_{11}^{(n)} & \mathbf{R}_{12}^{(n)} & \mathbf{R}_{13}^{(n)} \\ \mathbf{R}_{21}^{(n)} & \mathbf{R}_{22}^{(n)} & \mathbf{R}_{23}^{(n)} \\ \mathbf{R}_{31}^{(n)} & \mathbf{R}_{32}^{(n)} & \mathbf{R}_{33}^{(n)} \end{bmatrix},$$

where $\mathbf{R}_{33}^{(n)}$ is an $l \times l$ matrix, $\mathbf{R}_{31} = \mathbf{R}_{13}^T, \mathbf{R}_{32} = \mathbf{R}_{23}^T$ are $l \times 1$ matrices, and $\mathbf{R}_{11}, \mathbf{R}_{12} = \mathbf{R}_{21}, \mathbf{R}_{22}$ are 1×1 matrices (scalars).

Assume that the parameter V of the distribution (2) is unknown but we know its prior distribution D. For the given D and control strategy $\mathbf{U}$ we define the *Bayes risk* connected with D and $\mathbf{U}$ by

$$r(D, \mathbf{U}) = E_D R(V, \mathbf{U}),$$

where $E_D(\ )$ denotes the expectation w.r.t. the distribution D of V.

The Bayes risk will be the cost function (loss function) in our problem.

Let $\mathcal{C}_D$ denote the class of control strategies $\mathbf{U}$ for which the Bayes risk $r(D, \mathbf{U})$ exists. Then a control strategy $\mathbf{U}^*$ which satisfies

$$r(D, \mathbf{U}^*) = \inf_{\mathbf{U} \in \mathcal{C}_D} r(D, \mathbf{U})$$

is called a *Bayes strategy* (w.r.t. D).

Sometimes we do not know D but we have some information about it, e.g. we may know some of its moments. Then we know that D belongs to some class $\mathcal{G}$ of distributions of the parameter V. Denote the class of control strategies $\mathbf{U}$ for which the Bayes risk $r(D, \mathbf{U})$ exists for each $D \in \mathcal{G}$ by $\Gamma_{\mathcal{G}}$.

A control strategy $\widetilde{\mathbf{U}}$ is called a *minimax control strategy* (w.r.t. $\mathcal{G}$) if

$$\sup_{D \in \mathcal{G}} r(D, \widetilde{\mathbf{U}}) = \inf_{\mathbf{U} \in \Gamma_{\mathcal{G}}} \sup_{D \in \mathcal{G}} r(D, \mathbf{U}).$$

Our principal aim is to find minimax control strategies w.r.t. some classes of prior distributions. We will use the following lemma.

LEMMA. *Let $\{D_k\}_{k=1}^{\infty}$, $D_k \in \mathcal{G}$, be a sequence of distributions of V and let $\{\mathbf{U}_k\}_{k=1}^{\infty}$ and $\{r(D_k, \mathbf{U}_k)\}_{k=1}^{\infty}$ be the corresponding sequences of Bayes control strategies and Bayes risks. If $\widetilde{\mathbf{U}}$ is a control strategy for which*

$$\sup_{D \in \mathcal{G}} r(D, \widetilde{\mathbf{U}}) \le \limsup_{k \to \infty} r(D_k, \mathbf{U}_k) \tag{4}$$

then it is a $\mathcal{G}$-minimax estimate.

This generalization of the well-known theorem 6.5.2 in [14] can be found in [12].

The following corollary is also well known.

COROLLARY. *If a control strategy $\mathbf{U}^*$ is Bayes w.r.t. some distribution belonging to $\mathcal{G}$ and satisfies*

$$\forall D \in \mathcal{G} \quad r(D, \mathbf{U}^*) = \text{ const.},$$

then it is a minimax control strategy w.r.t. $\mathcal{G}$.

4. Statement of the problems. In the sequel we consider the classes $\mathcal{G}_1$, $\mathcal{G}_2$ of prior distributions of V defined as follows:

$$D \in \mathcal{G}_1 \Leftrightarrow E_D V^2 \le a\,,$$
$$D \in \mathcal{G}_2 \Leftrightarrow E_D V = m_1 \wedge E_D V^2 = m_2\,,$$

where the constants a, m_1, m_2 are known.

Our task is to solve the following two problems.

PROBLEM A. For the stochastic system described in Section 2 find a minimax control strategy against the nature's choice of distribution in the class $\mathcal{G}_1$.

PROBLEM B. For the same system, find a minimax control strategy when the nature's choice of the distribution of V is confined to the class $\mathcal{G}_2$.

5. Conditional distributions. Suppose that the parameter V is known to be equal to v. Then by (3), in view of our assumptions, $\mathbf{Y}_n$ has the Gaussian distribution with mean and covariance matrix, respectively,

$$\hat{\mathbf{Y}}_n^v = qv\varepsilon_n \mathbf{e}_l, \quad \mathrm{Cov}(\mathbf{Y}_n^v) = q\varepsilon_n\varepsilon_n^T + \Sigma_n\,. \tag{5}$$

From (3) and (2), using the Bayes rule, one can obtain the distribution $P^v_{\mathbf{V}_n|\mathbf{Y}_n}$: it is normal with covariance matrix and mean

$$\begin{aligned} \mathrm{Cov}(\mathbf{V}_n^v \mid n+1) &= (\varepsilon_n^T \Sigma_n \varepsilon_n^{-1} + g^{-1}\mathbf{I}_l)^{-1}\,, \\ \hat{\mathbf{V}}^v_{n|n+1} &= [\mathrm{Cov}(\mathbf{V}_n^v \mid n+1)](\varepsilon_n^T \Sigma_n^{-1}\mathbf{Y}_n + v\mathbf{e}_l)\,. \end{aligned} \tag{6}$$

In order to shorten further formulae the two covariance matrices given in (5) and (6) will be denoted by Ξ_n and $\mathbf{F}_n$, respectively.

Now let the parameter V be a random variable having a prior distribution $N(rs^{-1}, s^{-1})$, with r, s being known constants, $s > 0$. This distribution will be denoted by $D_{s,r}$.

According to the Bayes rule, using (5), we find that the conditional distribution of V given $\mathbf{Y}^n$ is $N(r_n s_n^{-1}, s_n^{-1})$ where r_n, s_n can be obtained from the equations

$$r_{n+1} = r_n + \mathbf{t}_n^T \mathbf{Y}_n\,, \quad r_0 = r\,, \quad s_{n+1} = s + t_n\,, \quad s_0 = s\,, \tag{7}$$

with

$$\mathbf{t}_n = q\Xi_n^{-1}\varepsilon_n \mathbf{e}_l \quad \text{and} \quad t_n = q\mathbf{t}_n^T \varepsilon_n \mathbf{e}_l, \qquad n = 0, 1, \ldots$$

In view of our assumptions in the case where the parameter V is known the random vectors $\mathbf{V}_n$ and $\mathbf{Y}^{n-1}$ are independent. So, the conditional distribution $P^v_{\mathbf{V}_n|\mathbf{Y}^{n-1}}$ is then equal to $P^v_{\mathbf{V}_n}$. Using this fact, the formulae (2), (7) and the equation

$$f_{\mathbf{V}_n|\mathbf{Y}^{n-1}}(\mathbf{v}_n \mid \mathbf{y}^{n-1}) = \int\limits_{\mathbf{R}} f_{\mathbf{V}_n|\mathbf{Y}^{n-1},v}(\mathbf{v}_n \mid \mathbf{y}^{n-1}, v) f_{V|\mathbf{Y}^{n-1}}(v \mid \mathbf{y}^{n-1})\, dv$$

one can find that $P_{\mathbf{V}_n|\mathbf{Y}^{n-1}}$ is Gaussian with covariance matrix and mean

$$\operatorname{Cov}(\mathbf{V}_n \mid n) = [q^{-1}\mathbf{I}_l + (ql + s_n)^{-1}\mathbf{e}_l\mathbf{e}_l^T]^{-1},$$
$$\widehat{\mathbf{V}}_{n|n} = (ql + s_n)^{-1} \operatorname{Cov}(\mathbf{V}_n \mid n) r_n \mathbf{e}_l .$$

It is easy to verify that

$$[q^{-1}\mathbf{I}_l + (ql + s_n)^{-1}\mathbf{e}_l\mathbf{e}_l^T]^{-1} = q\mathbf{I}_l + q^2 s_n^{-1}\mathbf{e}_l\mathbf{e}_l^T .$$

Using this equation we obtain eventually

$$\widehat{\mathbf{V}}_{n|n} = q\mathbf{e}_l r_n s_n^{-1}, \quad \operatorname{Cov}(\mathbf{V}_n \mid n) = q\mathbf{I}_l + q^2 s_n^{-1}\mathbf{e}_l\mathbf{e}_l^T .$$

6. Filtering of the states. For our further investigations we need the conditional expectation of $\mathbf{X}_n$ given $\mathbf{W}_n$. It is well known (see e.g. [1]) that the expectation is the minimum mean squared error estimate for the state variable. In this section we consider the cases where the parameter V is fixed and where V has the prior distribution $D_{s,r}$. So we solve the problem of Bayesian filtering (w.r.t. $D_{s,r}$) as well.

First suppose that V is known to be equal to v. Since the distribution of the initial state is assumed to be Gaussian and since each of the random elements $\mathbf{V}_n$, $\mathbf{Z}_n$, β_n, $n = 0, 1, \ldots$, is also Gaussian, the distribution $P^v_{\mathbf{X}_n|\mathbf{W}_n}$ at each stage is again Gaussian. We shall obtain its parameters $\widehat{\mathbf{X}}^v_{n|n}$, $\operatorname{Cov}(\mathbf{X}^v_n \mid n)$. According to our assumptions we have the following initial conditions: $\widehat{\mathbf{X}}^v_{0|0} = \mathbf{m}$ and $\operatorname{Cov}(\mathbf{X}^v_0 \mid 0) = \Theta$. Suppose that $\widehat{\mathbf{X}}^v_{n|n}$ and $\operatorname{Cov}(\mathbf{X}^v_n \mid n)$ are known and $\mathbf{Y}_n$ has been observed. The independence of $\mathbf{X}_n$ and $\mathbf{Y}_n$ implies that after the value of $\mathbf{Y}_n$ has been observed the conditional distribution of $\mathbf{X}_n$ does not change. Then, using (1) and (6) we obtain

$$\widehat{\mathbf{X}}^v_{n+1|n+1} = \alpha_n \widehat{\mathbf{X}}^v_{n|n} + \mathbf{B}_n\mathbf{u}_n + \gamma_n \widehat{\mathbf{V}}^v_{n|n+1},$$
$$\operatorname{Cov}(\mathbf{X}^v_{n+1} \mid n+1) = \alpha_n \operatorname{Cov}(\mathbf{X}^v_n \mid n)\alpha_n^T + \Lambda_n^T \otimes \mathbf{u}_n^T \Delta_n \mathbf{u}_n + \gamma_n \mathbf{F}_n \gamma^T$$

(for the distribution of $\beta_n\mathbf{u}_n$ see e.g. [3]).

Now suppose that V has the distribution $D_{s,r}$. Then all random vectors having distributions depending on V become dependent and the filtering problem is more difficult. Moreover, the observation at the moment n influences the estimate for $\mathbf{X}_i$, $i < n$. Write the state $\mathbf{X}_{n+1}$ down emphasizing its dependence on the initial state $\mathbf{X}_0$ and the random vectors $\mathbf{V}_n$:

$$\mathbf{X}_{n+1} = \alpha_{n,0}\mathbf{X}_0 + \sum_{i=0}^{n} \alpha_{n,i+1}\beta_i \mathbf{u}_i + \sum_{i=0}^{n} \alpha_{n,i+1}\gamma_i \mathbf{V}_i . \tag{8}$$

In view of (8) the only distribution we need to find in order to obtain the conditional distribution of $\mathbf{X}_{n+1}$ is the conditional distribution of the $(n+1)l$-dimensional vector $\mathbf{V}^n = [\mathbf{V}_0^T, \mathbf{V}_1^T, \ldots, \mathbf{V}_n^T]^T$ given the vector $\mathbf{W}_{n+1}$. If we know the density functions $f_{\mathbf{Y}^n|\mathbf{V}^n}(\mathbf{y}^n \mid \mathbf{v}^n)$ and $f_{\mathbf{V}^n}(\mathbf{v}^n)$ this distribution can be obtained by the Bayes formula. By our assumption on independence of the random vectors $\mathbf{Z}_n$ the formula (3) implies that the distribution $P_{\mathbf{Y}^n|\mathbf{V}^n}$ is $N(\varepsilon^n \mathbf{V}^n, \Sigma^n)$, where the matrices ε^n and Σ^n are of the following block-diagonal forms:

$$\varepsilon^n = \lceil \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \rfloor \quad \text{and} \quad \Sigma^n = \lceil \Sigma_0, \Sigma_1, \ldots, \Sigma_n \rfloor .$$

The distribution of $\mathbf{V}^n$ can be obtained similarly to $P_{\mathbf{V}_n|\mathbf{Y}^{n-1}}$ (see Section 5). It is

$$N(qrs^{-1}\mathbf{e}_{(n+1)l}\,,\ q\mathbf{I}_{(n+1)l} + q^2 s^{-1}\mathbf{e}_{(n+1)l}\mathbf{e}_{(n+1)l}^T) .$$

Finally, using the Bayes rule we find that the required distribution is Gaussian with covariance matrix

$$\operatorname{Cov}(\mathbf{V}^n \mid n+1) = \left[(\varepsilon^n)^T(\Sigma^n)^{-1}\varepsilon^n + q^{-1}\mathbf{I}_{(n+1)l} - \frac{\mathbf{e}_{(n+1)l}\mathbf{e}_{(n+1)l}^T}{q(n+1)l+s}\right]^{-1}$$

and mean

$$\begin{aligned} E(\mathbf{V}^n \mid \mathbf{W}_{n+1}) = {} & \operatorname{Cov}(\mathbf{V}^n \mid n+1)\Big\{(\varepsilon^n)^T(\Sigma^n)^{-1}\mathbf{Y}^n \\ & + \left[q^{-1}\mathbf{I}_{(n+1)l} - \frac{\mathbf{e}_{(n+1)l}\mathbf{e}_{(n+1)l}^T}{q(n+1)l+s}\right] q\frac{r}{s}\mathbf{e}_{(n+1)l}\Big\} . \end{aligned}$$

The above expressions can be simplified with the help of the following equations:

$$\begin{aligned} q(n+1)l &= q^2\mathbf{e}_{(n+1)l}^T(\varepsilon^n)^T[q\varepsilon^n(\varepsilon^n)^T + \Sigma^n]^{-1}\varepsilon^n\mathbf{e}_{(n+1)l} \\ &\quad + \mathbf{e}_{(n+1)l}^T\mathbf{F}^n\mathbf{e}_{(n+1)l}^T , \\ q(\varepsilon^n)^T[q\varepsilon^n(\varepsilon^n)^T + \Sigma^n]^{-1} &= \mathbf{F}^n(\varepsilon^n)^T(\Sigma^n)^{-1} , \end{aligned} \tag{9}$$

where $\mathbf{F}^n = (\varepsilon^n)^T(\Sigma^n)^{-1}\varepsilon^n + q^{-1}\mathbf{I}_{(n+1)l}$. Note that $\mathbf{F}^n = \lceil \mathbf{F}_0, \mathbf{F}_1, \ldots, \mathbf{F}_n \rfloor$. Using (9) one can prove that

$$s + q(n+1)l = s_{n+1} + \mathbf{e}_{(n+1)l}^T\mathbf{F}^n\mathbf{e}_{(n+1)l} ,$$

where s_{n+1} is given in (7).

Now one can easily verify that

$$
(10) \qquad \begin{aligned} \mathrm{Cov}(\mathbf{V}^n \mid n+1) &= \mathbf{F}^n + s_{n+1}^{-1}\mathbf{F}^n \mathbf{e}_{(n+1)l}\mathbf{e}_{(n+1)l}^T \mathbf{F}^n , \\ E(\mathbf{V}^n \mid \mathbf{W}_{n+1}) &= \mathbf{F}^n[(\varepsilon^n)^T(\Sigma^n)^{-1}\mathbf{Y}^n + r_{n+1}s_{n+1}^{-1}\mathbf{e}_{(n+1)l}] , \end{aligned}
$$

where r_{n+1} is given in (7).

Using (8) and (10) one can check that

$$
(11) \qquad \begin{aligned} \widehat{\mathbf{X}}_{n+1|n+1} &= \alpha_n \widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n\mathbf{u}_n + (\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)\mathbf{Y}_n \\ &\qquad + (\gamma_n \mathbf{F}_n \mathbf{e}_l + \mathbf{w}_1^{(n)}) r_n s_n^{-1} , \\ \mathrm{Cov}(\mathbf{X}_{n+1} \mid n+1) &= \alpha_n \,\mathrm{Cov}(\mathbf{X}_n \mid n)\alpha_n^T + \Lambda_n^T \mathbf{u}_n^T \Delta_n \mathbf{u}_n + \mathbf{H}_n , \end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{J}_n &= \gamma_n \mathbf{F}_n \varepsilon_n^T \Sigma_n^{-1} , \\
\mathbf{N}_n &= \sum_{k=0}^{n} \alpha_{n,k+1}\gamma_k \mathbf{F}_k \mathbf{e}_l \mathbf{e}_l^T \mathbf{F}_n \varepsilon_n^T \Sigma_n^{-1} , \\
\mathbf{w}_1^{(n)} &= -t_n \sum_{k=0}^{n} \alpha_{n,k+1}\gamma_k \mathbf{F}_k \mathbf{e}_l , \\
\mathbf{H}_n &= -s_n^{-1}s_{n+1}^{-1}t_n \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_{n,k+1}\gamma_k \mathbf{F}_k \mathbf{e}_l \mathbf{e}_l^T \mathbf{F}_i \gamma_i^T \alpha_{n,i+1}^T \\
&\quad + s_{n+1}^{-1}\Big[\gamma_n \mathbf{F}_n \mathbf{e}_l \mathbf{e}_l^T \mathbf{F}_n \gamma_n^T + \sum_{k=0}^{n-1}(\gamma_n \mathbf{F}_n \mathbf{e}_l \mathbf{e}_l^T \mathbf{F}_k \gamma_k^T \alpha_{n,k+1}^T \\
&\quad + \alpha_{n,k+1}\gamma_k \mathbf{F}_k \mathbf{e}_l \mathbf{e}_l^T \mathbf{F}_n \gamma_n^T)\Big] + \gamma_n \mathbf{F}_n \gamma_n^T .
\end{aligned}
$$

Note that the first equation in (11) is the filter (one step prediction) equation for our model. The initial condition is $\widehat{\mathbf{X}}_{0|0} = \mathbf{m}$.

7. Conditional expectations. Now we write down formulae for conditional expectations which will be used in further calculations. They can be obtained with the help of (3), (7), (11) and the equation

$$
\begin{aligned} E(\mathbf{w}^T\mathbf{A}\mathbf{w} \mid \mathbf{W}_n) &= E(\mathbf{w}^T \mid \mathbf{W}_n)\mathbf{A}E(\mathbf{w} \mid \mathbf{W}_n) \\ &\qquad + \mathbf{e}_l^T[\mathbf{A} * \mathrm{Cov}(\mathbf{w} \mid n)]\mathbf{e}_l , \end{aligned}
$$

which holds for an arbitrary random vector $\mathbf{w}$ and a matrix $\mathbf{M}$ of appropriate dimension. Since the Hadamard product is, obviously, distributive with respect to the sum of matrices the above equation is very useful for us.

We obtain:

(12)
$$
\begin{aligned}
&E(r_{n+1}s_{n+1}^{-1} \mid \mathbf{W}_n, \mathbf{u}_n) = r_n s_n^{-1}\,,\\
&E(\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n) = \alpha_n \widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n\mathbf{u}_n + \gamma_n \mathbf{e}_l q r_n s_n^{-1}\,,\\
&E(r_{n+1}^2 s_{n+1}^{-2} \mid \mathbf{W}_n, \mathbf{u}_n) = r_n^2 s_n^{-2} + s_n^{-1} s_{n+1}^{-1} t_n\,,\\
&E(\widehat{\mathbf{X}}_{n+1|n+1} r_{n+1} s_{n+1}^{-1} \mid \mathbf{W}_n, \mathbf{u}_n)\\
&\qquad = r_n s_n^{-1}(\alpha_n \widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n\mathbf{u}_n + \gamma_n \mathbf{e}_l q r_n s_n^{-1}) + s_{n+1}^{-1}\mathbf{N}_n\mathbf{G}_n\mathbf{t}_n\,,\\
&E(\widehat{\mathbf{X}}_{n+1|n+1}^T \mathbf{A} \widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)\\
&= (\alpha_n \widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n\mathbf{u}_n + \gamma_n \mathbf{e}_l q r_n s_n^{-1})^T \mathbf{A} (\alpha_n \widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n\mathbf{u}_n + \gamma_n \mathbf{e}_l q r_n s_n^{-1})\\
&\quad + \mathbf{e}_l^T((\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)^T \mathbf{A} (\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n) * \mathbf{G}_n)\mathbf{e}_l\,,
\end{aligned}
$$

where $\mathbf{G}_n = \mathrm{Cov}(\mathbf{Y}_n \mid n) = \varepsilon_n(q\mathbf{I}_l + q^2 s_n^{-1}\mathbf{e}_l\mathbf{e}_l^T)\varepsilon_n^T + \Sigma_n$ and $\mathbf{A}$ in the last equation is an arbitrary $l \times l$ matrix.

8. Bayes strategies. Now we are looking for the Bayes control strategies w.r.t. $D_{s,r}$. Following Bellman's approach suppose that we are at the nth stage and we start to control our system. The data $\mathbf{W}_n$ are known and the Bayes risk is

$$r_n(D_{s,r}, \mathbf{U}_n) = E_N\Big\{E\Big[\sum_{i=n}^{N} L_i(\mathbf{X}_i, \mathbf{u}_i, v) \,\Big|\, \mathbf{W}_n\Big] \,\Big|\, N \ge n\Big\}\,,$$

where $\mathbf{U}_n = (\mathbf{u}_n, \ldots, \mathbf{u}_M)$. This risk can be transformed to the form

$$r_n(D_{s,r}, \mathbf{U}_n) = E\Big[\sum_{i=n}^{N} p_n^i L_i(\mathbf{X}_i, \mathbf{u}_i, v) \,\Big|\, \mathbf{W}_n\Big]$$

with $p_n^i = (\sum_{k=i}^M p_k)(\sum_{k=n}^M p_k)^{-1}$.

Let $w_n(\widehat{\mathbf{X}}_{n|n}) = \inf r_n(D_{s,r}, \mathbf{U}_n)$, where the infimum is taken over all control strategies $\mathbf{U}_n$ for which the Bayes risk exists. It is easy to see that

$$
\begin{aligned}
w_M(\widehat{\mathbf{X}}_{M|M}, r_M) = L_M(\widehat{\mathbf{X}}_{M|M}, \mathbf{o}_l, r_M s_M^{-1}) + \mathbf{e}_l^T[\mathbf{R}_{33}^{(M)} * \mathrm{Cov}(\mathbf{X}_M \mid M)]\mathbf{e}_l\\
+ \mathbf{R}_{22}^{(M)} s_M^{-1} + 2\mathbf{R}_{23}^{(M)}\mathbf{l}_M s_M^{-1}\,.
\end{aligned}
$$

For $n = 0, \ldots, M-1$, using the results of Section 5, we obtain the following Bellman equation:

(13)
$$
\begin{aligned}
w_n(\widehat{\mathbf{X}}_{n|n}, r_n) = \inf_{\mathbf{u}_n}\{L(\widehat{\mathbf{X}}_{n|n}, \mathbf{u}_n, r_n s_n^{-1}) + \mathbf{e}_l^T[\mathbf{R}_{33}^{(n)} * \mathrm{Cov}(\mathbf{X}_n \mid n)]\mathbf{e}_l\\
+\mathbf{R}_{22}^{(n)} s_n^{-1} + 2\mathbf{R}_{23}^{(n)}\mathbf{l}_n s_n^{-1} + p_n^{n+1} E[w_{n+1}(\widehat{\mathbf{X}}_{n+1|n+1}) \mid \mathbf{W}_n]\}\,.
\end{aligned}
$$

In the above equations $\mathbf{l}_n = \sum_{i=0}^{n-1} \alpha_{n-1,i+1}\gamma_i \mathbf{F}_i \mathbf{e}_l$.

According to Bellman's optimality principle, using (13) and (12), one can show by an inductive argument that for $n = 0, \dots, M-1$ the infimum w_n of the Bayes risks takes the form

$$w_n(\widehat{\mathbf{X}}_{n|n}) = (1, r_n s_n^{-1}, \widehat{\mathbf{X}}_{n|n}^T)\mathbf{A}_n(1, r_n s_n^{-1}, \widehat{\mathbf{X}}_{n|n}^T)^T + \mathbf{e}_l^T(\mathbf{A}_{44}^{(n)} * \mathrm{Cov}(\mathbf{X}_n \mid n))\mathbf{e}_l$$

and the Bayes control $\mathbf{u}_n^*$ satisfies

$$\begin{aligned}(14)\quad &\{\mathbf{K}_n + p_n^{n+1}[\mathbf{B}_n^T\mathbf{A}_{33}^{(n+1)}\mathbf{B}_n + \Delta_n\mathbf{e}_l^T(\mathbf{A}_{44}^{(n+1)} * \Lambda_n)\mathbf{e}_l]\}\mathbf{u}_n^* \\ &= -p_n^{n+1}[\mathbf{B}_n^T\mathbf{A}_{33}^{(n+1)}\alpha_n\widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n^T(\mathbf{A}_{33}^{(n+1)}q\gamma_n\mathbf{e}_l \\ &\qquad + \mathbf{A}_{32}^{(n+1)})r_n s_n^{-1} + \mathbf{B}_n^T\mathbf{A}_{31}^{(n+1)}],\end{aligned}$$

where the $(l+2) \times (l+2)$ matrices $\mathbf{A}_n$ have the following block form:

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{A}_{11}^{(n)} & \mathbf{A}_{12}^{(n)} & \mathbf{A}_{13}^{(n)} \\ \mathbf{A}_{21}^{(n)} & \mathbf{A}_{22}^{(n)} & \mathbf{A}_{23}^{(n)} \\ \mathbf{A}_{31}^{(n)} & \mathbf{A}_{32}^{(n)} & \mathbf{A}_{33}^{(n)} \end{bmatrix}$$

with $\mathbf{A}_{33}^{(n)}$ being an $l \times l$ matrix, $\mathbf{A}_{31} = \mathbf{A}_{13}^T$, $\mathbf{A}_{32} = \mathbf{A}_{23}^T$ being $l \times 1$ matrices, $\mathbf{A}_{11}$, $\mathbf{A}_{12} = \mathbf{A}_{21}$ and $\mathbf{A}_{22}$ being 1×1 matrices (scalars).

Throughout the paper we assume that a solution of the equation (14) exists. Then denoting the matrix in braces on the left-hand side of (14) by $\mathbf{M}_n$ we can write

$$\mathbf{u}_n^* = -\mathbf{P}_n\widehat{\mathbf{X}}_{n|n} - \mathbf{h}_n r_n s_n^{-1} - \mathbf{d}_n, \quad n = 0, \dots, M-1,$$

with

$$\begin{aligned} &\mathbf{P}_n = p_n^{n+1}\mathbf{M}_n^+\mathbf{B}_n^T\mathbf{A}_{33}^{(n+1)}\alpha_n, \\ (15)\quad &\mathbf{h}_n = p_n^{n+1}\mathbf{M}_n^+\mathbf{B}_n^T(\mathbf{A}_{33}^{(n+1)}q\gamma_n\mathbf{e}_l + \mathbf{A}_{32}^{(n+1)}), \\ &\mathbf{d}_n = p_n^{n+1}\mathbf{M}_n^+\mathbf{B}_n^T\mathbf{A}_{31}^{(n+1)}. \end{aligned}$$

The blocks of the matrix $\mathbf{A}_n$ can be obtained from the following equations:

$$\begin{aligned} \mathbf{A}_{11}^{(n)} &= \mathbf{R}_{11}^{(n)} + \mathbf{R}_{22}^{(n)}s_n^{-1} + 2\mathbf{R}_{23}^{(n)}\mathbf{l}_n s_n^{-1} - \mathbf{d}_n^T\mathbf{M}_n\mathbf{d}_n \\ &\quad + p_n^{n+1}[\mathbf{A}_{11}^{(n+1)} + \mathbf{e}_l^T(\mathbf{A}_{44}^{(n)} * \mathbf{H}_n + \mathbf{N}_n^T\mathbf{A}_{33}^{(n+1)}\mathbf{N}_n * \mathbf{G}_n)\mathbf{e}_l \\ &\quad + \mathbf{A}_{22}^{(n+1)}t_n s_n^{-1}s_{n+1}^{-1} + 2\mathbf{A}_{23}^{(n+1)}\mathbf{N}_n\mathbf{G}_n\mathbf{t}_n s_{n+1}^{-1}], \\ \mathbf{A}_{21}^{(n)} &= \mathbf{R}_{21}^{(n)} - \mathbf{h}_n^T\mathbf{M}_n\mathbf{d}_n + p_n^{n+1}(q\mathbf{e}_l^T\gamma_n^T\mathbf{A}_{31}^{(n+1)} + \mathbf{A}_{21}^{(n+1)}), \\ (16)\quad \mathbf{A}_{22}^{(n)} &= \mathbf{R}_{22}^{(n)} - \mathbf{h}_n^T\mathbf{M}_n\mathbf{h}_n \\ &\quad + p_n^{n+1}[q\mathbf{e}_l^T\gamma_n^T(q\mathbf{A}_{33}^{(n+1)}\gamma_n\mathbf{e}_l + 2\mathbf{A}_{32}^{(n+1)}) + \mathbf{A}_{22}^{(n+1)}], \\ \mathbf{A}_{31}^{(n)} &= \mathbf{R}_{31}^{(n)} + p_n^{n+1}(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{A}_{31}^{(n+1)}, \end{aligned}$$

$$\mathbf{A}_{32}^{(n)} = \mathbf{R}_{32}^{(n)} + p_n^{n+1}(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)(q\mathbf{A}_{33}^{(n+1)}\gamma_n\mathbf{e}_l + \mathbf{A}_{32}^{(n+1)}),$$
$$\mathbf{A}_{33}^{(n)} = \mathbf{R}_{33}^{(n)} + p_n^{n+1}(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{A}_{33}^{(n+1)}\alpha_n,$$
$$\mathbf{A}_{44}^{(n)} = \mathbf{R}_{33}^{(n)} + p_n^{n+1}\langle\mathbf{A}_{44}^{(n+1)}/\alpha_n\rangle,$$

with the boundary condition:

$$\mathbf{A}_{11}^{(M)} = \mathbf{R}_{11}^{(M)} + \mathbf{R}_{22}^{(M)}s_M^{-1} + 2\mathbf{R}_{23}^{(M)}\mathbf{l}_M s_M^{-1},$$
$$\mathbf{A}_{21}^{(M)} = \mathbf{R}_{21}^{(M)}, \quad \mathbf{A}_{22}^{(M)} = \mathbf{R}_{22}^{(M)}, \quad \mathbf{A}_{31}^{(M)} = \mathbf{R}_{31}^{(M)},$$
$$\mathbf{A}_{32}^{(M)} = \mathbf{R}_{32}^{(M)}, \quad \mathbf{A}_{33}^{(M)} = \mathbf{R}_{33}^{(M)}, \quad \mathbf{A}_{44}^{(M)} = \mathbf{R}_{33}^{(M)},$$

and each of the remaining blocks is the zero matrix of appropriate dimensions.

For given $k \times k$ matrices $\mathbf{A}$ and $\mathbf{B}$ the symbol $\langle\mathbf{A}/\mathbf{B}\rangle$ (which appears in the last equation) denotes the $k \times k$ matrix whose elements are determined by

$$\langle\mathbf{A}/\mathbf{B}\rangle[i,j] = \mathbf{e}_k^T(\mathbf{A} * \mathbf{B}[i,\cdot]^T\mathbf{B}[j,\cdot])\mathbf{e}_k$$

where for an arbitrary matrix $\mathbf{A}$ the symbols $\mathbf{A}[i,j]$ and $\mathbf{A}[i,\cdot]$ denote the (i,j)th element and ith row of the matrix, respectively.

The results we have just obtained are summarized in the following proposition:

PROPOSITION 1. *The Bayes control strategy* $\mathbf{U}^*(s,r) = (\mathbf{u}_0^*, \ldots, \mathbf{u}_M^*)$ *w.r.t. the distribution* $D_{s,r}$ *for the system described in Section* 2 *is given by*

$$\mathbf{u}_M^* = \mathbf{o}_l, \quad \mathbf{u}_n^* = -\mathbf{P}_n\widehat{\mathbf{X}}_{n|n} - \mathbf{h}_n r_n s_n^{-1} - \mathbf{d}_n, \quad n = 0, \ldots, M-1,$$

with $\mathbf{P}_n$, $\mathbf{h}_n$, $\mathbf{d}_n$ *being given by* (15) *and* (16).

9. Risk functions for the Bayes strategies. Consider the case where the parameter V equals v and a control strategy $\mathbf{U}$ is given. We are looking for the risk $R(v, \mathbf{U})$. Define the *truncated risk* $R_n(v, \mathbf{U})$ by

$$R_n(v,\mathbf{U}) = E_N\Big\{E_v\Big[\sum_{i=n}^{N} L_i(\mathbf{X}_i, \mathbf{u}_i, v) \,\Big|\, \mathbf{W}_n\Big] \,\Big|\, N \ge n\Big\}.$$

Note that $R_0(v, \mathbf{U}) = R(v, \mathbf{U})$.

The truncated risk can be written as follows:

$$R_n(v,\mathbf{U}) = E_v\Big[\sum_{i=n}^{N} p_n^{n+1} L_i(\mathbf{X}_i, \mathbf{u}_i, v) \,\Big|\, \mathbf{W}_n\Big].$$

One can easily derive the following recursive equations for the risk:

$$R_n(v,\mathbf{U}) = L_n(\widehat{\mathbf{X}}_{n|n}^v, \mathbf{u}_n, v) + \mathbf{e}_l^T(\mathbf{R}_{33}^{(n)} * \mathrm{Cov}(\mathbf{X}_n^v \mid n))\mathbf{e}_l + p_n^{n+1}E_v[R_{n+1}(v,\mathbf{U}) \mid \mathbf{W}_n], \quad n = 0, \ldots, M-1,$$

$$R_M(v,\mathbf{U}) = L_M(\widehat{\mathbf{X}}^v_{M|M}, \mathbf{u}_M, v) + \mathbf{e}_l^T(\mathbf{R}_{33}^{(M)} * \mathrm{Cov}(\mathbf{X}^v_M \mid M))\mathbf{e}_l\,.$$

Using these relations we find the explicit form of the risk function. We need the following conditional expectations:

$$E_v(\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n) = \alpha_n \widehat{\mathbf{X}}^v_{n|n} + \mathbf{B}_n\mathbf{u}_n + q\gamma_n \mathbf{e}_l v\,,$$

$$E_v(\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n) = \alpha_n \widehat{\mathbf{X}}_{n|n} + \mathbf{B}_n\mathbf{u}_n + q(\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)\varepsilon_n \mathbf{e}_l v$$
$$+ (\gamma_n \mathbf{F}_n \mathbf{e}_l + s_{n+1}^{-1}\mathbf{w}_1^{(n)}) r_n s_n^{-1}\,,$$

$$E_v(r_{n+1} \mid \mathbf{W}_n, \mathbf{u}_n) = r_n + t_n v\,,$$

$$E_v(r_{n+1}^2 \mid \mathbf{W}_n, \mathbf{u}_n) = (r_n + t_n v)^2 + t_n\,,$$

$$E_v(\widehat{\mathbf{X}}^T_{n+1|n+1}\mathbf{A}\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$= E_v^T(\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)\mathbf{A}E_v(\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$+ \mathbf{e}_l^T[(\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)\Xi_n(\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)^T * \mathbf{A}]\mathbf{e}_l\,,$$

$$E_v(\widehat{\mathbf{X}}^T_{n+1|n+1}\mathbf{A}\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$= E_v^T(\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)\mathbf{A}E_v(\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$+ \mathbf{e}_l^T[(\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)\Xi_n\mathbf{J}_n^T * \mathbf{A}]\mathbf{e}_l\,,$$

$$E_v[(\widehat{\mathbf{X}}^v_{n+1|n+1})^T\mathbf{A}\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n]$$
$$= E_v^T(\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)\mathbf{A}E_v(\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$+ \mathbf{e}_l^T(\mathbf{J}_n\Xi_n\mathbf{J}_n^T * \mathbf{A})\mathbf{e}_l\,,$$

$$E_v(r_{n+1}\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$= E_v(r_{n+1} \mid \mathbf{W}_n, \mathbf{u}_n)E_v(\widehat{\mathbf{X}}_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$+ (\mathbf{J}_n + s_{n+1}^{-1}\mathbf{N}_n)\Xi_n\mathbf{t}_n\,,$$

$$E_v(r_{n+1}\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n)$$
$$= E_v(r_{n+1} \mid \mathbf{W}_n, \mathbf{u}_n)E_v(\widehat{\mathbf{X}}^v_{n+1|n+1} \mid \mathbf{W}_n, \mathbf{u}_n) + \mathbf{J}_n\Xi_n\mathbf{t}_n\,,$$

where $\mathbf{A}$ is an arbitrary matrix of appropriate dimensions.

With the help of the above equations one can prove that

$$\text{(17)}\quad R_n(v,\mathbf{U}^*(s,r)) = \left[1, v, \frac{r_n}{s_n}, \widehat{\mathbf{X}}^T_{n|n}, (\widehat{\mathbf{X}}^v_{n|n})^T\right]$$
$$\times \mathbf{C}_n\left[1, v, \frac{r_n}{s_n}, \widehat{\mathbf{X}}^T_{n|n}, (\widehat{\mathbf{X}}^v_{n|n})^T\right]^T + \mathbf{e}_l^T(\mathbf{C}_{66}^{(n)} * \mathrm{Cov}(\mathbf{X}^v_n \mid n))\mathbf{e}_l\,,$$

where the $(2l+3)\times(2l+3)$ matrix $\mathbf{C}_n$ has the following block form:

$$\mathbf{C}_n = \begin{bmatrix} \mathbf{C}_{11}^{(n)} & \mathbf{C}_{12}^{(n)} & \cdots & \mathbf{C}_{15}^{(n)} \\ \mathbf{C}_{21}^{(n)} & \mathbf{C}_{22}^{(n)} & \cdots & \mathbf{C}_{25}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{C}_{51}^{(n)} & \mathbf{C}_{52}^{(n)} & \cdots & \mathbf{C}_{55}^{(n)} \end{bmatrix}$$

with $\mathbf{C}_{44}^{(n)}$, $\mathbf{C}_{45}^{(n)} = \mathbf{C}_{54}^{(n)T}$, $\mathbf{C}_{55}^{(n)}$ being $l\times l$ matrices, $\mathbf{C}_{41}^{(n)} = \mathbf{C}_{14}^{(n)T}$, $\mathbf{C}_{51}^{(n)} = \mathbf{C}_{15}^{(n)T}$, $\mathbf{C}_{42}^{(n)} = \mathbf{C}_{24}^{(n)T}$, $\mathbf{C}_{52}^{(n)} = \mathbf{C}_{25}^{(n)T}$, $\mathbf{C}_{43}^{(n)} = \mathbf{C}_{34}^{(n)T}$, $\mathbf{C}_{53}^{(n)} = \mathbf{C}_{35}^{(n)T}$ being $l\times 1$ matrices, and $\mathbf{C}_{11}^{(n)}$, $\mathbf{C}_{12}^{(n)} = \mathbf{C}_{21}^{(n)}$, $\dot{\mathbf{C}}_{22}$, $\mathbf{C}_{13}^{(n)} = \mathbf{C}_{31}^{(n)}$, $\mathbf{C}_{33}^{(n)}$, $\mathbf{C}_{32}^{(n)} = \mathbf{C}_{23}^{(n)}$ being 1×1 matrices.

The blocks can be obtained from the following equations:

$$\begin{aligned}
\mathbf{C}_{11}^{(n+1)} &= \mathbf{R}_{11}^{(n)} + \mathbf{d}_n^T\mathbf{M}_n\mathbf{d}_n + p_n^{n+1}\{\operatorname{tr}\mathbf{J}_n\Xi_n\mathbf{J}_n^T\mathbf{C}_{55}^{(n+1)} + \operatorname{tr}\gamma_n\mathbf{F}_n\gamma_n^T\mathbf{C}_{66}^{(n+1)} \\
&\quad + \operatorname{tr}\mathbf{J}_n\Xi_n\mathbf{J}_n^T\mathbf{C}_{44}^{(n+1)} + 2\operatorname{tr}\mathbf{J}_n\Xi_n\mathbf{J}_n^T\mathbf{C}_{45}^{(n+1)} + \mathbf{C}_{11}^{(n)} \\
&\quad - 2\mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{C}_{41}^{(n+1)} + \mathbf{C}_{51}^{(n+1)}) + [2(\mathbf{C}_{34}^{(n+1)} + \mathbf{C}_{35}^{(n+1)})\mathbf{J}_n\Xi_n\mathbf{t}_n \\
&\quad + \operatorname{tr}(\mathbf{N}_n\Xi_n\mathbf{J}_N^T + \mathbf{J}_n\Xi_n\mathbf{N}_n^T)\mathbf{C}_{44}^{(n+1)} + \operatorname{tr}\mathbf{N}_n^T\Xi_n\mathbf{J}_n\mathbf{C}_{45}^{(n+1)}]s_{n+1}^{-1} \\
&\quad + [2\mathbf{C}_{34}^{(n+1)}\mathbf{N}_n\Xi_n\mathbf{t}_n + t_n + \operatorname{tr}\mathbf{N}_n\Xi_n\mathbf{N}_n^T\mathbf{C}_{44}^{(n+1)}]s_{n+1}^{-2}\}, \\
\mathbf{C}_{31}^{(n)} &= \mathbf{d}_n^T\mathbf{M}_n\mathbf{h}_n + p_n^{n+1}\{[\mathbf{C}_{31}^{(n+1)} - \mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{C}_{53}^{(n+1)} + \mathbf{C}_{43}^{(n+1)})]s_ns_{n+1}^{-1} \\
&\quad + [\mathbf{C}_{14}^{(n+1)} - \mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{C}_{54}^{(n+1)} + \mathbf{C}_{44}^{(n+1)})]\mathbf{w}_1^{(n)}s_{n+1}^{-1} \\
&\quad + [\mathbf{C}_{14}^{(n+1)} - \mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{C}_{54}^{(n+1)} + \mathbf{C}_{44}^{(n+1)})]\gamma_n\mathbf{F}_n\mathbf{e}_l \qquad (18) \\
&\quad - \mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{C}_{41}^{(n+1)} + \mathbf{C}_{51}^{(n+1)})\}, \\
\mathbf{C}_{21}^{(n)} &= \mathbf{A}_{21}^{(n)} - \mathbf{C}_{31}^{(n)}, \\
\mathbf{C}_{51}^{(n)} &= \mathbf{R}_{31}^{(n)} + p_n^{n+1}\alpha_n^T[\mathbf{C}_{51}^{(n+1)} - (\mathbf{C}_{54}^{(n+1)} + \mathbf{C}_{55}^{(n+1)})\mathbf{B}_n\mathbf{d}_n], \\
\mathbf{C}_{41}^{(n)} &= \mathbf{A}_{31}^{(n)} - \mathbf{C}_{51}^{(n)}, \\
\mathbf{C}_{22}^{(n)} &= \mathbf{R}_{22}^{(n)} + p_n^{n+1}\{\mathbf{C}_{22}^{(n+1)} + \mathbf{w}_2^{(n)T}(\mathbf{C}_{44}^{(n+1)}\mathbf{w}_2^{(n)} + 2\mathbf{C}_{42}^{(n+1)} \\
&\quad + 2\mathbf{C}_{45}^{(n+1)}q\gamma_n\mathbf{e}_l) + q\mathbf{e}_l^T\gamma_n^T(\mathbf{C}_{55}^{(n+1)}q\gamma_n\mathbf{e}_l + 2\mathbf{C}_{52}^{(n+1)}) + [2\mathbf{C}_{23}^{(n+1)}t_n \\
&\quad + 2\mathbf{w}_3^{(n)T}(\mathbf{C}_{44}^{(n+1)}\mathbf{w}_2^{(n)} + \mathbf{C}_{42}^{(n+1)} + \mathbf{C}_{45}^{(n+1)}q\gamma_n\mathbf{e}_l) \\
&\quad + 2\mathbf{w}_2^{(n)T}\mathbf{C}_{43}^{(n+1)}t_n + 2q\mathbf{e}_l^T\gamma_n^T\mathbf{C}_{53}^{(n+1)}t_n]s_{n+1}^{-1} \\
&\quad + [\mathbf{w}_3^{(n)T}(\mathbf{C}_{44}^{(n+1)}\mathbf{w}_3^{(n)} + 2\mathbf{C}_{43}^{(n+1)}t_n) + \mathbf{C}_{33}^{(n+1)}t_n^2]s_{n+1}^{-2}, \\
\mathbf{C}_{33}^{(n)} &= \mathbf{h}_n^T\mathbf{M}_n\mathbf{h}_n + p_n^{n+1}\{\mathbf{C}_{33}^{(n+1)}s_n^2s_{n+1}^{-2} + 2\mathbf{w}_1^{(n)T}\mathbf{C}_{43}^{(n+1)}s_ns_{n+1}^{-2} \\
&\quad + [2\mathbf{C}_{34}^{(n+1)}\gamma_n\mathbf{F}_n\mathbf{e}_l - 2\mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{C}_{53}^{(n+1)} + \mathbf{C}_{43}^{(n+1)})]s_ns_{n+1}^{-1} \\
&\quad + [2\mathbf{e}_l^T\mathbf{F}_n^T\gamma_n^T\mathbf{C}_{44}^{(n+1)}\mathbf{w}_1^{(n)} - 2\mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{C}_{54}^{(n+1)} + \mathbf{C}_{44}^{(n+1)})\mathbf{w}_1^{(n)}]s_{n+1}^{-1}
\end{aligned}$$

$$+ \mathbf{w}_1^{(n)T}\mathbf{C}_{44}^{(n+1)}\mathbf{w}_1^{(n)}s_{n+1}^{-2} + \mathbf{e}_l^T\mathbf{F}_n^T\gamma_n^T\mathbf{C}_{44}^{(n+1)}\gamma_n\mathbf{F}_n\mathbf{e}_l$$
$$- 2\mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{C}_{54}^{(n+1)} + \mathbf{C}_{44}^{(n+1)})\gamma_n\mathbf{F}_n\mathbf{e}_l\},$$
$$\mathbf{C}_{23}^{(n)} = 2^{-1}(\mathbf{A}_{22}^{(n)} - \mathbf{C}_{33}^{(n)} - \mathbf{C}_{22}^{(n)}),$$
$$\mathbf{C}_{43}^{(n)} = \mathbf{P}_n^T\mathbf{M}_n\mathbf{h}_n + p_n^{n+1}\{[(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{C}_{43}^{(n+1)} - \mathbf{P}_n^T\mathbf{B}_n^T\mathbf{C}_{53}^{(n+1)}]s_n s_{n+1}^{-1}$$
$$+ [(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{C}_{44}^{(n+1)} - \mathbf{P}_n^T\mathbf{B}_n^T\mathbf{C}_{54}^{(n+1)}]\mathbf{w}_1^{(n)}s_{n+1}^{-1}$$
$$+ [(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{C}_{44}^{(n+1)} - \mathbf{P}_n^T\mathbf{B}_n^T\mathbf{C}_{54}^{(n+1)}]\gamma_n\mathbf{F}_n\mathbf{e}_l$$
$$- \alpha_n^T(\mathbf{C}_{44}^{(n+1)} + \mathbf{C}_{45}^{(n+1)})\mathbf{B}_n\mathbf{h}_n\},$$
$$\mathbf{C}_{52}^{(n)} = \mathbf{R}_{23}^{(n)} + p_n^{n+1}\alpha_n^T[\mathbf{C}_{52}^{(n+1)} + (\mathbf{C}_{53}^{(n+1)}t_n + \mathbf{C}_{54}^{(n+1)}\mathbf{w}_3^{(n)})s_{n+1}^{-1}$$
$$+ \mathbf{C}_{54}^{(n+1)}\mathbf{w}_2^{(n)} + \mathbf{C}_{55}^{(n+1)}q\gamma_n\mathbf{e}_l],$$
$$\mathbf{C}_{53}^{(n)} = p_n^{n+1}\alpha_n^T[(\mathbf{C}_{53}^{(n+1)}s_n s_{n+1}^{-1} + \mathbf{C}_{54}^{(n+1)}\mathbf{w}_1^{(n)}s_{n+1}^{-1} + \mathbf{C}_{54}^{(n+1)}\gamma_n\mathbf{F}_n\mathbf{e}_l$$
$$- (\mathbf{C}_{54}^{(n+1)} + \mathbf{C}_{55}^{(n+1)})\mathbf{B}_n\mathbf{h}_n],$$
$$\mathbf{C}_{42}^{(n)} = \mathbf{A}_{32}^{(n)} - \mathbf{C}_{43}^{(n)} - \mathbf{C}_{53}^{(n)} - \mathbf{C}_{52}^{(n)},$$
$$\mathbf{C}_{45}^{(n)} = p_n^{n+1}\alpha_n^T[\mathbf{C}_{54}^{(n+1)}(\alpha_n - \mathbf{B}_n\mathbf{P}_n) - \mathbf{C}_{55}^{(n+1)}\mathbf{B}_n\mathbf{P}_n],$$
$$\mathbf{C}_{55}^{(n)} = \mathbf{R}_{33}^{(n)} + p_n^{n+1}\alpha_n^T\mathbf{C}_{55}^{(n+1)}\alpha_n,$$
$$\mathbf{C}_{44}^{(n)} = \mathbf{A}_{33}^{(n)} - 2\mathbf{C}_{45}^{(n)} - \mathbf{C}_{55}^{(n)},$$
$$\mathbf{C}_{66}^{(n)} = \mathbf{A}_{44}^{(n)},$$

where $\mathbf{w}_2^{(n)} = q\mathbf{J}_n\varepsilon_n\mathbf{e}_l$ and $\mathbf{w}_3^{(n)} = q\mathbf{N}_n\varepsilon_n\mathbf{e}_l$.

The boundary condition is as follows: $\mathbf{C}_{11}^{(M)} = \mathbf{R}_{11}^{(M)}$, $\mathbf{C}_{22}^{(M)} = \mathbf{R}_{22}^{(M)}$, $\mathbf{C}_{12}^{(M)} = \mathbf{R}_{12}^{(M)}$, $\mathbf{C}_{15}^{(M)} = \mathbf{R}_{13}^{(M)}$, $\mathbf{C}_{25}^{(M)} = \mathbf{R}_{23}^{(M)}$, $\mathbf{C}_{55}^{(M)} = \mathbf{R}_{33}^{(M)}$, $\mathbf{C}_{66}^{(M)} = \mathbf{R}_{33}^{(M)}$ and the remaining elements vanish.

Note that all of the quantities which appear on the right-hand sides of (18) except, obviously, s_n, s_{n+1} and some of $\mathbf{C}_{ij}^{(n+1)}$, $i,j = 1,\ldots,5$, do not depend on s.

It follows from the above considerations that the risk function $R(v, \mathbf{U}^*(s,r))$ is given by (17) with $n = 0$.

10. Extended Bayes strategies and their risk. Let $\widetilde{\mathbf{U}}(m) = (\widetilde{\mathbf{u}}_0,\ldots,\widetilde{\mathbf{u}}_M)$ denote the control strategy given by

$$\widetilde{\mathbf{u}}_M = 0, \quad \widetilde{\mathbf{u}}_n = -\mathbf{P}_n\widetilde{\mathbf{X}}_{n|n}^m - \mathbf{h}_n m - d_n, \quad n = 0,\ldots,M-1,$$

where $\mathbf{P}_n$, $\mathbf{h}_n$, $\mathbf{d}_n$ are given in (15) and for $n = 0,\ldots,M-1$ the vector $\widetilde{\mathbf{X}}_{n|n}^m$ is defined by the recursive equations

$$\widetilde{\mathbf{X}}_{n+1|n+1}^m = \alpha_n\widetilde{\mathbf{X}}_{n|n}^m + \mathbf{B}_n\mathbf{u}_n + \gamma_n\mathbf{F}_n\mathbf{e}_l m + \gamma_n\mathbf{J}_n Y_n, \quad \widetilde{\mathbf{X}}_{0|0}^m = \mathbf{m}.$$

In the way presented in the previous section one can obtain the risk function $R(v, \widetilde{\mathbf{U}}(m))$. The truncated risk is

$$(19) \quad R_n(v, \widetilde{\mathbf{U}}(m)) = [1, v, m, (\widetilde{\mathbf{X}}^m_{n|n})^T, (\widehat{\mathbf{X}}^v_{n|n})^T]\mathbf{S}_n[1, v, m, (\widetilde{\mathbf{X}}^m_{n|n})^T, (\widehat{\mathbf{X}}^v_{n|n})^T]^T + \mathbf{e}_l^T(\mathbf{S}^{(n)}_{66} * \mathrm{Cov}(\mathbf{X}^v_n \mid n))\mathbf{e}_l .$$

where the $(2l+3) \times (2l+3)$ matrix $\mathbf{S}_n$ has the same block form

$$\mathbf{S}_n = \begin{bmatrix} \mathbf{S}^{(n)}_{11} & \mathbf{S}^{(n)}_{12} & \cdots & \mathbf{S}^{(n)}_{15} \\ \mathbf{S}^{(n)}_{21} & \mathbf{S}^{(n)}_{22} & \cdots & \mathbf{S}^{(n)}_{25} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{S}^{(n)}_{51} & \mathbf{S}^{(n)}_{52} & \cdots & \mathbf{S}^{(n)}_{55} \end{bmatrix}$$

as $\mathbf{C}_n$ in (17). The blocks can be obtained from the following equations:

$$\begin{aligned}
\mathbf{S}^{(n)}_{11} &= \mathbf{R}^{(n)}_{11} + \mathbf{d}_n^T\mathbf{M}_n\mathbf{d}_n + p_n^{n+1}[\mathrm{tr}\,\mathbf{J}_n\Xi_n\mathbf{J}_n^T\mathbf{S}^{(n+1)}_{55} + \mathrm{tr}\,\gamma_n\mathbf{F}_n\gamma_n^T\mathbf{S}^{(n+1)}_{66} \\
&\quad + \mathrm{tr}\,\mathbf{J}_n\Xi_n\mathbf{J}_n^T\mathbf{S}^{(n+1)}_{44} + \mathrm{tr}\,\mathbf{J}_n\Xi_n\mathbf{J}_n^T\mathbf{S}^{(n+1)}_{45} + \mathbf{S}^{(n+1)}_{11} \\
&\quad - 2\mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{S}^{(n+1)}_{41} + \mathbf{S}^{(n+1)}_{51})], \\
\mathbf{S}^{(n)}_{31} &= \mathbf{d}_n^T\mathbf{M}_n\mathbf{h}_n + p_n^{n+1}\{\mathbf{S}^{(n+1)}_{31} - \mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{S}^{(n+1)}_{53} + \mathbf{S}^{(n+1)}_{43}) \\
&\quad + [\mathbf{S}^{(n+1)}_{14} - \mathbf{d}_n^T\mathbf{B}_n^T(\mathbf{S}^{(n+1)}_{54} + \mathbf{S}^{(n+1)}_{44})]\gamma_n\mathbf{F}_n\mathbf{e}_l \\
&\quad - \mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{S}^{(n+1)}_{41} + \mathbf{S}^{(n+1)}_{51})\}, \\
\mathbf{S}^{(n)}_{21} &= \mathbf{A}^{(n)}_{21} - \mathbf{S}^{(n)}_{31}, \\
\mathbf{S}^{(n)}_{51} &= \mathbf{R}^{(n)}_{31} + p_n^{n+1}\alpha_n^T[\mathbf{S}^{(n+1)}_{51} - (\mathbf{S}^{(n+1)}_{54} + \mathbf{S}^{(n+1)}_{55})\mathbf{B}_n\mathbf{d}_n], \\
\mathbf{S}^{(n)}_{41} &= \mathbf{A}^{(n)}_{31} - \mathbf{S}^{(n)}_{51}, \\
\mathbf{S}^{(n)}_{22} &= \mathbf{R}^{(n)}_{22} + p_n^{n+1}[\mathbf{S}^{(n+1)}_{22} + \mathbf{w}_2^{(n)T}(\mathbf{S}^{(n+1)}_{44}\mathbf{w}_2^{(n)} + 2\mathbf{S}^{(n+1)}_{42} \\
(20) \qquad &\quad + 2\mathbf{S}^{(n+1)}_{45}q\gamma_n\mathbf{e}_l) + q\mathbf{e}_l^T\gamma_n^T(\mathbf{S}^{(n+1)}_{55}q\gamma_n\mathbf{e}_l + 2\mathbf{S}^{(n+1)}_{52})], \\
\mathbf{S}^{(n)}_{33} &= \mathbf{h}_n^T\mathbf{M}_n\mathbf{h}_n + p_n^{n+1}[\mathbf{S}^{(n+1)}_{33} - 2\mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{S}^{(n+1)}_{53} + \mathbf{S}^{(n+1)}_{43}) \\
&\quad + \mathbf{e}_l^T\mathbf{F}_n^T\gamma_n^T\mathbf{S}^{(n+1)}_{44}\gamma_n\mathbf{F}_n\mathbf{e}_l - 2\mathbf{h}_n^T\mathbf{B}_n^T(\mathbf{S}^{(n+1)}_{54} + \mathbf{S}^{(n+1)}_{44})\gamma_n\mathbf{F}_n\mathbf{e}_l], \\
\mathbf{S}^{(n)}_{23} &= 2^{-1}(\mathbf{A}^{(n)}_{22} - \mathbf{S}^{(n)}_{33} - \mathbf{S}^{(n)}_{22}), \\
\mathbf{S}^{(n)}_{43} &= \mathbf{P}_n^T\mathbf{M}_n\mathbf{h}_n + p_n^{n+1}\{(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{S}^{(n+1)}_{43} - \mathbf{P}_n^T\mathbf{B}_n^T\mathbf{S}^{(n+1)}_{53} \\
&\quad + [(\alpha_n^T - \mathbf{P}_n^T\mathbf{B}_n^T)\mathbf{S}^{(n+1)}_{44} - \mathbf{P}_n^T\mathbf{B}_n^T\mathbf{S}^{(n+1)}_{54}]\gamma_n\mathbf{F}_n\mathbf{e}_l \\
&\quad - \alpha_n^T(\mathbf{S}^{(n+1)}_{44} + \mathbf{S}^{(n+1)}_{45})\mathbf{B}_n\mathbf{h}_n\}, \\
\mathbf{S}^{(n)}_{52} &= \mathbf{R}^{(n)}_{32} + p_n^{n+1}\alpha_n^T(\mathbf{S}^{(n+1)}_{52} + \mathbf{S}^{(n+1)}_{54}\mathbf{w}_2^{(n)} + \mathbf{S}^{(n+1)}_{55}q\gamma_n\mathbf{e}_l), \\
\mathbf{S}^{(n)}_{53} &= p_n^{n+1}\alpha_n^T[\mathbf{S}^{(n+1)}_{53} + \mathbf{S}^{(n+1)}_{54}\gamma_n\mathbf{F}_n\mathbf{e}_l - (\mathbf{S}^{(n+1)}_{54} + \mathbf{S}^{(n+1)}_{55})\mathbf{B}_n\mathbf{h}_n], \\
\mathbf{S}^{(n)}_{42} &= \mathbf{A}^{(n)}_{32} - \mathbf{S}^{(n)}_{43} - \mathbf{S}^{(n)}_{53} - \mathbf{S}^{(n)}_{52},
\end{aligned}$$

$$\mathbf{S}_{45}^{(n)} = p_n^{n+1}\alpha_n^T[\mathbf{S}_{54}^{(n+1)}(\alpha_n - \mathbf{B}_n\mathbf{P}_n) - \mathbf{S}_{55}^{(n+1)}\mathbf{B}_n\mathbf{P}_n],$$
$$\mathbf{S}_{55}^{(n)} = \mathbf{R}_{33}^{(n)} + p_n^{n+1}\alpha_n^T\mathbf{S}_{55}^{(n+1)}\alpha_n,$$
$$\mathbf{S}_{44}^{(n)} = \mathbf{A}_{33}^{(n)} - 2\mathbf{S}_{45}^{(n)} - \mathbf{S}_{55}^{(n)},$$
$$\mathbf{S}_{66}^{(n)} = \mathbf{A}_{44}^{(n)},$$

with the following boundary conditions: $\mathbf{S}_{11}^{(M)} = \mathbf{R}_{11}^{(M)}$, $\mathbf{S}_{22}^{(M)} = \mathbf{R}_{22}^{(M)}$, $\mathbf{S}_{12}^{(M)} = \mathbf{R}_{12}^{(M)}$, $\mathbf{S}_{15}^{(M)} = \mathbf{R}_{13}^{(M)}$, $\mathbf{S}_{25}^{(M)} = \mathbf{R}_{23}^{(M)}$, $\mathbf{S}_{55}^{(M)} = \mathbf{R}_{33}^{(M)}$, $\mathbf{S}_{66}^{(M)} = \mathbf{R}_{33}^{(M)}$ and the remaining elements vanish.

By an inductive argument, using (17)–(20) one can prove that in the case where $\widetilde{\mathbf{X}}_{n|n}^m = \widehat{\mathbf{X}}_{n|n}$ we have

$$R_n(v, \widetilde{\mathbf{U}}(m)) = \lim_{\substack{s\to\infty \\ rs^{-1}\to m}} R_n(v, \mathbf{U}^*(s,r)), \quad n = 0, \ldots, M.$$

Since $\widetilde{\mathbf{X}}_{0|0}^m = \widehat{\mathbf{X}}_{0|0} = \mathbf{m}$ we obtain

$$R(v, \widetilde{\mathbf{U}}(m)) = \lim_{\substack{s\to\infty \\ rs^{-1}\to m}} R(v, \mathbf{U}^*(s,r)).$$

By the definition of the Bayes risk and (17)–(20), this yields that for every prior distribution of V,

$$r(D, \widetilde{\mathbf{U}}(m)) = \lim_{\substack{s\to\infty \\ rs^{-1}\to m}} r(D, \mathbf{U}^*(s,r)). \tag{21}$$

Note that this means that for each m, $\widetilde{\mathbf{U}}(m)$ is an extended Bayes strategy.

11. Minimax control strategies. In view of (17) the Bayes risk $r(D, \mathbf{U}^*(s,r))$ can be expressed as follows:

$$r(D, \mathbf{U}^*(s,r)) = Z_2(s)E_DV^2 + Z_1(s,r)E_DV + Z_0(s,r),$$

where

$$Z_2(s) = \mathbf{C}_{22}^{(0)},$$
$$Z_1(s,r) = 2[\mathbf{m}^T(\mathbf{C}_{42}^{(0)} + \mathbf{C}_{52}^{(0)}) + \mathbf{C}_{12}^{(0)} + \mathbf{C}_{32}^{(0)}rs^{-1}],$$
$$Z_0(s,r) = 2[\mathbf{C}_{13}^{(0)}\mathbf{C}_{14}^{(0)}\mathbf{C}_{15}^{(0)}](rs^{-1}, \mathbf{m}^T, \mathbf{m}^T)^T + \operatorname{tr}\mathbf{C}_{66}^{(0)}\Theta + \mathbf{C}_{11}^{(0)}$$
$$+ (rs^{-1}, \mathbf{m}^T, \mathbf{m}^T)\begin{bmatrix} \mathbf{C}_{33}^{(0)} & \mathbf{C}_{34}^{(0)} & \mathbf{C}_{35}^{(0)} \\ \mathbf{C}_{43}^{(0)} & \mathbf{C}_{44}^{(0)} & \mathbf{C}_{45}^{(0)} \\ \mathbf{C}_{53}^{(0)} & \mathbf{C}_{54}^{(0)} & \mathbf{C}_{55}^{(0)} \end{bmatrix}(rs^{-1}, \mathbf{m}^T, \mathbf{m}^T)^T.$$

Note that $Z_2(s)$ is always a positive number.

The following propositions provide minimax control strategies for Problems A and B.

PROPOSITION 2. *A solution of Problem* A *does exist and*:

(i) *if* $\forall s \geq a^{-1}$, $Z_1[s, \sqrt{s^2 a - s}] > 0$ *then the control strategy* $\widetilde{\mathrm{U}}(\sqrt{a})$ *is minimax w.r.t.* $\mathcal{G}_1$,

(ii) *if* $\forall s \geq a^{-1}$ $Z_1[s, -\sqrt{s^2 a - s}] < 0$ *then the control strategy* $\widetilde{\mathrm{U}}(-\sqrt{a})$ *is minimax w.r.t.* $\mathcal{G}_1$,

(iii) *if* $\exists s^*, r^*$ *such that* $s^* > 0 \wedge Z_1(s^*, r^*) = 0 \wedge (r^*)^2 (s^*)^{-2} + (s^*)^{-1} = a$ *then the control strategy* $\mathrm{U}^*(s^*, r^*)$ *is minimax w.r.t.* $\mathcal{G}_1$.

PROPOSITION 3. *Let* $\overline{s} = (m_2 - m_1^2)^{-1}$ *and* $\overline{r} = m_1(m_2 - m_1^2)^{-1}$. *The control strategy* $\mathrm{U}^*(\overline{s}, \overline{r})$ *is minimax w.r.t.* $\mathcal{G}_2$.

Proof of Proposition 2. First, suppose that the condition in (i) is fulfilled. In view of (21), for each $D \in \mathcal{G}_1$ we can write

$$
\begin{aligned}
r(D, \widetilde{\mathrm{U}}(\sqrt{a})) &= \lim_{s\to\infty} [Z_2^{(n)}(s) E_D V^2 + Z_1^{(n)}(s, \sqrt{s^2 a - s}) E_D V \\
&\qquad\qquad + Z_0^{(n)}(s, \sqrt{s^2 a - s})] \\
&\leq \lim_{s\to\infty} [Z_2^{(n)}(s) a + Z_1^{(n)}(s, \sqrt{s^2 a - s}) \sqrt{a} \\
&\qquad\qquad + Z_0^{(n)}(s, \sqrt{s^2 a - s})] \\
&= \lim_{s\to\infty} [Z_2^{(n)}(s) a + Z_1^{(n)}(s, \sqrt{s^2 a - s}) s^{-1} \sqrt{s^2 a - s} \\
&\qquad\qquad + Z_0^{(n)}(s, \sqrt{s^2 a - s})] \\
&= \lim_{s\to\infty} r(D_{s, \sqrt{s^2 a - s}}, \mathrm{U}^*(s, \sqrt{s^2 a - s}))\,.
\end{aligned}
$$

This inequality, in view of our Lemma, implies that $\widetilde{\mathrm{U}}(\sqrt{a})$ is a $\mathcal{G}_1$-minimax control strategy. (ii) can be proved similarly.

Now we consider (iii). For each $D \in \mathcal{G}_1$ we obtain

$$
\begin{aligned}
r(D, \mathrm{U}^*(s^*, r^*)) &= Z_2^{(n)}(s^*) E_D V^2 + Z_1^{(n)}(s^*, r^*) E_D V + Z_0^{(n)}(s^*, r^*) \\
&\leq Z_2^{(n)}(s^*) a + Z_0^{(n)}(s^*, r^*) = r(D_{r^*, s^*}, \mathrm{U}^*(s^*, r^*))\,.
\end{aligned}
$$

Setting $D_k = D_{s^*, r^*}$ and $\mathrm{U}_k = \mathrm{U}^*(s^*, r^*)$ for each k in the Lemma we find that the control strategy $\mathrm{U}^*(s^*, r^*)$ fulfils the condition (4), so it is minimax w.r.t. $\mathcal{G}_1$.

One can easily verify that one of the three conditions given in (i)–(iii) must be fulfilled. It follows that a solution for our problem always exists and our proof is complete.

Proof of Proposition 3. Notice that $D_{\overline{s}, \overline{r}} \in \mathcal{G}_2$. For each $D \in \mathcal{G}_2$ we have

$$
\begin{aligned}
r(D, \mathrm{U}^*(\overline{s}, \overline{r})) &= Z_2^{(n)}(\overline{s}) E_D V^2 + Z_1^{(n)}(\overline{s}, \overline{r}) E_D V + Z_0^{(n)}(\overline{s}, \overline{r}) \\
&= Z_2^{(n)}(\overline{s}) m_2 + Z_1^{(n)}(\overline{s}, \overline{r}) m_1 + Z_0^{(n)}(\overline{s}, \overline{r}) = \text{const.}
\end{aligned}
$$

Hence, in view of the Corollary the proposition is valid.

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