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RELATIONS BETWEEN OPTIMUM CHEMICAL BALANCE WEIGHING DESIGNS FOR v AND $v+1$ OBJECTS

Abstract. The incidence matrices of BIB designs for v treatments have been used to construct optimum chemical balance weighing designs for $p = v$ or $p = v+1$ objects. Conditions under which the existence of an optimum chemical balance weighing design for v objects implies the existence of an optimum chemical balance weighing design for $v+1$ objects are given. The existence of an optimum chemical balance weighing design for $v+1$ objects implies the existence of an optimum chemical balance weighing design for any $p < v+1$ objects.

1. Introduction. Consider the problem of weighing p objects in n weighings on a chemical balance; then the design matrix X has the elements $-1, 1$ or 0 if the object is placed on the left pan, right pan, or is not included in the particular weighing, respectively. The least square estimate of the vector of the true weights is given by

$$\underline{\widehat{w}} = (X'X)^{-1}X'\underline{y},$$

provided $(X'X)$ is nonsingular, where $\underline{w}$ and $\underline{y}$ are the column vectors of the unknown weights of p objects and of the results recorded in the n weighings, respectively. The covariance matrix of $\underline{\widehat{w}}$ is

$$\mathrm{Var}(\underline{\widehat{w}}) = \sigma^2(X'X)^{-1}.$$

DEFINITION 1. A weighing design is said to be *optimal* if $X'X = nI_p$.

The problem is to choose X in such a way that the variance factors are minimized. Several methods of constructing X are available in the literature. Dey [2], Nigam [4], Saha [5], Kageyama and Saha [3] and others have shown

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how optimum chemical balance weighing designs can be constructed from the incidence matrices of balanced incomplete block (BIB) designs for $p = v$ objects. Saha and Kageyama [6] have constructed optimum chemical balance weighing designs for $p = v + 1$ objects from incidence matrices of BIB designs for v treatments.

In the present paper we study some relations between an optimum chemical balance weighing design for $p = v$ and $p = v + 1$ objects.

2. Main results. Consider two BIB designs with parameters $v, b_i, r_i, k_i, \lambda_i$, $i = 1,2$. Let N_i^* denote the $v \times b_i$ binary incidence matrix, and let $N_i = 2N_i^* - \underline{1}_v\underline{1}_{b_i}'$, where $\underline{1}_a$ is the $a \times 1$ vector of ones. Then

$$X' = [N_1 \,\vdots\, N_2] \tag{1}$$

is the design matrix of a chemical balance weighing design for $p = v$ objects in $n = b_1 + b_2$ weighings.

LEMMA 1 (Ceranka and Katulska [1]). *The chemical balance weighing design with X given by* (1) *is optimal if and only if*

$$\alpha = 0\,, \tag{2}$$

where $\alpha = b_1 + b_2 - 4[(r_1 - \lambda_1) + (r_2 - \lambda_2)]$.

Now consider the design matrices X of the chemical balance weighing designs in the form

$$X_i = \begin{bmatrix} N_1' & \underline{1}_{b_1} \\ N_2' & (-1)^i\underline{1}_{b_2} \end{bmatrix}, \quad i = 1,2\,. \tag{3}$$

In these designs we have $p = v + 1$ and $n = b_1 + b_2$.

THEOREM 1. *The chemical balance weighing design with X_i given by* (3) *is optimal if and only if* (2) *holds and*

$$b_1 + (-1)^i b_2 = 2[r_1 + (-1)^i r_2]\,, \quad i = 1,2\,. \tag{4}$$

Proof. For the design matrix X_i given by (3) we have

$$X_i'X_i = \begin{bmatrix} (b_1 + b_2 - \alpha)I_v + \alpha\underline{1}_v\underline{1}_v' & a_i\underline{1}_v \\ a_i\underline{1}_v' & b_1 + b_2 \end{bmatrix},$$

where $a_i = -[b_1 + (-1)^i b_2] + 2[r_1 + (-1)^i r_2]$, $i = 1,2$. Since a chemical balance weighing design is optimal if and only if $X'X = nI_p$ conditions (2) and (4) follow, which completes the proof.

From Lemma 1 and Theorem 1 we have the following corollary.

COROLLARY 1. *If a chemical balance weighing design with X given by* (1) *is optimal and* (4) *holds, then a chemical balance weighing design with X_i given by* (3) *is optimal.*

THEOREM 2. *If a chemical balance weighing design with X_i given by (3) is optimal, then any $p < v+1$ columns of this matrix constitute an optimum chemical balance weighing design for p objects in $b_1 + b_2$ weighings.*

Proof. A chemical balance weighing design with X_i given by (3) is optimal if and only if $X_i'X_i = (b_1 + b_2)I_{v+1}$, $i = 1,2$. This means that a chemical balance weighing design with X_i, $i = 1,2$, is optimal if it is a $(b_1 + b_2) \times (v+1)$ matrix of ± 1 whose columns are orthogonal, which yields the assertion of the theorem.

References

[1] B. Ceranka and K. Katulska, *A relation between* BIB *designs and chemical balance weighing designs*, Statist. Probab. Lett. 5 (1987), 339–341.

[2] A. Dey, *On some chemical balance weighing designs*, Austral. J. Statist. 13 (1971), 137–141.

[3] S. Kageyama and G. M. Saha, *Note on the construction of optimum chemical balance weighing designs*, Ann. Inst. Statist. Math. 35A (1983), 447–452.

[4] A. K. Nigam, *A note on optimum chemical balance weighing designs*, Austral. J. Statist. 16 (1974), 50–52.

[5] G. M. Saha, *A note on relations between incomplete block and weighing designs*, Ann. Inst. Statist. Math. 27 (1975), 387–390.

[6] G. M. Saha and S. Kageyama, *Balanced arrays and weighing designs*, Austral. J. Statist. 26 (1984), 119–124.

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