ZASTOSOWANIA MATEMATYKI
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L. MARZEC (Wrocław)

ROBUSTNESS OF TESTS BASED ON SPACINGS IN THE EXPONENTIAL MODEL

1. Abstract. We consider the problem of testing a scale parameter of the exponential distribution function. In the class of unbiased tests based on spacings we find the most robust test under the violations generated by the convex, starshaped and dispersive ordering, respectively.

2. Introduction and preliminaries. Throughout the paper we identify a probability distribution with its distribution function and assume that all considered distributions F are absolutely continuous (with respect to Lebesgue measure) and $F(0) = 0$. The *failure rate function* of F is defined as

$$r_F(t) = f(t)/\overline{F}(t),$$

where f denotes the density function of F and $\overline{F} = 1 - F$. We use the notations:

$$F_\lambda(t) = F(t/\lambda),$$
$$F^{-1}(x) = \inf\{t : F(t) \geq x\},$$
$$r_F(\infty) = \lim_{t\to\infty} r_F(t)$$

if the limit exists.

Let a random variable X have the distribution F, and let $X_{0:n} = 0$, $X_{1:n}, \ldots, X_{n:n}$ be order statistics of a sample from the distribution F. Then the random variable

$$D_{i:n} = (n - i + 1)(X_{i:n} - X_{(i-1):n}) \quad \text{for } i = 1, \ldots, n$$

is called the *i-th normalized spacing*.

Denote by $F_{\underline{a}}$ the distribution of $\sum_{i=1}^n a_i D_{i:n}$, where $\underline{a} = (a_1, \ldots, a_n)$, and by K the unit mean exponential distribution. Let $\underline{X} = (X_1, \ldots, X_n)$ be

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a sample of size $n \geq 2$ from a population with the exponential distribution K_λ whose scale parameter λ is unknown. For a given significance level $\alpha \in (0,1)$ consider the problem of testing the hypothesis $H_0 : \lambda \leq \lambda_0$ $(\lambda_0 > 0)$ versus $H_A : \lambda > \lambda_0$ in the class of the following unbiased tests:

$$\varphi_{\underline{a}}(\underline{X}, \alpha) = \begin{cases} 1 & \text{if } \sum_{i=1}^n a_i D_{i:n} \geq c_{\underline{a},\alpha}, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

$$\sup_{\lambda \leq \lambda_0} E_{K_\lambda} \varphi_{\underline{a}}(\underline{X}, \alpha) = \alpha, \quad \text{where } \underline{a} \geq \underline{0},\ \underline{a} \neq \underline{0}. \tag{2}$$

Note that for each $\alpha \in (0,1)$ the test $\varphi_{\underline{b}}(\underline{X}, \alpha)$, where $b_i = 1/n$ for $i = 1, \ldots, n$, is based on the sample mean $\overline{X}$ and is uniformly most powerful for testing the above hypotheses.

Assume that due to measurement errors the observations are slightly disturbed and the deviation from the distribution K is described by a specified set of distributions $\pi(K)$ such that $K \in \pi(K)$. The natural question to ask about the robustness of a test concerns the behaviour of the power function. Let

$$\beta_F^{\underline{a},\alpha}(\lambda) = E_{F_\lambda} \varphi_{\underline{a}}(\underline{X}, \alpha), \quad F \in \pi(K) \setminus \{K\},$$

denote the "violated" power function of the test $\varphi_{\underline{a}}(\underline{X}, \alpha)$ at the point $\lambda > 0$, i.e. when a sample comes from the distribution F_λ. Hence, to each test $\varphi_{\underline{a}}(\underline{X}, \alpha)$ there corresponds the set of functions $M(\underline{a}, \alpha) = \{\beta_F^{\underline{a},\alpha}(\cdot) : F \in \pi(K)\}$.

Let $(M(\underline{a}, \alpha), d)$ be a metric space for each $\alpha \in (0,1)$ and all $\underline{a} \geq \underline{0}$, $\underline{a} \neq \underline{0}$. Then the robustness of the test $\varphi_{\underline{a}}(\underline{X}, \alpha)$ under the violation $\pi(K)$ is described in the metric d by the quantity

$$R_{\underline{a},\alpha} = \sup\{d(\beta_F^{\underline{a},\alpha}, \beta_S^{\underline{a},\alpha}) : F, S \in \pi(K)\}. \tag{3}$$

DEFINITION 1. We say that the test $\varphi_{\underline{a}}(\underline{X}, \alpha)$ is *more robust* than the test $\varphi_{\underline{b}}(\underline{X}, \alpha)$ if $R_{\underline{a},\alpha} < R_{\underline{b},\alpha}$.

DEFINITION 2. We call $\varphi_{\underline{a}}(\underline{X}, \alpha)$ *most robust* in a specified class V_α of level-α tests if $\varphi_{\underline{a}}(\underline{X}, \alpha)$ is more robust than any other test in V_α.

The above concept of robustness is connected with the general approach presented in [8] and [9].

3. Violations of the exponential model. Let F and G be two distributions with density functions f and g. Recall the definitions of three well-known partial orderings and of two classes of monotone failure rate distributions.

DEFINITION 1. (i) $F <_c G$ if $G^{-1} \circ F$ is convex (*convex ordering*),
(ii) $F <_* G$ if $G^{-1} \circ F$ is starshaped (*starshaped ordering*),
(iii) $F <_d G$ if $G^{-1} - F^{-1}$ is nondecreasing (*dispersive ordering*).

DEFINITION 2. (i) F is an *increasing failure rate distribution* (F is IFR) if $F <_c K$.

(ii) F is an *decreasing failure rate distribution* (F is DFR) if $K <_c F$.

LEMMA 1. (i) *If* $F <_* G$ *and* $f(0) \geq g(0) > 0$ *then* $F <_d G$.

(ii) *If* $r_F(t) \leq r_G(t)$ *for every* $t \geq 0$ *and* F *or* G *is DFR then* $G <_d F$.

For the proof see [3] and [6].

LEMMA 2. *If* $F <_c G$ *and* $\lim_{u\to 1} r_G(G^{-1}(u))/r_F(F^{-1}(u)) \geq 1$ *then* $G <_d F$.

Proof. If $F <_c G$ then $G^{-1} \circ F$ is convex. Hence for every $v \in (0,1)$

$$g(G^{-1}(v))/f(F^{-1}(v)) \geq \lim_{u\to 1} g(G^{-1}(u))/f(F^{-1}(u)) \geq 1.$$

Thus the lemma follows from Definition 1(iii).

Let H and G be two fixed absolutely continuous distributions with density functions h and g such that

1) $H(0) = G(0) = 0$,

2) $r_G(t) \leq 1 \leq r_H(t)$ for every $t \geq 0$ and $r_G(t)/r_H(t) \neq \text{const}$,

and either

3) H is DFR and G is IFR, or

3′) H is IFR and G is DFR.

If H and G satisfy the conditions 1), 2), 3) then let $\pi(K) \in \Pi = \{\pi_1, \pi_2, \pi_3\}$, where

$$\pi_1 = \{F : H <_d F <_d G\},$$
$$\pi_2 = \{F : G <_c F <_c H,\ r_G(\infty) \leq r_F(\infty) \leq r_H(\infty)\},$$
$$\pi_3 = \{F : 1 \leq r_F(t) \leq r_H(t) \text{ for every } t \geq 0\}.$$

By Lemma 1(ii) and Lemma 2 we obtain

$$\pi_2 \subset \pi_1 \quad \text{and} \quad \pi_3 \subset \pi_1. \tag{4}$$

If H and G satisfy the conditions 1), 2), 3′) then let $\pi(K) \in \widetilde{\Pi} = \{\pi_1, \widetilde{\pi}_2, \widetilde{\pi}_3, \widetilde{\pi}_4\}$, where

$$\widetilde{\pi}_2 = \{F : H <_* F <_* G,\ g(0) \leq f(0) \leq h(0)\},$$
$$\widetilde{\pi}_3 = \{F : H <_c F <_c G,\ g(0) \leq f(0) \leq h(0)\},$$
$$\widetilde{\pi}_4 = \{F : r_G(t) \leq r_F(t) \leq 1 \text{ for every } t \geq 0\}.$$

Then from Lemma 1 it follows that

$$\widetilde{\pi}_3 \subset \widetilde{\pi}_2 \subset \pi_1 \quad \text{and} \quad \widetilde{\pi}_4 \subset \pi_1. \tag{5}$$

Violations generated by the ordering relations were considered in [4]. Now we give some parametric examples.

a) Let $H = \Gamma_{1,p}$ for $0 < p \le 1$, $G = \Gamma_{1,q}$ for $q > 1$, where $\Gamma_{1,s}, s > 0$, denotes the gamma distribution with density function $x^{s-1}\exp(-x)/\Gamma(s)$, $x \ge 0$. From the well-known property of gamma distributions which says that $\Gamma_{1,s} <_c \Gamma_{1,t}$ for $t < s$, and the fact that $r_F(\infty) = 1$ for $F = \Gamma_{1,s}$, we obtain

$$\{\Gamma_{1,s} : p \le s \le q\} \subset \pi_2.$$

b) Let $H = \Gamma^*_{1,p}$ for $1 < p \le 2.16$ and $G = \Gamma^*_{1,q}$ for $0 < q \le 1$, where $\Gamma^*_{1,s}, s > 0$, denotes the exponential power distribution with density function $\exp(-x^s)/\Gamma(1+1/s), x \ge 0$. Then H is IFR, G is DFR and $\Gamma^*_{1,t} <_d \Gamma^*_{1,s}$ for $0 < s < t \le 2.16$. Consequently,

$$\{\Gamma^*_{1,s} : q \le s \le p\} \subset \pi_1.$$

c) Let $G = S_{q,p}$ for $0 < q < p$, where $S_{q,p}(x) = 1 - (1 + x/p)^{-q}$ is the Pareto type distribution. Then G is DFR. Denote by $r_{s,t}$ the failure rate function of $S_{s,t}$. Using the relations $r_{s,t}(0) = s/t$, $r_G(x) \le r_{s,t}(x)$, $x \ge 0$, and $S_{s,t} <_c S_{q,p}$ for $s \ge q$ and $q/p \le s/t \le 1$, we conclude that

$$\{S_{s,t} : q/p \le s/t \le 1,\ q \le s\} \subset \widetilde{\pi}_3 \cap \widetilde{\pi}_4.$$

4. Results. Let the assumptions of Sections 2, 3 be satisfied and $E_G X < \infty$. Consider the metric defined in some "suitably restricted" class of functions $\xi, \eta : \mathbf{R}_+ \to \mathbf{R}$ as follows:

$$d(\xi,\eta) = \int_0^\infty x^{-2}|\xi(x) - \eta(x)|\,dx. \tag{6}$$

We prove the following

THEOREM 1. *Let $\alpha \in (0,1)$ and let $\underline{a} = (a_1, \ldots, a_n) \ge \underline{0}$ be a vector such that $a_i \ne a_j$ for some i and j. Denote by $a_{1:n} \le \ldots \le a_{n:n}$ the ordered coordinates of $\underline{a}$ and by P_n the permutation group on $\{1, \ldots, n\}$. Then the following tests are most robust in the class $V_\alpha = \{\varphi_{(a_{\tau(1)},\ldots,a_{\tau(n)})}(\underline{X},\alpha) : \tau \in P_n\}$:*

(i) $\varphi_{(a_{1:n},a_{2:n},\ldots,a_{n:n})}(\underline{X},\alpha)$ *if* $\pi(K) \in \Pi$,
(ii) $\varphi_{(a_{n:n},a_{n-1:n},\ldots,a_{1:n})}(\underline{X},\alpha)$ *if* $\pi(K) \in \widetilde{\Pi}$.

Proof. Fix $\alpha \in (0,1)$. From (1), (2) we find that $c_{\underline{a},\alpha} = \lambda_0 K_{\underline{a}}^{-1}(1-\alpha)$, whence $\beta_F^{\underline{a},\alpha}(\lambda) = \overline{F}_{\underline{a}}(\lambda_0 K_{\underline{a}}^{-1}(1-\alpha)/\lambda)$, $\lambda > 0$. Then from (6)

$$d(\beta_F^{\underline{a},\alpha}, \beta_S^{\underline{a},\alpha}) = \int_0^\infty \lambda^{-2}|\beta_F^{\underline{a},\alpha}(\lambda) - \beta_S^{\underline{a},\alpha}(\lambda)|\,d\lambda$$

$$= (\lambda_0 K_{\underline{a}}^{-1}(1-\alpha))^{-1} \int_0^\infty |F_{\underline{a}}(t) - S_{\underline{a}}(t)|\, dt.$$

Since $H <_d F <_d G$ implies $G_{\underline{a}}(t) \le F_{\underline{a}}(t) \le H_{\underline{a}}(t)$ for every $t \ge 0$ (see [7]) and by (3)–(5) we have

$$\begin{aligned}(7)\qquad R_{\underline{a},\alpha} &= \sup\{d(\beta_F^{\underline{a},\alpha}, \beta_S^{\underline{a},\alpha}) : F, S \in \pi(K)\}\\ &\doteq (\lambda_0 K_{\underline{a}}^{-1}(1-\alpha))^{-1} \int_0^\infty (H_{\underline{a}}(t) - G_{\underline{a}}(t))\, dt\\ &= (\lambda_0 K_{\underline{a}}^{-1}(1-\alpha))^{-1} \sum_{i=1}^n a_i(E_G D_{i:n} - E_H D_{i:n}).\end{aligned}$$

From the well-known properties of the normalized spacings (see [1], [2]) it follows that if F is an absolutely continuous not exponential life distribution and F is IFR (DFR) then $E_F D_{i:n}$ is strictly decreasing (increasing) in $i = 1, \ldots, n$. Consequently, for $\pi(K) \in \Pi$ ($\widetilde{\Pi}$) we obtain

(8) $E_G D_{i:n} - E_H D_{i:n}$ is strictly decreasing (increasing) in $i = 1, \ldots, n$.

Moreover, $K_{\underline{a}}(x) = K_{(a_{\tau(1)},\ldots,a_{\tau(n)})}(x)$, $x \ge 0$, for every $\tau \in P_n$.

To complete the proof it suffices to show that for any numbers $x_1 > \cdots > x_n$ and $(a_{\tau(1)}, \ldots, a_{\tau(n)}) \ne (a_{1:n}, \ldots, a_{n:n})$ we have

$$\sum_{i=1}^n a_{i:n} x_i < \sum_{i=1}^n a_{\tau(i)} x_i.$$

Indeed, if we take $x_{n+1} = 0$, $y_j = x_j - x_{j+1}$ for $j = 1, \ldots, n$ then $y_j > 0$ for $j \ne n$ and

$$\begin{aligned}\sum_{i=1}^n a_{i:n} x_i &= \sum_{i=1}^n a_{i:n}\Big(\sum_{j=i}^n y_j\Big) = \sum_{j=1}^n y_j\Big(\sum_{i=1}^j a_{i:n}\Big)\\ &< \sum_{j=1}^n y_j\Big(\sum_{i=1}^j a_{\tau(i)}\Big) = \sum_{i=1}^n a_{\tau(i)} x_i\,.\end{aligned}$$

THEOREM 2. *Let $c_2 = \exp(-3/2)$ and $c_n = \exp(-(n+1))$ for $n > 2$. Then there exists $c \in (c_n, 1]$ such that for $\alpha \in (0, c)$ the following tests are most robust in the class $V_\alpha = \{\varphi_{\underline{a}}(\underline{X}, \alpha) : \underline{a} = (a_1, \ldots, a_n) \ge \underline{0},\ \underline{a} \ne \underline{0}\}$:*

(i) $\varphi_{(0,0,\ldots,0,1)}(\underline{X}, \alpha)$ *if* $\pi(K) \in \Pi$,

(ii) $\varphi_{(1,0,0,\ldots,0)}(\underline{X}, \alpha)$ *if* $\pi(K) \in \widetilde{\Pi}$.

Proof. It is easy to note that $\varphi_{\underline{a}}(\underline{X}, \alpha) = \varphi_{\underline{b}}(\underline{X}, \alpha)$, where $\underline{b} = (\sum_{i=1}^n a_i)^{-1}\underline{a}$, whence we conclude that $V_\alpha = \{\varphi_{\underline{a}}(\underline{X}, a) : \underline{a} \ge \underline{0},$

$\sum_{i=1}^{n} a_i = 1\}$. If $\pi(K) \in \Pi$ $(\widetilde{\Pi})$ then by (8) we obtain

$$\sum_{i=1}^{n} a_i(E_G D_{i:n} - E_H D_{i:n}) > E_G D_{n:n} - E_H D_{n:n} \quad (\text{resp. } > E_G D_{1:n} - E_H D_{1:n}) \tag{9}$$

for every $\underline{a} \geq \underline{0}$, $\sum_{i=1}^{n} a_i = 1$, $\underline{a} \neq (0,0,\ldots,0,1)$ (resp. $\neq (1,0,0,\ldots,0)$). It is well-known that normalized spacings of a sample from the exponential distribution are i.i.d. with the same exponential distribution. In view of the results given in [5] we have the Schur-concavity of $K_{\underline{a}}(t)$, $\underline{a} > \underline{0}$, for every fixed $t \geq (-\ln c_n) \sum_{i=1}^{n} a_i$. Consequently, if $\underline{a} \geq \underline{0}$ and $\sum_{i=1}^{n} a_i = 1$ one can deduce that $K(t) \leq K_{\underline{a}}(t)$ for $t \geq -\ln c_n$. Hence $y \leq K_{\underline{a}}(K^{-1}(y))$ for $y \geq 1 - c_n$ and

$$K_{\underline{a}}^{-1}(y) = \inf\{x : K_{\underline{a}}(x) \geq y\} \leq K^{-1}(y) \quad \text{for } y \geq 1 - c_n,$$

i.e.

$$K_{\underline{a}}^{-1}(1-\alpha) \leq K^{-1}(1-\alpha) \quad \text{for } \alpha \leq c_n.$$

The above inequality and (9) applied to (7) complete the proof.

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LESZEK MARZEC
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCLAW
PL. GRUNWALDZKI 3/4
50-384 WROCLAW, POLAND

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