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ON THE EXTREME GAP IN THE RENEWAL PROCESS

1. Introduction. Consider a sequence $X_1, X_2, \ldots$ of independent non-negative random variables with common distribution function F. Define the renewal process and the process of actual working time by

$$N(t) = \sum_{n=1}^{\infty} 1_{S_n \le t}, \quad \gamma(t) = t - S_{N(t)},$$

respectively, where $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n = 1, 2, \ldots$, $t \ge 0$.

Define the process of extreme gap by

$$M(t) = \sup_{0 \le u \le t} \gamma(u), \quad t \ge 0, \tag{1}$$

and the first passage time in the process γ from the state $\gamma(0) = 0$ to the state x by

$$T(x) = \inf\{t : M(t) > x\}, \quad x \ge 0. \tag{2}$$

Our purpose in the paper is to analyse the distribution of $M(t)$. In particular, we consider the distribution of the extreme gap in the Poisson process, for which we give a formula for the expected value of the extreme gap. In the general considerations we study the discrete part of the distribution F and its role in the limit theorem and estimation of the distribution.

Note that the distribution functions are right continuous; the discrete part of the distribution F will be denoted by

$$p_x = F(x) - F(x-).$$

Observe that the stochastic processes N, γ, T have right continuous realizations and the process M is continuous. The fundamental relation is obvious:

$$P(M(t) > x) = P(T(x) \le t), \quad t \ge 0, \ x \ge 0. \tag{3}$$

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Now we introduce a few general notations:

$$P(t,x) = P(M(t) \le x),$$
$$R(t,x) = 1 - P(t,x),$$
$$\mu(t) = EM(t), \quad t \ge 0,\ x \ge 0.$$

It is trivial and we do not repeat this in further considerations that $P(t,x) = 1$, $R(t,x) = 0$ for $x \ge t$.

2. The extreme gap in the Poisson process. Suppose that $F(x) = 1 - e^{-\lambda x}$, $x \ge 0$, $\lambda > 0$, i.e. N is a Poisson process with parameter λ. Now we characterize the distribution function of the extreme gap and find its expected value.

THEOREM 1. *For the Poisson process with parameter* λ *we have*

$$(4)\quad R(t,x) = e^{-\lambda x} + \int_0^x R(t-u,x)\lambda e^{-\lambda u}\,du, \quad 0 < x \le t,$$

$$(5)\quad R^*(s,x) = \int_0^\infty e^{-st}R(t,x)\,dt = (s+\lambda)e^{-(s+\lambda)x}\Big/\left(s^2\left(1+\frac{\lambda}{s}e^{-(s+\lambda)x}\right)\right),$$

$$(6)\quad R^{**}(s,u) = \int_0^\infty\int_0^\infty e^{-st-ux}R(t,x)\,dt\,dx$$
$$= \frac{s+\lambda}{s\lambda}\sum_{k=1}^\infty (-1)^{k-1}\left(\frac{\lambda}{s}\right)^k \frac{1}{k(s+\lambda)+u}, \quad \operatorname{Re} s > 0,\ \operatorname{Re} u \ge 0.$$

Proof. For each renewal process, from the total probability formula it follows that

$$R(t,x) = 1 - F(x) + \int_0^x R(t-u,x)\,dF(u), \quad 0 < x \le t.$$

For the exponential distribution function, this is (4). Hence for the Laplace transform we have

$$(7)\qquad sR^*(s,x) = \frac{s+\lambda}{s}e^{-(s+\lambda)x} - \lambda e^{-(s+\lambda)x}R^*(s,x),$$

which yields (5) for $\operatorname{Re} s > 0$.

Finally, from (7) we get

$$sR^{**}(s,u) = \frac{s+\lambda}{s}\frac{1}{s+\lambda+u} - \lambda R^{**}(s, s+\lambda+u).$$

The solution of this equation is (6).

COROLLARY 1. *For the Poisson process with parameter* λ *we have*

$$(8)\quad R(t,x) = e^{-\lambda x}\sum_{k=1}^{\infty}\frac{1}{(k-1)!}(-\lambda(t-kx)_+e^{-\lambda x})^{k-1}\left(1+\frac{1}{k}\lambda(t-kx)_+\right),$$

where $a_+ = \max(0,a)$, *and*

$$(9)\qquad \mu(t) = \frac{1}{\lambda}\int_0^{\lambda t}\frac{1}{u}(1-e^{-u})\,du.$$

Indeed, from (5) we get

$$R^*(s,x) = \left(\frac{1}{s}+\frac{\lambda}{s^2}\right)\sum_{k=1}^{\infty}\left(\frac{-\lambda}{s}\right)^{k-1}e^{-(s+\lambda)kx}.$$

The original for this function is

$$R(t,x) = \sum_{k=1}^{\infty}(-\lambda)^{k-1}e^{-\lambda kx}\left(\int_0^t\frac{(t-u)^{k-2}}{(k-2)!}1_{(kx,\infty)}(u)\,du - \int_0^t\frac{\lambda(t-u)^{k-1}}{(k-1)!}1_{(kx,\infty)}(u)\,du\right).$$

Hence, after a transformation we get (8).

Note that

$$\mu(t) = \int_0^{\infty}P(M(t)>x)\,dx = \int_0^t R(t,x)\,dx,$$

hence

$$\mu^*(s) = \int_0^{\infty}e^{-st}\mu(t)\,dt = \int_0^{\infty}e^{-st}\int_0^t R(t,x)\,dx\,dt = \int_0^{\infty}\int_x^{\infty}R(t,x)e^{-st}\,dx dt$$
$$= R^{**}(s,0) = \frac{1}{s\lambda}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\left(\frac{\lambda}{s}\right)^k = \frac{1}{s\lambda}\log\left(1+\frac{\lambda}{s}\right).$$

Using the properties of convolution and the Laplace transform we get

$$\mu(t) = \frac{1}{\lambda}\int_0^t\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}\frac{\lambda^k u^{k-1}}{(k-1)!}\,du = \frac{1}{\lambda}\int_0^t\frac{1}{z}(1-e^{-\lambda z})\,dz.$$

This proves (9).

COROLLARY 2. *For the Poisson process with parameter* λ *we have*

$$\mu(t) = \frac{1}{\lambda}(\log\lambda t + C) + o(1), \qquad t\to\infty,$$

where $C = 0.5772\ldots$ is the Euler constant.

Indeed, we have

$$\int_0^t \frac{1}{u}(1 - e^{-u})\,du$$

$$= \int_0^1 \frac{1}{u}(1 - e^{-u})\,du + \int_1^t \frac{1}{u}\,du - \int_1^\infty \frac{1}{u}e^{-u}\,du + \int_t^\infty \frac{1}{u}e^{-u}\,du$$

$$= \int_0^\infty (1 - \exp(-e^{-v}))\,dv + \log t - \int_{-\infty}^0 \exp(-e^{-v})\,dv + \int_t^\infty \frac{1}{u}e^{-u}\,du$$

$$= C + \log t + \int_t^\infty \frac{1}{u}e^{-u}\,du,$$

and

$$\int_t^\infty \frac{1}{u}e^{-u}\,du = o(1), \quad t \to \infty$$

(see for example [4]).

If the distribution function F is concentrated on the positive integers, then the processes N, γ, M and T may be considered in discrete time. Set $p_k = F(k) - F(k-1)$, $R(k) = p_{k+1} + p_{k+2} + \ldots$, $k = 1, 2, \ldots$ In these terms we present a recurrence formula for the distribution of the extreme gap.

COROLLARY 3. *If F is discrete, then*

$$R(t,k) = R(k+1) + \sum_{j=1}^{k+1} R(t-j,k)p_j, \quad t = k+1, k+2, \ldots,\ k = 0, 1, \ldots$$

If, in particular $p_k = q^{k-1}p$, $k = 1, 2, \ldots$, $q = 1 - p$, $0 < p < 1$, then the formula has the form (see [5], Theorem 1)

$$(10) \quad R(t+1,k) - R(t,k) = q^{k+1}p(1 - R(t-k-1,k)), \qquad t = k+1, k+2, \ldots,\ k = 0, 1, \ldots$$

COROLLARY 4. *For the Poisson process with parameter λ we have*

$$(11) \qquad R(x,x) = e^{-\lambda x},$$

$$(12) \qquad \frac{\partial}{\partial t}R(t,x) = \lambda e^{-\lambda x}(1 - R(t-x,x)), \quad x \le t.$$

Indeed, from (4) we get

$$R(t+h,x) - e^{-\lambda h}R(t,x) = e^{-\lambda x}(1 - e^{-\lambda h})$$

$$+\int_0^h R(t+h-u,x)\lambda e^{-\lambda u}\,du + e^{-\lambda h}\int_{x-h}^{x} R(t-u,x)\lambda e^{-\lambda u}\,du.$$

Since $R(t,x)$ is continuous for $t > x$, the derivative $\left(\frac{\partial}{\partial t}\right) R(t,x)$ exists and (12) holds. The formula (11) follows from the assumption that in a Poisson process there are no signals in an interval of length x.

The formula (12) is a continuous version of the discrete formula (10).

3. Limit theorem and estimation. The analysis of the limiting distribution of the extreme gap is strictly related to the analysis of extremes in sequences of random variables. The latter theory concentrates on the existence and form of the limiting distributions for which some regularity of F at $a_F = \sup\{x : F(x) < 1\}$ is required. The next theorem is a simple consequence of Theorem 1.5.1 of [7].

THEOREM 2. *Let $X_1, \ldots, X_n$ be independent random variables with common distribution function F and let $M_n = \max(X_1, \ldots, X_n)$. Assume the existence of two sequences of real constants u_n, τ_n with $\tau_n \geq 0$ and τ_n bounded, such that*

$$n(1 - F(u_n)) - \tau_n \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Then

$$P(M_n \leq u_n) - \exp(-\tau_n) \to 0 \quad \text{as } n \to \infty. \tag{14}$$

If the distribution function F is continuous, then we may take $u_n = F^{-1}(1 - t/n) = \inf\{x : F(x) > 1 - t/n\}$ and $\tau_n = n(1 - F(u_n))$. However, in this case not the estimation but the limit convergence is established. If F has a discrete part then for the limit theorem to hold it is necessary to make some additional assumptions (see [7], Theorem 1.7.13).

THEOREM 3. *Assume that F has the discrete part $p_{x_k} > 0$, $k = 1, 2, \ldots$, for which $x_k \to a_F$ as $k \to \infty$. Then if*

$$\delta = \limsup_{k\to\infty} p_{x_k}/(1 - F(x_k)) < \infty, \tag{15}$$

then the condition (13) *is satisfied for some sequences u_n and $\tau_n \geq 0$.*

Proof. Fix t and let $u_n = F^{-1}(1 - t/n)$, $\tau_n = n(1 - F(u_n))$. Then $F(u_n) - p_{u_n} \leq 1 - t/n \leq F(u_n)$, which implies that $t(1 - \delta) \leq \tau_n \leq t$, showing that the sequence τ_n is bounded and (13) holds.

Now we pass to the analysis of the limiting properties of the extreme gap. We formulate the main theorem and in addition analyse the asymptotics of T. Denote by μ_F the expected value in distribution F.

THEOREM 4. *Assume that $\mu_F < \infty$ and that $x \to a_F$ implies $F(x) \to 1$, and let $u(t)$, $\tau(t)$ be functions such that*

$$\frac{1}{\mu_F} t(1 - F(u(t))) - \tau(t) \to 0\,, \quad u(t) \to a_F \quad \text{as } t \to \infty, \tag{16}$$

where $\tau(t)$ is bounded. Then

$$P(t, u(t)) - \exp(-\tau(t)) \to 0 \quad \text{as } t \to \infty. \tag{17}$$

Remark. The condition (16) holds if (15) is fulfilled (in particular, if F is continuous). If $\tau(t) \to \tau$, $0 \le \tau \le \infty$, as $t \to \infty$, then $P(t, u(t)) \to e^{-\tau}$ as $t \to \infty$.

Now we pass to the asymptotics of T. Fix x such that $x < a_F$. Define a sequence of random variables by

$$X_i(x) = \begin{cases} X_i & \text{if } X_i \le x, \\ +\infty & \text{if } X_i > x, \end{cases}$$

and introduce a renewal process $N(x, t)$ as follows:

$$N(x, t) = \sum_{n=1}^{\infty} 1_{S_n(x) \le t}, \quad t \ge 0, \tag{18}$$

where $S_0(x) = 0$, $S_n(x) = S_{n-1}(x) + X_n(x)$, $n = 1, 2, \ldots$

Note that this is a transitive process with the total number of renewals $N(x, \infty)$, and the endpoint $S(x) = S_{N(x,\infty)}(x)$.

LEMMA. *If $\mu_F < \infty$, and if $x \to a_F$ implies $F(x) \to 1$, then*

$$P\left(\frac{T(x)}{ES(x)} \le t\right) \to 1 - e^{-t} \quad \text{as } x \to a_F.$$

Proof. We have $P(N(x, \infty) = n) = F^n(x)(1 - F(x))$, $n = 0, 1, \ldots$, and so

$$\begin{aligned} P(S(x) \le y) &= \sum_{n=0}^{\infty} P(N(x, \infty) = n) P(S_n(x) \le y) \\ &= \sum_{n=0}^{\infty} F^n(x)(1 - F(x)) F^{*n}(x, y) \end{aligned}$$

where $F^{*n}(x, y)$ is the n-fold convolution (with respect to y) of the function

$$F(x, y) = \begin{cases} F(y) & \text{if } y \le x, \\ F(x) & \text{if } y > x. \end{cases}$$

Hence

$$ES(x) = \frac{F(x)}{1 - F(x)} \mu_F(x) \tag{19}$$

where

(20) $$\mu_F(x) = \int_0^x t\,dF(t) = \mu_F + o(1), \quad x \to a_F.$$

Let $f^*(x,s)$ denote the Laplace–Stieltjes transform of $F(x,y)$:

$$f^*(x,s) = \int_0^x e^{-st}\,dF(t) = 1 - s\int_0^x t\,dF(t) + o(s) \quad \text{as } s \to 0.$$

Hence the Laplace–Stieltjes transform of the distribution function of the random variable $S(x)$ has the form

$$s^*(x,s) = \int_0^\infty e^{-sy}\,dP(S(x) \le y) = \sum_{n=0}^\infty (1 - F(x))(F(x)f^*(x,s))^n$$
$$= \frac{1 - F(x)}{1 - F(x)f^*(x,s)} = \left(1 + \frac{sF(x)}{1 - F(x)}\int_0^x t\,dF(t) + o(1)\right)^{-1}$$
$$\text{as } s \to 0.$$

For the random variable $S(x)/ES(x)$ we have, for fixed s,

$$s^*\left(x, \frac{s}{ES(x)}\right) = \left(1 + s + o\left(\frac{s}{ES(x)}\right)\frac{1}{1 - F(x)}\right)^{-1} \to \frac{1}{1+s} \quad \text{as } x \to a_F.$$

But $T(x) = S(x) + x$, $ET(x)/ES(x) = 1 + o(1)$ as $x \to a_F$ and $\mu_F < \infty$ implies $x(1 - F(x)) \to 0$ as $x \to a_F$. Hence $P(T(x)/ES(x) > t) = P(S(x)/ES(x) > t) = e^{-t} + o(1)$ as $x \to a_F$.

Proof of Theorem 4. From (16) it follows that $t/ES(u(t)) = \tau(t) + o(1)$ as $t \to \infty$. Hence, from (3) and the Lemma it follows that $P(M(t) > u(t)) = P(T(u(t)) \le t) = \exp(-\tau(t)) + o(1)$ as $t \to \infty$.

COROLLARY 5. *In the Poisson process with parameter λ the extreme gap has asymptotically an exponential distribution function*:

$$P\left(t, \frac{1}{\lambda}(\log \lambda t + x)\right) \to \exp(-e^{-x}) \quad \textit{as } t \to \infty.$$

Indeed, for $u(t) = (1/\lambda)(\log \lambda t + x)$, $\tau(t) = e^{-x}$, (16) holds.

4. The extreme gap in an alternating process. Consider the alternating process generated by a sequence of pairs of random variables $(X_1, Y_1), (X_2, Y_2), \ldots$ independent with joint distribution function H and with boundary distributions F and G for working and breakdown times, respectively. Define the process of actual working time by

$$\gamma(t) = \begin{cases} 0 & \text{if } S_n + S'_n < t \le S_n + S'_{n+1}, \\ t - S_n - S'_{n+1} & \text{if } S_n + S'_{n+1} < t \le S_{n+1} + S'_{n+1}, \end{cases}$$

where $S_0 = 0$, $S_n = S_{n-1} + X_n$, $S'_0 = 0$, $S'_n = S'_{n-1} + Y_n$, $n = 1, 2, \ldots$

As previously, we define the process of the extreme gap by (1).

THEOREM 5. *Assume that* $\mu_F + \mu_G < \infty$, *and let* $u(t)$ *and* $\tau(t)$ *be such that*

$$(21) \qquad (\mu_F + \mu_G)^{-1} t(1 - F(u(t))) - \tau(t) \to 0 \quad \text{as } t \to \infty,$$

where $\tau(t)$ *is bounded. Then*

$$(22) \qquad P(t, u(t)) - \exp(-\tau(t)) \to 0 \quad \text{as } t \to \infty.$$

Proof. Set $S(x) = S_{N(x,\infty)}(x) + S'_{N(x,\infty)}(x)$, where $N(x,t)$ is defined by (18). Then

$$ES(x) = (\mu_F(x) + \mu_G(x))F(x)/(1 - F(x))$$

where $\mu_F(x)$ is defined by (20) and

$$\mu_G(x) = \int_0^x \int_0^\infty y H(dx, dy) = \mu_G + o(1) \quad \text{as } x \to a_F.$$

As previously, we can prove that $P(S(x)/ES(x) \le t) \to 1 - e^{-t}$ as $x \to a_F$. Hence using (3) we get

$$P(t(\mu_F + \mu_G)/(1 - F(x)), x) \to e^{-t} \quad \text{as } x \to a_F.$$

Thus, (21) implies (22).

5. Estimation of the expected value. In the case of a Poisson process we have proved an elegant formula for the expected value of the extreme gap. For a general renewal process we give an estimation using the method of Lai and Robbins [6].

THEOREM 6. *If* F *is continuous, then*

$$EM(t) \le a(t) + (H(t) + 1) \int_{a(t)}^\infty (1 - F(u))\, du,$$

where $a(t) = F^{-1}\left(1 - \dfrac{1}{H(t) + 1}\right)$.

Proof. For each a we have

$$M(t) = \max(X_1, \ldots, X_{N(t)}, \gamma(t)) \le \max(X_1, \ldots, X_{N(t)+1})$$

$$\le a + \max((X_1 - a)_+, \ldots, (X_{N(t)+1} - a)_+) \le a + \sum_{i=1}^{N(t)+1} (X_i - a)_+.$$

From Wald's equality (see e.g. Ross [8], p. 59), after taking the most favourable a we get

$$EM(t) \le a + E(N(t)+1)E(X_1 - a)_+ = a + (H(t)+1)\int_a^\infty (1-F(u))\,du,$$

and therefore

$$EM(t) \le \min_a \left(a + (H(t)+1)\int_a^\infty (1-F(u))\,du\right)$$

$$= a(t) + (H(t)+1)\int_{a(t)}^\infty (1-F(u))\,du.$$

COROLLARY 6. *In the case of a Poisson process with parameter* λ *we have* $H(t) = \lambda t$, $a(t) = (1/\lambda)\log(\lambda t + 1)$, *hence*

$$EM(t) \le \frac{1}{\lambda}(\log(\lambda t+1)+1) \quad \textit{for } t \ge 0.$$

6. Maximal success-run in Bernoulli trials. Let $X_1, X_2, \ldots$ be independent random variables with the geometric distribution function $P(X_1 = k) = q^{k-1}p$, $k = 1, 2, \ldots$ This is a well known example in which the limiting distribution of the extreme value does not exist but an asymptotic estimation does. Referring to Theorem 2 we set, for fixed x,

$$u_n = [\log_q(x/n)], \quad \tau_n = xq \uparrow (-\{\log_q(x/n)\}),$$

where $[a]$ is the integer part of a and $\{a\} = a - [a]$. Then the condition (13) is fulfilled and from (14) we get

$$P(M_n \le u_n) = \exp(-xq \uparrow (-\{\log_q(x/n)\})) + o(1) \quad \text{as } n \to \infty. \tag{23}$$

For $p = 1/2$ and fixed k the last relation yields

$$P(M_n - [\log_2 n] \le k) = \exp(-2^{-k+\{\log_2 n\}}) + o(1) \quad \text{as } n \to \infty.$$

Referring to Theorem 4 we set for fixed k

$$u(t) = k + [\log_q pt], \quad \tau(t) = q \uparrow (\{\log_q pt\} + k).$$

Then the condition (16) is fulfilled and from (17) we get

$$P(M(t) + 1 \le k - [\log_q pt]) = \exp(-q \uparrow (\{\log_q pt\} + k)) + o(1) \quad \text{as } t \to \infty. \tag{24}$$

For $p = 1/2$ and fixed k the last relation yields (see Főldes [2])

$$P(M(n) \le k + [\log_2 n]) = \exp(-2^{-(k+2-\{\log_2 n\})}) + o(1) \quad \text{as } n \to \infty.$$

Referring to Theorem 5 let $P(X_1 = k, Y_1 = l) = p^k q^l$, $k, l = 1, 2, \ldots$ Then for fixed k we set

$$u(t) = k + [\log_q pt], \qquad \tau(t) = q \uparrow (k + 1 - \{\log_q pt\}).$$

Then the condition (21) is fulfilled and (22) is equivalent to (24).

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