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A REMARK ON SELF-CENTROIDAL GRAPHS

Abstract. The concept of centroid of a graph, related to the concept of convexity, is studied. A class of graphs G in which the centroid is equal to the whole vertex set is shown.

The notions of monophonical convexity and of geodesical convexity were studied e.g. in [1]. By means of these concepts the centroid of a graph is defined. At the end of [2] W. Piotrowski poses three problems, including the problem which graphs are self-centroidal and the problem of describing centroids in particular classes of graphs. In [3] these problems were investigated for graphs without a separating set of vertices inducing a clique and in particular for chordal, Halin, series-parallel and outerplanar graphs. Here we shall show a considerably wide class of self-centroidal graphs. We consider finite undirected graphs without loops and multiple edges.

A path P in a graph G is called *chordless* if no two of its vertices are joined by an edge not belonging to P. A subset M of the vertex set $V(G)$ of G is called *monophonically* (resp. *geodesically*) *convex* if for any two vertices u, v of M the set M contains all vertices lying on chordless (resp. shortest) paths connecting u and v in M. Instead of monophonically convex we write shortly m-convex, instead of geodesically convex we write g-convex.

Obviously any shortest path connecting two vertices is chordless. Therefore each m-convex set in G is g-convex in G; the converse assertion is not true.

For any type of convexity we may define the weight of a vertex in G. The *m-weight* (resp. *g-weight*) of a vertex v in G is the maximum number of vertices of an m-convex (resp. g-convex) set in G which does not contain v. Then the *m-centroid* (resp. *g-centroid*) of G is the set of vertices of G whose m-weight (resp. g-weight) in G is minimum.

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The graph G is called *m-self-centroidal* (resp. *g-self-centroidal*) if the m-centroid (resp. g-centroid) of G is $V(G)$.

Obviously all vertex-transitive graphs are m-self-centroidal and g-self-centroidal.

A *clique* in a graph G is a subgraph of G which is a complete graph and is not a proper subgraph of another complete subgraph of G. The maximum number of vertices of a clique in G is called the *clique number* of G and is denoted by $\omega(G)$. A clique in G having $\omega(G)$ vertices is called *maximal*.

The *Zykov sum* $G_1 \oplus G_2$ of two graphs G_1, G_2 is the graph obtained from vertex-disjoint graphs G_1, G_2 by joining each vertex of G_1 with each vertex of G_2 by an edge.

Note that a graph is the Zykov sum of some graphs if and only if it is the complement of a disconnected graph.

THEOREM 1. *Let G be the Zykov sum of two non-empty non-complete graphs G_1 and G_2. If the intersection $J(G)$ of all maximal cliques of G is non-empty, then $J(G)$ is the m-centroid and the g-centroid of G. Otherwise G is m-self-centroidal and g-self-centroidal.*

Proof. Obviously each subset of $V(G)$ inducing a complete subgraph is m-convex and g-convex in G. Now let M be an m-convex set in G and let M contain two non-adjacent vertices u, v. As G is the Zykov sum $G_1 \oplus G_2$, u and v are either both in G_1, or both in G_2. Without loss of generality suppose that u and v are in G_1. Then the distance between u and v in G is 2 and each vertex of G_2 is an inner vertex of a path of length 2 connecting u and v in G; such a path is obviously chordless. Hence $V(G_2) \subset M$. As G_1, G_2 are not complete, there exist two non-adjacent vertices u' and v' in G_2 and $u' \in M$, $v' \in M$. The distance between u' and v' in G is 2 and each vertex of G_1 is an inner vertex of a path of length 2 connecting u' and v' in G. Hence $V(G_1) \subset M$ and $M = V(G)$. If we suppose M only to be g-convex, we obtain the same result. Therefore m-convex sets and g-convex ones coincide in G and we may just speak about convex sets. We see that a non-empty subset M of $V(G)$ is convex in G if and only if either $M = V(G)$, or M induces a complete subgraph of G. The weight of any vertex v is the maximum number of vertices of a subset of $V(G) - \{v\}$ which induces a complete subgraph of G.

If the intersection $J(G)$ of all maximal cliques of G is empty, then for each vertex v of G there exists a maximal clique of G which does not contain v. Hence the weight of each vertex v is $\omega(G)$ and G is m-self-centroidal and g-self-centroidal. If $J(G) \neq \emptyset$, then the weight of each $v \in J(G)$ is $\omega(G) - 1$, while for each $v \in V(G) - J(G)$ it is $\omega(G)$ (obviously $V(G) - J(G) \neq \emptyset$); this implies that $J(G)$ is the m-centroid and the g-centroid of G. ∎

Corollary 1. *Let G be the Zykov sum of two non-empty non-complete graphs G_1, G_2. The graph G is m-self-centroidal and g-self-centroidal if and only if in both G_1, G_2 the intersection of all maximal cliques is empty.*

Proof. A subgraph of G is a maximal clique in G if and only if it is the Zykov sum of a maximal clique in G_1 and a maximal clique in G_2. Also the intersection $J(G)$ of all maximal cliques of G is the Zykov sum of the intersections $J(G_1)$, $J(G_2)$ of maximal cliques of G_1 and G_2 respectively. It is empty if and only if both $J(G_1)$ and $J(G_2)$ are empty. This implies the assertion. ∎

In Theorem 1 we have supposed that both the graphs G_1, G_2 are non-complete. Now the case remains when at least one of them is complete. Evidently a graph G is the Zykov sum of two graphs G_1, G_2, at least one of which is complete, if and only if it contains saturated vertices; a *saturated vertex* is a vertex adjacent to all other vertices. Let $S(G)$ denote the set of all saturated vertices of G.

Theorem 2. *Let G be a graph, and let its set $S(G)$ of saturated vertices be non-empty. Then $S(G)$ is a g-centroid of G.*

Proof. Let $u \in S(G)$. Let x, y be two non-adjacent vertices of G. Then the distance between x and y in G is 2 and u is the inner vertex of a path of length 2 connecting x and y in G. Therefore each g-convex set in G containing two non-adjacent vertices contains u. The g-convex sets in G not containing u are exactly those which induce complete subgraphs of G and do not contain u. A saturated vertex of G is contained in each clique of G, therefore the weight of u is $\omega(G) - 1$.

Now let $v \in V(G) - S(G)$. If v is not contained in a maximal clique of G, then evidently its g-weight is at least $\omega(G)$. If v belongs to a maximal clique of G, let w be a vertex of G non-adjacent to v. Consider the set $M = (V(C) - \{v\}) \cup \{w\}$. We have $u \in M$ and w is adjacent to u. Therefore the distance between w and any other vertex of M is at most 2, and the distance between any two vertices of $M - \{w\}$ is 1. As v is not adjacent to w, it belongs to no shortest path from w to a vertex of $M - \{w\}$. Hence the least g-convex set containing M (the g-convex hull of M) does not contain v. Its number of vertices is at least $\omega(G)$, therefore the weight of v is at least $\omega(G)$ and v does not belong to the g-centroid of G. This implies the assertion. ∎

An analogous assertion for the m-centroid does not hold, as the following example shows.

Example. Let $k \geq 5$, let $V(G) = \{u_1, \ldots, u_k, v, w\}$. Let the edges of G be $u_i u_j$ for any two distinct numbers i, j from the set $\{1, \ldots, k\}$ and further $u_1 v$, $u_1 w$, vw, $u_2 w$. The vertex u_1 is saturated in G and its weight is $k - 1$.

Now let M be an m-convex set with at least k vertices; it must contain at least one of the vertices u_2, v, w and at least one of the vertices u_i for $i \geq 3$. There exists a chordless path of length 3 between u_i and v over u_2 and w and a chordless path of length 2 between u_i and w over u_2. Hence in any case M contains u_2. This implies that the m-weight of u_2 is at most $k-1$ and it does not exceed the m-weight of u_1, while u_1 is saturated and u_2 is not.

It can be easily proved that $S(G)$ is a subset of the m-centroid of G.

A *complete n-partite graph* is a graph G with the property that there exists a partition (called *n-partition*) of its vertex set $V(G)$ into n classes $V_1, \ldots, V_n$ with the property that two vertices of G are adjacent if and only if they belong to different classes of this partition. In other words, it is the complement of a graph consisting of n connected components which are complete graphs.

The following corollary immediately follows from Theorem 1 and Theorem 2.

COROLLARY 2. *Let G be a complete n-partite graph. The graph G is m-self-centroidal and g-self-centroidal if and only if either G is a complete graph, or each class of the n-partition of G has at least two vertices.*

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