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A REMARK ON SELF-CENTROIDAL GRAPHS

Abstract. The concept of centroid of a graph, related to the concept of convexity, is studied. A class of graphs $G$ in which the centroid is equal to the whole vertex set is shown.

The notions of monophonical convexity and of geodesical convexity were studied e.g. in [1]. By means of these concepts the centroid of a graph is defined. At the end of [2] W. Piotrowski poses three problems, including the problem which graphs are self-centroidal and the problem of describing centroids in particular classes of graphs. In [3] these problems were investigated for graphs without a separating set of vertices inducing a clique and in particular for chordal, Halin, series-parallel and outerplanar graphs. Here we shall show a considerably wide class of self-centroidal graphs. We consider finite undirected graphs without loops and multiple edges.

A path $P$ in a graph $G$ is called chordless if no two of its vertices are joined by an edge not belonging to $P$. A subset $M$ of the vertex set $V(G)$ of $G$ is called monophonically (resp. geodesically) convex if for any two vertices $u, v$ of $M$ the set $M$ contains all vertices lying on chordless (resp. shortest) paths connecting $u$ and $v$ in $M$. Instead of monophonically convex we write shortly m-convex, instead of geodesically convex we write g-convex.

Obviously any shortest path connecting two vertices is chordless. Therefore each m-convex set in $G$ is g-convex in $G$; the converse assertion is not true.

For any type of convexity we may define the weight of a vertex in $G$. The $m$-weight (resp. $g$-weight) of a vertex $v$ in $G$ is the maximum number of vertices of an m-convex (resp. g-convex) set in $G$ which does not contain $v$. Then the $m$-centroid (resp. $g$-centroid) of $G$ is the set of vertices of $G$ whose $m$-weight (resp. $g$-weight) in $G$ is minimum.

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The graph $G$ is called \textit{m-self-centroidal} (resp. \textit{g-self-centroidal}) if the \textit{m-centroid} (resp. \textit{g-centroid}) of $G$ is $V(G)$.

Obviously all vertex-transitive graphs are \textit{m-self-centroidal} and \textit{g-self-centroidal}.

A \textit{clique} in a graph $G$ is a subgraph of $G$ which is a complete graph and is not a proper subgraph of another complete subgraph of $G$. The maximum number of vertices of a clique in $G$ is called the \textit{clique number} of $G$ and is denoted by $\omega(G)$. A clique in $G$ having $\omega(G)$ vertices is called \textit{maximal}.

The \textit{Zykov sum} $G_1 \oplus G_2$ of two graphs $G_1$, $G_2$ is the graph obtained from vertex-disjoint graphs $G_1$, $G_2$ by joining each vertex of $G_1$ with each vertex of $G_2$ by an edge.

Note that a graph is the Zykov sum of some graphs if and only if it is the complement of a disconnected graph.

\textbf{Theorem 1.} Let $G$ be the Zykov sum of two non-empty non-complete graphs $G_1$ and $G_2$. If the intersection $J(G)$ of all maximal cliques of $G$ is non-empty, then $J(G)$ is the \textit{m-centroid} and the \textit{g-centroid} of $G$. Otherwise $G$ is \textit{m-self-centroidal} and \textit{g-self-centroidal}.

\textbf{Proof.} Obviously each subset of $V(G)$ inducing a complete subgraph is \textit{m-convex} and \textit{g-convex} in $G$. Now let $M$ be an \textit{m-convex} set in $G$ and let $M$ contain two non-adjacent vertices $u$, $v$. As $G$ is the Zykov sum $G_1 \oplus G_2$, $u$ and $v$ are either both in $G_1$, or both in $G_2$. Without loss of generality suppose that $u$ and $v$ are in $G_1$. Then the distance between $u$ and $v$ in $G$ is 2 and each vertex of $G_2$ is an inner vertex of a path of length 2 connecting $u$ and $v$ in $G$; such a path is obviously chordless. Hence $V(G_2) \subset M$. As $G_1$, $G_2$ are not complete, there exist two non-adjacent vertices $u'$ and $v'$ in $G_2$ and $u' \in M$, $v' \in M$. The distance between $u'$ and $v'$ in $G$ is 2 and each vertex of $G_1$ is an inner vertex of a path of length 2 connecting $u'$ and $v'$ in $G$. Hence $V(G_1) \subset M$ and $M = V(G)$. If we suppose $M$ only to be \textit{g-convex}, we obtain the same result. Therefore \textit{m-convex} sets and \textit{g-convex} ones coincide in $G$ and we may just speak about convex sets. We see that a non-empty subset $M$ of $V(G)$ is convex in $G$ if and only if either $M = V(G)$, or $M$ induces a complete subgraph of $G$. The weight of any vertex $v$ is the maximum number of vertices of a subset of $V(G) - \{v\}$ which induces a complete subgraph of $G$.

If the intersection $J(G)$ of all maximal cliques of $G$ is empty, then for each vertex $v$ of $G$ there exists a maximal clique of $G$ which does not contain $v$. Hence the weight of each vertex $v$ is $\omega(G)$ and $G$ is \textit{m-self-centroidal} and \textit{g-self-centroidal}. If $J(G) \neq \emptyset$, then the weight of each $v \in J(G)$ is $\omega(G) - 1$, while for each $v \in V(G) - J(G)$ it is $\omega(G)$ (obviously $V(G) - J(G) \neq \emptyset)$; this implies that $J(G)$ is the \textit{m-centroid} and the \textit{g-centroid} of $G$. \rule{1.5em}{1.5em}
COROLLARY 1. Let $G$ be the Zykov sum of two non-empty non-complete graphs $G_1$, $G_2$. The graph $G$ is m-self-centroidal and g-self-centroidal if and only if in both $G_1$, $G_2$ the intersection of all maximal cliques is empty.

Proof. A subgraph of $G$ is a maximal clique in $G$ if and only if it is the Zykov sum of a maximal clique in $G_1$ and a maximal clique in $G_2$. Also the intersection $J(G)$ of all maximal cliques of $G$ is the Zykov sum of the intersections $J(G_1)$, $J(G_2)$ of maximal cliques of $G_1$ and $G_2$ respectively. It is empty if and only if both $J(G_1)$ and $J(G_2)$ are empty. This implies the assertion. ■

In Theorem 1 we have supposed that both the graphs $G_1$, $G_2$ are non-complete. Now the case remains when at least one of them is complete. Evidently a graph $G$ is the Zykov sum of two graphs $G_1$, $G_2$, at least one of which is complete, if and only if it contains saturated vertices; a saturated vertex is a vertex adjacent to all other vertices. Let $S(G)$ denote the set of all saturated vertices of $G$.

THEOREM 2. Let $G$ be a graph, and let its set $S(G)$ of saturated vertices be non-empty. Then $S(G)$ is a g-centroid of $G$.

Proof. Let $u \in S(G)$. Let $x$, $y$ be two non-adjacent vertices of $G$. Then the distance between $x$ and $y$ in $G$ is 2 and $u$ is the inner vertex of a path of length 2 connecting $x$ and $y$ in $G$. Therefore each g-convex set in $G$ containing two non-adjacent vertices contains $u$. The g-convex sets in $G$ not containing $u$ are exactly those which induce complete subgraphs of $G$ and do not contain $u$. A saturated vertex of $G$ is contained in each clique of $G$, therefore the weight of $u$ is $\omega(G) - 1$.

Now let $v \in V(G) - S(G)$. If $v$ is not contained in a maximal clique of $G$, then evidently its g-weight is at least $\omega(G)$. If $v$ belongs to a maximal clique of $G$, let $w$ be a vertex of $G$ non-adjacent to $v$. Consider the set $M = (V(G) - \{v\}) \cup \{w\}$. We have $u \in M$ and $w$ is adjacent to $u$. Therefore the distance between $w$ and any other vertex of $M$ is at most 2, and the distance between any two vertices of $M - \{w\}$ is 1. As $v$ is not adjacent to $w$, it belongs to no shortest path from $w$ to a vertex of $M - \{w\}$. Hence the least g-convex set containing $M$ (the g-convex hull of $M$) does not contain $v$. Its number of vertices is at least $\omega(G)$, therefore the weight of $v$ is at least $\omega(G)$ and $v$ does not belong to the g-centroid of $G$. This implies the assertion. ■

An analogous assertion for the m-centroid does not hold, as the following example shows.

EXAMPLE. Let $k \geq 5$, let $V(G) = \{u_1, \ldots, u_k, v, w\}$. Let the edges of $G$ be $u_iu_j$ for any two distinct numbers $i$, $j$ from the set $\{1, \ldots, k\}$ and further $u_1v$, $u_1w$, $vw$, $u_2w$. The vertex $u_1$ is saturated in $G$ and its weight is $k - 1$. 
Now let $M$ be an $m$-convex set with at least $k$ vertices; it must contain at least one of the vertices $u_2, v, w$ and at least one of the vertices $u_i$ for $i \geq 3$. There exists a chordless path of length 3 between $u_i$ and $v$ over $u_2$ and $w$ and a chordless path of length 2 between $u_i$ and $w$ over $u_2$. Hence in any case $M$ contains $u_2$. This implies that the $m$-weight of $u_2$ is at most $k - 1$ and it does not exceed the $m$-weight of $u_1$, while $u_1$ is saturated and $u_2$ is not.

It can be easily proved that $S(G)$ is a subset of the $m$-centroid of $G$.

A complete $n$-partite graph is a graph $G$ with the property that there exists a partition (called $n$-partition) of its vertex set $V(G)$ into $n$ classes $V_1, \ldots, V_n$ with the property that two vertices of $G$ are adjacent if and only if they belong to different classes of this partition. In other words, it is the complement of a graph consisting of $n$ connected components which are complete graphs.

The following corollary immediately follows from Theorem 1 and Theorem 2.

**Corollary 2.** Let $G$ be a complete $n$-partite graph. The graph $G$ is $m$-self-centroidal and $g$-self-centroidal if and only if either $G$ is a complete graph, or each class of the $n$-partition of $G$ has at least two vertices.

**References**


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