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A SILENT DUEL UNDER ARBITRARY MOVING

In the paper a silent duel is considered in which the players have one bullet each, the accuracy functions are arbitrary and the players can move as they like.

1. Introduction. Consider a game which will be called the *game* (1, 1). Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$. The players have one bullet each and this fact is known to both of them. It is also known that the duel is *silent*: at a given moment neither player knows whether or not his opponent has fired.

At the beginning of the duel the players are at distance 1 from each other. Let $P_1(s)$ ($P_2(s)$) be the probability of succeeding (destroying the opponent) by Player I (II) when the distance between the players is $1-s$. The functions $P_1(s)$, $P_2(s)$ will be called the *accuracy functions*. It is assumed that they are increasing and continuous in $[0,1]$, have continuous second derivatives in $(0,1)$ and that $P_i(s) = 0$ for $s \leq 0$, $P_i(1) = 1$, $i = 1,2$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero-sum game.

As will be seen from the sequel, without loss of generality we can suppose that $v_1 = 1$ and that Player II is motionless. It is also assumed that at the beginning of the duel Player I is at the point 0 and Player II is at the point 1.

For definitions and results in the theory of games of timing see [3]–[5], [7], [9], [10], [14], [16].

2. Auxiliary duel. To solve the game (1, 1) presented in the pre-

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vious section, it will be necessary to determine equalizer strategies in the following auxiliary game $(1,1)^*$. Consider a one-bullet silent duel with accuracy functions $P_1(s)$, $P_2(s)$ in which Player I approaches Player II with constant velocity $v = 1$ all the time, even after firing his bullet. Player I gains 1 if only he succeeds etc., similarly to the duel defined in the previous section.

Denote by $K_0(s;t)$ the expected gain of Player I if he fires at time $s \in [0,1]$ and if Player II fires at time $t \in [0,1]$. It is assumed that

$$K_0(s;t) = \begin{cases} P_1(s) & \text{if } s < t, \\ P_1(s) - P_2(s) & \text{if } s = t, \\ -P_2(t) + (1 - P_2(t))P_1(s) & \text{if } s > t. \end{cases}$$

As is easy to see $K_0(s;t)$ is the expected payoff in the duel in which Player II is not allowed to fire after the shot of Player I.

Denote by ξ_0^a the strategy of Player I in the game $(1,1)^*$ in which he fires at a random moment s distributed according to a density $pf_1(s)$ in the interval $[a,1]$, $0 < a < 1$, and according to probability $1-p$, $0 < p < 1$, at the point 1. This distribution is chosen in such a way that if $t \in [a,1)$ then

$$(1) \qquad K_0(\xi_0^a;t) = p\Big[\int_a^t P_1(s)f_1(s)\,ds + \int_t^1 (-P_2(t) + (1 - P_2(t))P_1(s))f_1(s)\,ds\Big] + (1-p)(1-2P_2(t)) = \text{const.}$$

In the above formula $K_0(\xi_0^a;t)$ is the expected gain of Player I if he applies the strategy ξ_0^a and Player II fires at time t.

We obtain

$$(2) \qquad \frac{\partial K_0(\xi_0^a;t)}{\partial t} = p\Big[(1 + P_1(t))P_2(t)f_1(t) - P_2'(t)\int_t^1 (1 + P_1(s))f_1(s)\,ds\Big] - 2(1-p)P_2'(t) = 0\,,$$

$$(3) \qquad \frac{\partial^2 K_0(\xi_0^a;t)}{\partial t^2} = p\Big[(P_2'(t) + P_1'(t)P_2(t) + P_1(t)P_2'(t))f_1(t) + (1 + P_1(t))P_2(t)f_1'(t) - P_2''(t)\int_t^1 (1 + P_1(s))f_1(s)\,ds + (1 + P_1(t))P_2'(t)f_1(t)\Big] - 2(1-p)P_2''(t) = 0\,.$$

Eliminating the integral from (2) and (3) we obtain

$$(1+P_1(t))P_2(t)f_1'(t)+[2(1+P_1(t))P_2'(t)+P_2(t)P_1'(t)]f_1(t) - \frac{P_2''(t)}{P_2'(t)}(1+P_1(t))P_2(t)=0\,,$$

which gives

$$f_1(t)=C\frac{P_2'(t)}{P_2^2(t)(1+P_1(t))} \tag{4}$$

where the constant C satisfies

$$C\int_a^1 \frac{P_2'(t)\,dt}{P_2^2(t)(1+P_1(t))}=1\,. \tag{5}$$

Moreover, from (1) and (4) we obtain

$$\begin{aligned}(6)\quad & K_0(\xi_0^a;t)\\ &=p\left[C\int_a^1 \frac{P_1(s)P_2'(s)\,ds}{P_2^2(s)(1+P_1(s))}+C(P_2(t)-1)\right]+(1-p)(1-2P_2(t))\\ &=pC\left[\int_a^1 \frac{P_1(s)P_2(s)\,ds}{P_2^2(s)(1+P_1(s))}-1\right]+1-p=\text{const}\end{aligned}$$

if

$$pC=2(1-p)\,. \tag{7}$$

Let η_0^a be the strategy of Player II in the game $(1,1)^*$ in which he chooses at random a moment t for his shot according to the density $f_2(t)$ in $[a,1]$ to obtain

$$K_0(s;\eta_0^a)=\int_a^s(-P_2(t)+(1-P_2(t))P_1(s))f_2(t)\,dt + \int_s^1 P_1(s)f_2(t)\,dt=\text{const}$$

if $s\in[a,1]$, where $K_0(s;\eta_0^a)$ is the expected gain of Player I if Player II applies the strategy η_0^a and Player I fires at time s.

In the same way as before we obtain

$$f_2(t)=D\frac{P_1'(t)}{P_2(t)(1+P_1(t))^2}\,, \tag{8}$$

$$D\int_a^1 \frac{P_1'(s)\,ds}{P_2(s)(1+P_1(s))^2}=1\,, \tag{9}$$

$$D = 1 + P_1(a)\,, \tag{10}$$

$$K_0(s;\eta_0^a) = D\frac{P_1(a)}{1+P_1(a)} = P_1(a)\,. \tag{11}$$

Assuming that $K_0(\xi_0^a;t) = K_0(s;\eta_0^a) = \text{const}$ for $s,t \in [a,1)$ from (6) and (11) we obtain the additional equation

$$pC\left[\int_a^1 \frac{P_1(s)P_2'(s)\,ds}{P_2^2(s)(1+P_1(s))} - 1\right] + 1 - p = P_1(a)\,. \tag{12}$$

From (5), (7), (9), (10), (12) we determine the unknown parameters C, D, a, p. Notice that we have five equations but only four unknown quantities.

Eliminating from these equations the parameters C and D, we obtain the system of equations

$$2(1-p)\int_a^1 \frac{P_2'(t)\,dt}{P_2^2(t)(1+P_1(t))} = p\,, \tag{13}$$

$$(1+P_1(a))\int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2} = 1\,, \tag{14}$$

$$(1-p)\left[2\int_a^1 \frac{P_1(t)P_2'(t)\,dt}{P_2^2(t)(1+P_1(t))} - 1\right] = P_1(a)\,, \tag{15}$$

with unknown quantities p and a.

From (13) and (15) we obtain

$$2(1-p)\int_a^1 \frac{P_2'(t)\,dt}{P_2^2(t)} = 1 + P_1(a)$$

or, on computing the integral,

$$2(1-p) = \frac{P_2(a)(1+P_1(a))}{1-P_2(a)}\,. \tag{16}$$

On the other hand, integration by parts leads to

$$\int_a^1 \frac{P_2'(t)\,dt}{P_2^2(t)(1+P_1(t))} = -\frac{1}{2} + \frac{1}{P_2(a)(1+P_1(a))} - \int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2}\,, \tag{17}$$

$$\int_a^1 \frac{P_1(t)P_2'(t)\,dt}{P_2^2(t)(1+P_1(t))}$$
$$= -\frac{1}{2} + \frac{P_1(a)}{P_2(a)(1+P_1(a))} + \int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2}.$$

Then from (13) and (15) we obtain

$$(18)\qquad 2(1-p)\left[-\frac{1}{2} + \frac{1}{P_2(a)(1+P_1(a))} - \int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2}\right] = p,$$

$$(19)\qquad 2(1-p)\left[-1 + \frac{P_1(a)}{P_2(a)(1+P_1(a))} + \int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2}\right] = P_1(a).$$

Assume that equations (14) and (16) have a solution. Substituting the values p and $\int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2}$ obtained from (14) and (16) into (18) and (19) we obtain identities. Thus the considered system of five equations has a solution C, D, p, a, $C > 0$, $D > 0$, $0 < p < 1$, $0 < a < 1$, provided equations (14) and (16) have a solution p, a, $0 < p < 1$, $0 < a < 1$.

Consider the function

$$\varphi(a) = (1+P_1(a)) \int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2}.$$

We obtain

$$\varphi'(a) = \left[\int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1+P_1(t))^2} - \frac{1}{P_2(a)(1+P_1(a))}\right] P_1'(a)$$
$$\overset{(17)}{=} -\left[\int_a^1 \frac{P_1'(t)\,dt}{P_2^2(t)(1+P_1(t))} + \frac{1}{2}\right] P_1'(a) < 0.$$

It follows that there exists at most one solution a, $0 < a < 1$, of the equation $\varphi(a) = 1$.

We prove that if there exists a solution a, $0 < a < 1$, of (14) then there exists a solution p, $0 < p < 1$, of (16). Since the integral on the left side of (17) is positive, a being a solution of (14) implies

$$-\frac{1}{2} + \frac{1}{P_2(a)(1+P_1(a))} - \frac{1}{1+P_1(a)} > 0$$

or

$$\frac{P_2(a)(1+P_1(a))}{1-P_2(a)} < 2.$$

Thus a solution C, D, p, a of the five equations exists, $C > 0$, $D > 0$, $0 < p < 1$, $0 < a < 1$, provided there exists a solution a, $0 < a < 1$, of (14).

To see an example, let $P_1(t) = t$, $P_2(t) = t^\alpha$, $\alpha > 0$. We obtain

$$(1 + P_1(a)) \int_a^1 \frac{P_1'(t)\,dt}{P_2(t)(1 + P_1(t))^2} = (1 + a) \int_{1/2}^{1/(1+a)} \left(\frac{x}{1-x}\right)^\alpha dx$$

$$\leq (1 + a) \int_{1/2}^{1/(1+a)} \frac{dx}{(1-x)^\alpha} \underset{\alpha\to 0}{\to} \frac{1}{2} - \frac{a}{2} < 1.$$

Thus (14) has no solution for these $P_1(t)$, $P_2(t)$ when α is small.

LEMMA. *If there exists a solution a, $0 < a < 1$, of (14), then for this a the strategy ξ_0^a is maximin and the strategy ξ_0^a is minimax in the game $(1,1)^*$. The value of the game is $v_{11}^0 = P_1(a)$.*

Proof. We have proved that $K_0(\xi_0^a; t) = P_1(a)$ for $a \leq t < 1$. Moreover,

$$K_0(\xi_0^a; 1) = p \int_a^1 P_1(s) f_1(s)\,ds$$

$$> p \int_a^1 P_1(s) f_1(s)\,ds + (1-p)(1 - 2P_2(1))$$

$$= \lim_{t\to 1-} K_0(\xi_0^a; t) = P_1(a)$$

since $K_0(\xi_0^a; t) = \text{const} = P_1(a)$ for $a \leq t < 1$.

Finally, if $t < a$ we have

$$K_0(\xi_0^a; t)$$

$$= p \int_a^1 (-P_2(t) + (1 - P_2(t))P_1(s)) f_1(s)\,ds + (1-p)(1 - 2P_2(t))$$

$$> p \int_a^1 (-P_2(a) + (1 - P_2(a))P_1(s)) f_1(s)\,ds + (1-p)(1 - 2P_2(a))$$

$$= K_0(\xi_0^a; a) = P_1(a).$$

Thus $K_0(\xi_0^a; \eta) \geq P_1(a)$ for any strategy η of Player II.

On the other hand, $K_0(s; \eta_0^a) = P_1(a)$ for $a \leq s < 1$, and if $s < a$ then $K_0(s; \eta_0^a) = P_1(s) < P_1(a)$. Therefore $K_0(\xi; \eta_0^a) \leq P_1(a)$ for any strategy ξ of Player I. The lemma is proved.

3. Main result. Let us return to the duel $(1,1)$ defined at the beginning of the paper. Assume that there exists a solution a, $0 < a < 1$, of (14).

For a given natural n, let constants a_k be defined as follows:

$$a_0 = a\,, \quad p \int\limits_{a_{k-1}}^{a_k} f_1(s)\,ds = \frac{1}{n}\,, \quad k = 1,\ldots,n_0\,,\ a_{n_0+1} = 1\,,$$

where n_0 is defined from the inequalities

$$p > p \int\limits_{a}^{a_{n_0}} f_1(s)\,ds \geq p - \frac{1}{n}\,.$$

Define the strategy ξ^ε of Player I in the game $(1,1)$ as follows: If there exists a solution a of the equation (14) (*case* 1) Player I moves back and forth with maximal speed in the following manner: at first between 0 and a_1, then between 0 and $a_2,\ldots$, finally between 0 and a_{n_0+1}. At the kth step, $k = 1,\ldots,n_0+1$, he can fire his shot at random only if he is between the points a_{k-1} and a_k and goes forward, and he fires it with probability density $pf_1(s)$. If he has fired at the kth step, he reaches the point a_k, escapes to 0 and never approaches Player II. If Player I has not fired between the points 0 and 1 and survives, he fires when he is at 1, as soon as possible.

If no solution a, $0 < a < 1$, of (14) exists (*case* 2), Player I, following ξ^ε, does not approach Player II.

The strategy η^0 of Player II is defined in case 1 as follows: If Player I reaches the point t the first time and his velocity is $v_1(\tau)$, τ the time, fire at random with density $v_1(\tau)f_2(t(\tau))$. Otherwise do not fire.

It is assumed that the function $v_1(\tau)$ is piecewise continuous.

In case 2, when equation (14) has no solution a, $0 < a < 1$, the strategy η^0 is defined similarly but the firing has probability density $v_1(\tau)f_2^0(t(\tau))$ where the function $f_2^0(t)$ is defined in (8), for $a = 0$ and D satisfying (9).

THEOREM. *The strategy ξ^ε is ε-maximin and the strategy η^0 is minimax in the game $(1,1)$. The value of the game is $v_{11} = P_1(a)$ if there is a solution a, $0 < a < 1$, of (14), and $v_{11} = 0$ otherwise.*

Proof. Assume that Player I applies the strategy ξ^ε and that (14) has a solution a, $0 < a < 1$. We say that Player II fires at (k, a') if he fires when Player I is at the point a' and if this happens during the first player's approach to a_k or his escape from a_{k-1}.

Denote also by (k, a') the strategy of Player II similarly defined. We obtain

$$K(\xi^\varepsilon; k, a')$$
$$\geq p\Big[\int\limits_{a}^{a_{k-1}} P_1(s)f_1(s)\,ds + \int\limits_{a_k}^{1} (-P_2(a') + (1 - P_2(a'))P_1(s))f_1(s)\,ds\Big]$$

$$+ (1-p)(1-2P_2(a')) - \frac{1}{n}$$
$$\geq p\Big[\int_a^{a_{k-1}} P_1(s)f_1(s)\,ds + \int_{a_k}^1 (-P_2(a_k) + (1-P_2(a_k))P_1(s))f_1(s)\,ds\Big]$$
$$+ (1-p)(1-2P_2(a_k)) - \frac{1}{n}$$
$$\geq p\Big[\int_a^{a_k} P_1(s)f_1(s)\,ds + \int_{a_k}^1 (-P_2(a_k) + (1-P_2(a_k))P_1(s))f_1(s)\,ds\Big]$$
$$+ (1-p)(1-2P_2(a_k)) - \varepsilon$$
$$= P_1(a) - \varepsilon\,,$$

where $\varepsilon = 2/n$, $k = 1,\ldots,n_0+1$.

If Player II fires only when Player I reaches 1, the best for him is to fire as soon as possible. For such a strategy (call it η)

$$K(\xi^\varepsilon;\eta) \geq p\int_a^1 P_1(s)f_1(s)\,ds$$
$$\geq p\int_a^1 P_1(s)f_1(s)\,ds + (1-p)(1-2P_2(1)) = P_1(a)\,.$$

From the above it follows that $K(\xi^\varepsilon;\eta) \geq P_1(a) - \varepsilon$ for any strategy η of Player II.

On the other hand, suppose that Player I has fired from the point a' and later escaped. Assume that he reached this point for the first time. For such a strategy (denote it by a') we have, if $a \leq a' < 1$,

$$K(a';\eta^0) = \int_a^{a'} (-P_2(t) + (1-P_2(t))P_1(a'))f_2(t)\,dt + \int_{a'}^1 P_1(a')f_2(t)\,dt = P_1(a). \tag{20}$$

Suppose that the farthest point reached by Player I is a' but that he fires later from $a'' < a'$. For such a strategy, say ξ, we have, if $a \leq a' \leq 1$,

$$K(\xi;\eta^0) = \int_a^{a'} (-P_2(t) + (1-P_2(t))P_1(a''))f_2(t)\,dt + \int_{a'}^1 P_1(a'')f_2(t)\,dt \leq P_1(a)$$

by (20), and, if $0 \le a' < a$,

$$K(\xi;\eta^0) = P_1(a'') \le P_1(a).$$

Since approaching Player II after having fired is for Player I no better than escape when Player II applies η^0, we have $K(\xi;\eta^0) \le P_1(a)$ for any strategy ξ of Player I.

Suppose now that (14) has no solution a, $0 < a < 1$. In this case Player I ensures himself gain 0 simply by escape.

As we remember, in this case Player II applies the distribution $f_2^0(t)$, defined similarly to (8), for $a = 0$ and D satisfying (9), i.e. $D > 1+P_1(a) = 1$.

Suppose that Player I fires from the point a' and escapes (assume that he reaches this point for the first time). We obtain, for $0 \le a' \le 1$,

$$\begin{aligned} K(a';\eta^0) &= \int_0^{a'} (-P_2(t) + (1-P_2(t))P_1(a'))f_2^0(t)\,dt + \int_{a'}^1 P_1(a')f_2^0(t)\,dt \\ &= P_1(a')(1-D) \le 0. \end{aligned}$$

Suppose that the farthest point reached by Player I is a' but he fires later from $a'' \le a'$. For such a strategy, say ξ, we obtain

$$\begin{aligned} K(\xi;\eta^0) &= \int_0^{a'} (-P_2(t) + (1-P_2(t))P_1(a''))f_2^0(t)\,dt + \int_{a'}^1 P_1(a'')f_2^0(t)\,dt \\ &\le \int_0^{a'} (-P_2(t) + (1-P_2(t))P_1(a'))f_2^0(t)\,dt + \int_{a'}^1 P_1(a')f_2^0(t)\,dt \le 0. \end{aligned}$$

Since also here approaching Player II after having fired is for Player I no better than escape when Player II applies η^0, we have $K(\xi;\eta^0) \le 0$ for any strategy ξ of Player I. This ends the proof of the theorem.

When $P_1(s) = P_2(s) \stackrel{\text{def}}{=} P(s)$ we obtain from (14)

$$(1+P(a))\left[\log\frac{1+P(a)}{2P(a)} + \frac{1}{2}\right] = 2.$$

This equation has a solution a for which $P(a) \cong 0.177655$ and we obtain from (16), (7) and (10)

$$p \cong 0.872793, \qquad C \cong 0.291494, \qquad D \cong 1.177655.$$

Duels under arbitrary moving, as far as the author knows, were never considered before, except in the papers of the author (see [13]).

For other results in the theory of games of timing see [1], [2], [6], [8], [11], [12], [15].

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