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A NOISY DUEL UNDER ARBITRARY MOVING. VI

1. Introduction. In the papers [19]–[23] of the author and in this paper an m-versus-n-bullets noisy duel is considered in which duelists can move at will. The cases $m \le 25$, $n \le 6$, and $n = 1$ for any m are solved. Also an idea is given how to solve the duel for any (m, n) using the computer.

In this paper we consider the cases $n = 6$, $m = 4, 5, 6$ and $n < m \le 25$, $n = 1, \ldots, 6$.

Let us define a game which will be called the *game* (m, n). Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is supposed that $v_1 > v_2 \ge 0$. Player I has m bullets (or rockets), Player II has n bullets (rockets).

Assume that at time $t = 0$ the players are at distance 1 and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability (the same for both players) that a player succeeds (destroys his opponent) if he fires at distance $1 - s$. It is assumed that $P(s)$ is increasing and continuous in $[0, 1]$, has a continuous second derivative in $(0, 1)$, $P(s) = 0$ for $s \le 0$ and $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. The duel is a zero-sum game.

The duel is *noisy* — the player hears the shot of his opponent.

Without loss of generality we can assume that Player II is motionless. Then $v_1 = 1$, $v_2 = 0$.

We suppose that between successive shots of the same player there has to pass a time $\hat{\varepsilon} > 0$.

We also assume that the reader knows the papers [19]–[23] and remembers the notations, assumptions and results given there.

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For definitions and notions in the theory of games of timing see [1], [5], [24]. For results see [2], [3], [6], [8], [9], [11]–[13], [15], [25].

2. Duel $(4,6)$, $\langle a\rangle$. In this section we solve the duel in which Player I has 4 bullets, Player II has 6 bullets and at the beginning the players are at distance $1-a$ from each other. It is assumed that a is not too big.

Let the number $\langle\hat{a}\rangle$ denote the earliest moment when Player I reaches the point $\hat{a}$.

For a given moment t let $\rangle t\langle$ denote the point where Player I is at time t.

For a given $\hat{a}$ denote by $\hat{a}^{\varepsilon}$ a random variable with an absolutely continuous distribution in $[\langle\hat{a}\rangle, \langle a\rangle + \alpha(\varepsilon)]$, where $\alpha(\varepsilon)\to 0$ as $\varepsilon\to 0$ (and as $\hat{\varepsilon}\to 0$, see [20]).

Case 1. We define the strategies ξ and η of Players I and II. We prove that for some a these strategies are optimal in limit (i.e. optimal as $\hat{\varepsilon}\to 0$, see [20] for the precise definition).

STRATEGY OF PLAYER I: If Player II has not fired before, reach the point a_{35}, fire at a_{35}^{ε} and play optimally the resulting duel $(3,6)$, $\langle 1, \rangle a_{35}^{\varepsilon}\langle \wedge c, \rangle a_{35}^{\varepsilon}\langle\rangle$. If he has fired, play optimally the duel $(4,5)$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a_{46}\rangle$ and play optimally the resulting duel $(4,5)$ or $(3,5)$, $\langle 2, a_{46}, a_{46}\wedge c\rangle$. If he fired (say at a'), play optimally the duel $(3,6)$, $\langle 1, a'\wedge c, a'\rangle$. If he has not reached the point a_{46}, do not fire.

We have $Q(a_{35})\cong 0.980064$ (see [22]). The number a_{46} is determined from the equations

$$v_{46}^{a} = P(a_{35}) + Q(a_{35})\overset{1}{v}{}_{36}^{a_{35}} = -P(a_{46}) + Q(a_{46})v_{45}^{a_1} \overset{\mathrm{df}}{=} v_{46}^{a_1}\,, \tag{1}$$

where v_{mn}^{a}, $\overset{1}{v}{}_{mn}^{a}$, $\overset{2}{v}{}_{mn}^{a}$ are the limit values of the game (as $\hat{\varepsilon}\to 0$, see [20]) for the duels (m,n), $\langle a\rangle$, (m,n), $\langle 1, a\wedge c, a\rangle$ and (m,n), $\langle 2, a, a\wedge c\rangle$, respectively, and $Q(s) = 1 - P(s)$.

The duels (m,n), $\langle 1, a\wedge c, a\rangle$ and (m,n), $\langle 2, a, a\wedge c\rangle$ are defined and discussed in [20], Section 5; see also Section 3 in this paper.

"Play optimally" means: apply a strategy optimal in limit (i.e. as $\hat{\varepsilon}\to 0$).

Taking into account that

$$\overset{1}{v}{}_{36}^{a_{35}} = -1 + (1 + P^2(a_{24}))Q(a_{35})\,,$$

where $P(a_{24})\cong 0.013571$, we get

$$v_{46}^{a_1} = 1 - 2Q(a_{35}) + (1 + P^2(a_{24}))Q^2(a_{35}) \cong 0.000574\,. \tag{2}$$

Then from (1)

$$(3) \qquad Q(a_{46}) = \frac{1 + v_{46}^{a_1}}{1 + v_{45}^{a_1}} \cong 0.977254\,,$$

since $v_{45}^{a_1} \cong 0.023863$ (see [22]).

We now prove that if $a \le a_{35}$ then the strategies ξ and η are optimal in limit.

Suppose that Player I fighting against η fires at $a' \le a_{46}$ and then applies a strategy $\hat{\xi}_0$. Call this strategy $(a', \hat{\xi}_0)$; we have

$$\begin{aligned} K(a', \hat{\xi}_0; \eta) &\le P(a') + Q(a') \overset{1}{v}{}_{36}^{a'} + k(\hat{\varepsilon}) \\ &= \begin{cases} 1 - Q(a') + k(\hat{\varepsilon}) & \text{if } a' \le a_{36}, \\ 1 - 2Q(a') + (1 + P^2(a_{24}))Q^2(a') + k(\hat{\varepsilon}) & \text{if } a_{36} \le a' \le a_{35}, \\ 1 - 2Q(a') + (1 + v_{34}^{a_1})Q^3(a') + k(\hat{\varepsilon}) & \text{if } a_{35} \le a' \le a_{46}, \end{cases} \end{aligned}$$

where $K(\cdot;\cdot)$ denotes the payoff function (expected gain of Player I), $Q(a_{36}) \cong 0.999816$, $k(\hat{\varepsilon}) \to 0$ as $\hat{\varepsilon} \to 0$.

The first two functions on the right hand side are increasing in a', the third one is decreasing. Therefore

$$K(a', \hat{\xi}_0; \eta) \le 1 - 2Q(a_{35}) + (1 + P^2(a_{24}))Q(a_{35}) + k(\hat{\varepsilon}) = v_{46}^{a_1} + k(\hat{\varepsilon})\,.$$

Suppose now that Player I fighting against η does not fire before or at $\langle a_{46} \rangle$. For such a strategy, say $\hat{\xi}$, we obtain

$$K(\hat{\xi}; \eta) \le -P(a_{46}) + Q(a_{46}) \overset{2}{v}{}_{45}^{a_{46}} + k(\hat{\varepsilon}) = v_{46}^{a_1} + k(\hat{\varepsilon})\,,$$

since $\overset{2}{v}{}_{45}^{a_{46}} = v_{45}^{a_1} \cong 0.023863$ (see [22]).

Suppose that Player I fighting against η fires at $\langle a_{46} \rangle$. For such a strategy $\hat{\xi}$ we have

$$K(\hat{\xi}; \eta) \le Q^2(a_{46}) v_{35}^{a_{46}} + k(\hat{\varepsilon}) \cong -0.002563 + k(\hat{\varepsilon}) < v_{46}^{a_1} + k(\hat{\varepsilon})\,.$$

Thus Player II applying η assures that he does not loose (on the average) more than $v_{46}^{a_1} + k(\hat{\varepsilon})$ for properly chosen $k(\hat{\varepsilon}) \to 0$ as $\hat{\varepsilon} \to 0$. We then say that Player II *assures in limit the value* $v_{46}^{a_1}$.

It is now sufficient to prove that Player I applying ξ does not gain (on the average) in limit less than $v_{46}^{a_1}$.

Suppose that Player II fires before a_{35} ($a' < a_{35}$). For such a strategy (call it $(a', \hat{\eta}_0)$) we obtain

$$\begin{aligned} K(\xi; a', \hat{\eta}_0) &\ge -P(a') + Q(a') v_{45}^{a_1} - k(\hat{\varepsilon}) \ge -1 + Q(a_{35})(1 + v_{45}^{a_1}) - k(\hat{\varepsilon}) \\ &> -1 + Q(a_{46})(1 + v_{45}^{a_1}) - k(\hat{\varepsilon}) = v_{46}^{a_1} - k(\hat{\varepsilon})\,. \end{aligned}$$

If Player II applying $\hat{\eta}$ fires after $\langle a_{35} \rangle + \alpha(\varepsilon)$ then

$$K(\xi; \hat{\eta}) \ge P(a_{35}) + Q(a_{35}) \overset{1}{v}{}_{36}^{a_{35}} - k(\hat{\varepsilon}) = v_{46}^{a_1} - k(\hat{\varepsilon})\,.$$

Thus

$$K(\xi;\hat{\eta}) \geq v_{46}^{a_1} - k(\hat{\varepsilon})$$

for any $\hat{\eta}$ under properly chosen $k(\hat{\varepsilon})$, which ends the proof of the assertion.

It is easy to see that is the above proof it is sufficient to consider only nonrandom strategies $\hat{\xi}$, $\hat{\eta}$ (and $(a', \hat{\xi}_0)$, $(a', \hat{\eta}_0)$).

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at a_{35}^{ε} and play optimally the resulting duel $(3,6)$, $\langle 1, \rangle a_{35}^{\varepsilon} \langle \wedge c, \rangle a_{35}^{\varepsilon} \langle \rangle$. If he has fired, play optimally the duel $(4,5)$.

STRATEGY OF PLAYER II: If Player I escapes and has not fired, do not fire either. If he fired (say at a'), play optimally the resulting duel $(3,6)$, $\langle 1, a' \wedge c, a' \rangle$. If he comes nearer to you, fire at $\langle a_{46} \rangle$ and play optimally afterwards.

These strategies are optimal in limit and

$$a_{46}^{a} = v_{46}^{a_1} \tag{4}$$

for $a_{35} \leq a \leq a_{46}$.

To prove this assume that Player II fires before $\langle a_{35} \rangle$ at the point a'. We have

$$\begin{aligned} K(\xi; a', \hat{\eta}_0) &\geq -P(a') + Q(a')v_{45}^{a_1} - k(\hat{\varepsilon}) \\ &\geq -P(a_{46}) + Q(a_{46})v_{45}^{a_1} - k(\hat{\varepsilon}) \geq v_{46}^{a_1} - k(\hat{\varepsilon}). \end{aligned}$$

Assume that Player II fighting against a strategy ξ has no intention to fire before $\langle a_{35} \rangle + \alpha(\varepsilon)$. We then have

$$K(\xi;\hat{\eta}) \geq P(a_{35}) + Q(a_{35}) \overset{1}{v}{}_{36}^{a_{35}} - k(\hat{\varepsilon}) = v_{46}^{a_1} - k(\hat{\varepsilon}).$$

On the other hand, suppose that Player I fighting against η fires at $a' < a_{46}$. We obtain

$$\begin{aligned} K(a', \hat{\xi}_0; \eta) &\leq P(a') + Q(a') \overset{1}{v}{}_{36}^{a'} + k(\hat{\varepsilon}) \\ &\leq 1 - 2Q(a_{46}) + (1 + P^2(a_{24}))Q^2(a_{46}) + k(\hat{\varepsilon}) \leq v_{46}^{a_1} + k(\hat{\varepsilon}) \end{aligned}$$

by the corresponding inequality in Case 1.

When Player I has no intention to fire before or at a_{46} if Player II does not fire we obtain

$$K(\hat{\xi};\eta) \leq -P(a_{46}) + Q(a_{46}) \overset{2}{v}{}_{45}^{a_{46}} + k(\hat{\varepsilon}) = v_{46}^{a_1} + k(\hat{\varepsilon}).$$

When Player I fires at a_{46} we get

$$K(\hat{\xi};\eta) \leq Q^2(a_{46}) v_{35}^{a_{46}} + k(\hat{\varepsilon}) < k(\hat{\varepsilon}) < v_{46}^{a_1} + k(\hat{\varepsilon})$$

(see [22]).

When Player I never reaches the point a_{46} and never fires we obtain

$$K(\hat{\xi};\eta) = 0 < v_{46}^{a_1}.$$

The assertion is proved.

Case 3. We define ξ and η.

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at a_{35}^{ε} and play optimally the resulting duel $(3,6)$, $\langle 1,\rangle a_{35}^{\varepsilon}\langle\wedge c,\rangle a_{35}^{\varepsilon}\langle\rangle$. If he has fired, play optimally the duel $(4,5)$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally the resulting duel $(4,5)$ or $(3,5)$, $\langle a_1\rangle$, $a_1 = \rangle\langle a\rangle + \hat{\varepsilon}\langle$.

Now

$$v_{46}^{a} = -P(a) + Q(a)\overset{2}{v}{}_{45}^{a} = -1 + (1 + v_{45}^{a_1})Q(a) \tag{5}$$

for $a_{46} \le a \le \hat{a}_{46}$. The number $\hat{a}_{46}$ satisfies the equation

$$(1 + v_{34}^{a_1})Q^3(\hat{a}_{46}) - Q^2(\hat{a}_{46}) - (1 + v_{45}^{a_1})Q(\hat{a}_{46}) + 1 = 0\,, \tag{6}$$

$Q(\hat{a}_{46}) \cong 0.948815$, $v_{34}^{a_1} \cong 0.020530$, $v_{45}^{a_1} \cong 0.023863$.

Proof of optimality of ξ and η. Suppose that Player II fires at a', $a_{35} < a' \le a \le \hat{a}_{46}$, and fires before he reaches the point a_{35} for the first time. We have

$$K(\xi; a', \hat{\eta}_0) \ge -P(a') + Q(a')v_{45}^{a_1} - k(\hat{\varepsilon}) \ge -P(a) + Q(a)v_{45}^{a_1} - k(\hat{\varepsilon})\,.$$

If Player II fires after $\langle a_{35}\rangle + \alpha(\hat{\varepsilon})$ we have

$$\begin{aligned} K(\xi;\hat{\eta}) &\ge P(a_{35}) + Q(a_{35})\overset{1}{v}{}_{36}^{a_{35}} - k(\hat{\varepsilon}) \\ &= 1 - 2Q(a_{35}) + (1 + P^2(a_{24}))Q^2(a_{35}) - k(\hat{\varepsilon}) \\ &= v_{46}^{a_1} - k(\hat{\varepsilon}) \ge -1 + (1 + v_{45}^{a_1})Q(a) - k(\hat{\varepsilon}) \end{aligned}$$

provided

$$Q(a) \le \frac{1 + v_{46}^{a_1}}{1 + v_{45}^{a_1}} = Q(a_{46})$$

(see (3)).

On the other hand, to prove that Player II assures in limit the value $-1 + (1 + v_{45}^{a_1})Q(a)$ for $a_{46} \le a \le \hat{a}_{46}$ assume that Player I also fires at $\langle a\rangle$. We obtain for these a

$$\begin{aligned} K(\hat{\xi};\eta) &\le Q^2(a)v_{35}^{a} + k(\hat{\varepsilon}) \\ &= -Q^2(a) + (1 + v_{34}^{a_1})Q^3(a) + k(\hat{\varepsilon}) \le -1 + (1 + v_{45}^{a_1})Q(a) + k(\hat{\varepsilon}) \end{aligned}$$

provided $S(Q(a)) \le 0$, where $S(Q(\hat{a}_{46}))$ is the left hand side of (6). This function is increasing in the considered interval and $S(Q(\hat{a}_{46})) = 0$. Thus the inequality holds.

If Player I does not fire at $\langle a\rangle$ the proof is simple. We just have

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}^{a}_{45} + k(\hat{\varepsilon})$$

for any $\hat{\xi}$, since Player II fires at $\langle a\rangle$.

Case 4. We define ξ and η.

STRATEGY OF PLAYER I: Fire at $\langle a\rangle$ and play optimally the resulting duel $(3,6)$, $\langle 1, a\wedge c, a\rangle$ or $(3,5)$, $\langle a_1\rangle$, $a_1 = \rangle\langle a\rangle + \hat{\varepsilon}\langle$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally the resulting duel $(4,5)$, $\langle 2, a, a\wedge c\rangle$ or $(3,5)$, $\langle a_1\rangle$.

Now

$$v^a_{46} = Q^2(a)v^a_{35} = \begin{cases} -Q^2(a) + (1+v^{a_1}_{34})Q^3(a) & \text{if } \hat{a}_{46} \le a \le \hat{a}_{35}, \\ -Q^4(a) + (1+v^{a_1}_{23})Q^5(a) & \text{if } \hat{a}_{35} \le a \le \hat{a}_{24}, \\ -Q^6(a) + Q^7(a) & \text{if } \hat{a}_{24} \le a \le a_{34}, \end{cases} \tag{7}$$

$Q(\hat{a}_{35}) \cong 0.935980$ (see [22]), $Q(\hat{a}_{24}) \cong 0.918836$, $Q(a_{34}) \cong 0.903576$ (see [21]).

Proof. Suppose that Player II does not fire at $\langle a\rangle$; call his strategy $\hat{\eta}$. We have

$$K(\xi;\hat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}^{a}_{36} - k(\hat{\varepsilon})$$
$$= \begin{cases} 1 - 2Q(a) + (1+v^{a_1}_{34})Q^3(a) & \text{if } \hat{a}_{46} \le a \le \check{a}_{35}, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + (1+v^{a_1}_{23})Q^5(a) & \\ \qquad \text{if } \check{a}_{35} \le a \le \check{a}_{24}, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + 2Q^4(a) - 2Q^5(a) + Q^7(a) & \\ \qquad \text{if } \check{a}_{24} \le a \le a_{34}, \end{cases}$$

$Q(\check{a}_{35}) \cong 0.948807$ (see [22]), $Q(\check{a}_{24}) \cong 0.933827$ (see [21]).

(i) Let $\hat{a}_{46} \le a \le \check{a}_{35}$. We need the inequality

$$1 - 2Q(a) + (1+v^{a_1}_{34})Q^3(a) \ge -Q^2(a) + (1+v^{a_1}_{34})Q^3(a),$$

which is always satisfied.

(ii) Let $\check{a}_{35} \le a \le \hat{a}_{35}$. We need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + (1+v^{a_1}_{23})Q^5(a) \ge -Q^2(a) + (1+v^{a_1}_{34})Q^3(a)$$

or

$$S(Q(a)) = (1+v^{a_1}_{23})Q^5(a) - (3+v^{a_1}_{34})Q^3(a) + 3Q^2(a) - 2Q(a) + 1 \ge 0.$$

This function is increasing in the considered interval and $S(Q(\check{a}_{35})) \cong S(0.948867) \cong 0.002621$. Thus the inequality holds.

(iii) Let $\hat{a}_{35} \le a \le \check{a}_{24}$. We need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + (1+v^{a_1}_{23})Q^5(a) \ge -Q^4(a) + (1+v^{a_1}_{23})Q^5(a).$$

This inequality is satisfied for any a.

(iv) Let $\check{a}_{24} \le a \le \hat{a}_{24}$. We require

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + 2Q^4(a) - 2Q^5(a) + Q^7(a) \\ \ge -Q^4(a) + (1 + v_{23}^{a_1})Q^5(a).$$

The difference of the left and right hand sides is increasing in a and for $a = \hat{a}_{24}$ it is equal approximately 0.016845. Thus the inequality holds for $\check{a}_{24} \le a \le \hat{a}_{24}$.

(v) Finally, let $\hat{a}_{24} \le a \le a_{34}$. We need in this case

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + 2Q^4(a) - 2Q^5(a) + Q^7(a) \ge -Q^6(a) + Q^7(a),$$

which holds for any a.

Suppose that Player I does not fire at $\langle a \rangle$. We have

$$K(\hat{\xi}; \eta) \le -P(a) + Q(a)\overset{1}{v}{}_{35}^{a} + k(\hat{\varepsilon}) = -1 + (1 + v_{34}^{a_1})Q^2(a) + k(\hat{\varepsilon})$$

for $\hat{a}_{46} \le a \le a_{34}$.

(i) Assume that $\hat{a}_{46} \le a \le \hat{a}_{35}$. For these a we need the inequality

$$-1 + (1 + v_{34}^{a_1})Q^2(a) \le -Q^2(a) + (1 + v_{34}^{a_1})Q^3(a)$$

or

$$S(Q(a)) = (1 + v_{34}^{a_1})Q^3(a) - (2 + v_{34}^{a_1})Q^2(a) + 1 \ge 0.$$

This function is decreasing in the considered interval and $S(Q(\hat{a}_{35})) \cong S(0.935980) \cong 0.066705 > 0$. Thus the inequality is satisfied.

(ii) Assume that $\hat{a}_{35} \le a \le \hat{a}_{24}$. For these a we need

$$-1 + (1 + v_{34}^{a_1})Q^2(a) \le -Q^4(a) + (1 + v_{23}^{a_1})Q^5(a)$$

or

$$S(Q(a)) = (1 + v_{23}^{a_1})Q^5(a) - Q^4(a) - (1 + v_{34}^{a_1})Q^2(a) + 1 \ge 0.$$

This function is increasing in the considered interval and $S(Q(\hat{a}_{35})) \ge 0$. Thus the inequality holds.

(iii) Assume that $\hat{a}_{24} \le a \le a_{34}$. We need

$$-1 + (1 + v_{34}^{a_1})Q^2(a) \le -Q^6(a) + Q^7(a)$$

or

$$S(Q(a)) = Q^7(a) - Q^6(a) - (1 + v_{34}^{a_1})Q(a) + 1 \ge 0.$$

This function is also increasing. Therefore from the previous case it follows that the inequality holds.

From the above it follows that the strategies ξ and η are optimal in limit and v_{46}^{a} given by (7) is the limit value of the game.

3. Duel $(4,6)$, $\langle 1, a\wedge c, a\rangle$. Consider the duel $(4,6)$ in which Player I can fire from time $\langle a\rangle + c$ on and his opponent can fire from $\langle a\rangle$ on (but sometimes not at $\langle a\rangle$, see [20]).

Case 1: $a \le a_{35}$.
Case 2: $a_{35} \le a \le a_{46}$.

For these cases the strategies optimal in limit are the same as in the duel $(4,6)$, $\langle a\rangle$ and the limit values of the game are the same.

Case 3: $a_{46} \le a \le a_{45}$, $Q(a_{45}) \cong 0.919295$.

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at a_{35}^{ε} and play optimally the resulting duel $(3,6)$, $\langle 1,\rangle a_{35}^{\varepsilon}\langle\wedge c,\rangle a_{35}^{\varepsilon}\langle\rangle$. If he has fired, play optimally the duel $(4,5)$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the duel $(4,5)$.

We have

$$v_{46}^{a} = -1 + (1 + v_{45}^{a_1})Q(a)\,, \tag{8}$$

where $v_{45}^{a_1} \cong 0.023863$.

The proof of the limit optimality of the strategies for these a is omitted.

4. Duel $(4,6)$, $\langle 2, a, a\wedge c\rangle$

Case 1: $a \le a_{35}$.
Case 2: $a_{35} \le a \le a_{46}$.

Also here the strategies optimal in limit are the same as in the duel $(4,6)$, $\langle a\rangle$ and the limit values of the game are the same.

Case 3: $a_{46} \le a \le a_{46}^{(1)}$. The number $a_{46}^{(1)}$ satisfies the equation

$$(1 + v_{34}^{a_1})Q^3(a_{46}^{(1)}) - (3 + v_{45}^{a_1})Q(a_{46}^{(1)}) + 2 = 0\,, \quad Q(a_{46}^{(1)}) \cong 0.959955\,. \tag{9}$$

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at a_{35}^{ε} and play optimally the resulting duel $(3,6)$, $\langle 1,\rangle a_{35}^{\varepsilon}\langle\wedge c,\rangle a_{35}^{\varepsilon}\langle\rangle$. If he has fired, play optimally the duel $(4,5)$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the duel $(4,5)$ or $(3,5)$, $\langle a_2\rangle$, $a_2 = \rangle\langle a\rangle + c + \hat{\varepsilon}\langle$.

In the considered case

$$\overset{2}{v}{}_{46}^{a} = -1 + (1 + v_{45}^{a_1})Q(a)\,. \tag{8'}$$

Here also the proof of the optimality of the strategies is omitted.

Case 4: $a_{46}^{(1)} \le a \le a_{45}$. The strategies optimal in limit are:

STRATEGY OF PLAYER I: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(3,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$, $a_1 = \rangle\langle a\rangle + c\langle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel $(4,5)$ or $(3,5)$, $\langle a_2\rangle$, $a_2 = \rangle\langle a\rangle + c + \hat{\varepsilon}\langle$.

Now

$$\overset{2}{v}{}^a_{46} = P(a) + Q(a)\overset{1}{v}{}^a_{36} \tag{10}$$
$$= \begin{cases} 1 - 2Q(a) + (1 + v^{a_1}_{34})Q^3(a) & \text{if } a^{(1)}_{46} \le a \le \check{a}_{35}, \\ 1 - 2Q(a) + Q^2(a) - 2Q^3(a) + (1 + v^{a_1}_{23})Q^5(a) & \\ \qquad\qquad \text{if } \check{a}_{35} \le a \le \check{a}_{24}, & \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + 2Q^4(a) - 2Q^5(a) + Q^7(a) & \\ \qquad\qquad \text{if } \check{a}_{24} \le a \le a_{45}, & \end{cases}$$

$Q(\check{a}_{35}) \cong 0.948807$ (see [22]), $Q(\hat{a}_{24}) \cong 0.933827$ (see [21]).
The proof is omitted.

5. Results for the duels $(4,6)$

$$\overset{1}{v}{}^a_{46} = \begin{cases} v^{a_1}_{46} = 1 - 2Q(a_{35}) + (1 + P^2(a_{24}))Q^2(a_{35}) \cong 0.000574 & \\ \qquad\qquad \text{if } Q(a) \ge Q(a_{46}) \cong 0.977254, & \\ -1 + (1 + v^{a_1}_{45})Q(a) & \text{if } Q(a_{46}) \ge Q(a) \ge Q(a_{45}) \cong 0.919295, \end{cases}$$

$P(a_{24}) \cong 0.013571$ (see [21]), $Q(a_{35}) \cong 0.980064$ (see [22]),

$$v^a_{46} = \begin{cases} v^{a_1}_{46} & \text{if } Q(a) \ge Q(a_{46}), \\ -1 + (1 + v^{a_1}_{45})Q(a) & \text{if } Q(a_{46}) \ge Q(a) \ge Q(\hat{a}_{46}) \cong 0.948815, \\ -Q^2(a) + (1 + v^{a_1}_{34})Q^3(a) & \\ \qquad & \text{if } Q(\hat{a}_{46}) \ge Q(a) \ge Q(\hat{a}_{35}) \cong 0.935980, \\ -Q^4(a) + (1 + v^{a_1}_{23})Q^5(a) & \\ \qquad & \text{if } Q(\hat{a}_{35}) \ge Q(a) \ge Q(\hat{a}_{24}) \cong 0.918836, \\ -Q^6(a) + Q^7(a) & \text{if } Q(\hat{a}_{24}) \ge Q(a) \ge Q(a_{34}) \cong 0.903576, \end{cases}$$

$$\overset{2}{v}{}^a_{46} = \begin{cases} v^{a_1}_{46} & \text{if } Q(a) \ge Q(a_{46}), \\ -1 + (1 + v^{a_1}_{45})Q(a) & \text{if } Q(a_{46}) \ge Q(a) \ge Q(a^{(1)}_{46}) \cong 0.959955, \\ 1 - 2Q(a) + (1 + v^{a_1}_{34})Q^3(a) & \\ \qquad & \text{if } Q(a^{(1)}_{46}) \ge Q(a) \ge Q(\check{a}_{35}) \cong 0.948807, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + (1 + v^{a_1}_{23})Q^5(a) & \\ \qquad & \text{if } Q(\check{a}_{35}) \ge Q(a) \ge Q(\check{a}_{24}) \cong 0.933827, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + 2Q^4(a) - 2Q^5(a) + Q^7(a) & \\ \qquad & \text{if } Q(\check{a}_{24}) \ge Q(a) \ge Q(a_{45}). \end{cases}$$

6. Duel (5, 6). Consider the duel $(5,6)$, $\langle a\rangle$. We define the strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I: If Player II has not fired before, reach the point a_{56}, fire at a_{56}^{ε} and play optimally the resulting duel $(4,6)$, $\langle 1,\rangle a_{56}^{\varepsilon}\langle\wedge c,\rangle a_{56}\langle\rangle$. If he has fired, play optimally the duel $(5,5)$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a_{56}\rangle$ and play optimally the resulting duel $(5,5)$ or $(4,5)$. If he fired (say at a'), play optimally the duel $(4,6)$, $\langle 1, a'\wedge c, a'\rangle$. If he has not reached the point a_{56}, do not fire.

The numbers a_{56} and v_{56}^{a} are determined from the equations

$$(11)\qquad v_{56}^{a} = P(a_{56}) + Q(a_{56})\overset{1}{v}{}_{46}^{a_{56}} = -P(a_{56}) + Q(a_{56})v_{55} \overset{\mathrm{df}}{=} v_{56}^{a_1}\,.$$

If $0.919295 \cong Q(a_{45}) \le Q(a_{56}) \le Q(a_{46}) \cong 0.977254$ we have

$$P(a_{56}) + Q(a_{56})\overset{1}{v}{}_{46}^{a_{56}} = 1 - 2Q(a_{56}) + (1 + v_{45}^{a_1})Q^2(a_{56})\,.$$

Then from (11)

$$(12)\qquad (1 + v_{45}^{a_1})Q^2(a_{56}) - (3 + v_{55})Q(a_{56}) + 2 = 0\,.$$

Since $v_{45}^{a_1} \cong 0.023863$ (see [22]) and $v_{55} \cong 0.100470$ (see Section 7) we have

$$(13)\qquad Q(a_{56}) \cong 0.931760\,,\qquad v_{56}^{a_1} = -1 + (1 + v_{55})Q(a_{56}) \cong 0.025374\,.$$

To prove that the strategies ξ and η are optimal in limit for $a \le a_{56}$ assume that Player II fires at $a' \le a_{56}$ and then plays according to a strategy $\hat{\eta}_0$. We have

$$\begin{aligned} K(\xi; a', \hat{\eta}_0) &\ge -P(a') + Q(a')v_{55} - k(\hat{\varepsilon}) \\ &\ge -P(a_{56}) + Q(a_{56})v_{55} - k(\hat{\varepsilon}) = v_{56}^{a_1} - k(\hat{\varepsilon})\,. \end{aligned}$$

If Player II fighting against ξ fires after $\langle a_{56}\rangle + \alpha(\varepsilon)$ or does not fire at all we obtain

$$K(\xi;\hat{\eta}) \ge P(a_{56}) + Q(a_{56})\overset{1}{v}{}_{46}^{a_{56}} - k(\hat{\varepsilon}) = v_{56}^{a_1} - k(\hat{\varepsilon})\,.$$

On the other hand, if Player I fires at $a' < a_{56}$ then

$$\begin{aligned} K(a', \hat{\xi}_0; \eta) &\le P(a') + Q(a')\overset{1}{v}{}_{46}^{a'} + k(\hat{\varepsilon}) \\ &= \begin{cases} 1 - (1 - v_{46}^{a_1})Q(a') + k(\hat{\varepsilon}) & \text{if } a \le a_{46}, \\ 1 - 2Q(a') + (1 + v_{45}^{a_1})Q^2(a') + k(\hat{\varepsilon}) & \text{if } a_{46} \le a \le a_{56}. \end{cases} \end{aligned}$$

Denote the first function by $S_1(Q(a'))$ and the second by $S_2(Q(a'))$.

The first function is increasing and the second has one minimum in the considered interval. We have

$$\begin{aligned} &\max(S_1(Q(a_{46})), S_2(Q(a_{46})), S_2(Q(a_{56}))) \\ &\qquad = \max(0.023307,\ 0.023307,\ v_{56}^{a_1}) = v_{56}^{a_1}\,. \end{aligned}$$

Therefore

$$K(a', \hat{\xi}_0; \eta) \leq v_{56}^{a_1} + k(\hat{\varepsilon}).$$

If Player I fires at $\langle a_{56} \rangle$ we obtain

$$K(\hat{\xi}; \eta) \leq Q^2(a_{56}) v_{45}^{a_1} + k(\hat{\varepsilon}) \cong 0.020717 + k(\hat{\varepsilon}) < v_{56}^{a_1} + k(\hat{\varepsilon}).$$

If Player I does not fire before or at $\langle a_{56} \rangle$ but reaches this point we get

$$K(\hat{\xi}; \eta) \leq -P(a_{56}) + Q(a_{56}) v_{55} + k(\hat{\varepsilon}) = v_{56}^{a_1} + k(\hat{\varepsilon}).$$

If, finally, Player I neither reaches a_{56} nor fires then

$$K(\hat{\xi}; \eta) = 0 < v_{56}^{a_1}.$$

The assertion is proved.

The same strategies are optimal in limit for the duels $(5,6)$, $\langle 1, a \wedge c, a \rangle$ and $(5,6)$, $\langle 2, a, a \wedge c \rangle$ if $a \leq a_{56}$.

7. Duel $(6,6)$. Consider the duel $(6,6)$, $\langle a \rangle$. We define ξ and η.

STRATEGY OF PLAYER I: If Player II has not fired before, reach the point a_{66}, fire at a_{66}^{ε} and play optimally the resulting duel. If he has fired, play optimally the duel $(6,6)$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a_{66} \rangle$ and play optimally the resulting duel (6,5) or $(5,5)$. If he has fired, play optimally the duel $(5,6)$. If he has not reached the point a_{66}, do not fire.

The value of the game is

$$v_{66} = P(a_{66}) + Q(a_{66}) v_{56}^{a_{66}} = -P(a_{66}) + Q(a_{66}) v_{65}.$$

Since $v_{65} \cong 0.168089$ and $v_{56}^{a_{66}} = v_{56}^{a_1} \cong 0.025374$, from these equations we obtain

$$Q(a_{66}) = \frac{2}{2 + v_{65} + v_{56}^{a_{66}}} \cong 0.933395, \tag{14}$$

$$v_{66} = -1 + (1 + v_{65}) Q(a_{66}) \cong 0.090289. \tag{15}$$

The proof of the limit optimality of the above strategies for $a \leq a_{66}$ is omitted.

Let us compare the values $v_{m-1,m}^{a}$ and v_{mm} for small a:

$$\begin{array}{lll} v_{12}^{a} = 0, & v_{23}^{a_1} \cong 0.013757, & \\ v_{34}^{a_1} \cong 0.020530, & v_{45}^{a_1} \cong 0.023863, & v_{56}^{a_1} \cong 0.025374, \\ v_{11} \cong 0.171573, & v_{22} \cong 0.148461, & v_{33} \cong 0.129435, \\ v_{44} \cong 0.113748, & v_{55} \cong 0.100470, & v_{66} \cong 0.090289 \end{array}$$

(see [19]–[23]). From the above it follows that probably

$$\lim_{m\to\infty} v^{a_1}_{m-1,m} = \lim_{m\to\infty} v_{mm} \cong 0.035 .$$

Notice also that $v^{a_1}_{23}$ is positive though Player II has 3 bullets and Player I only 2!

8. Duel (m,n) for $m > n > 1$. Consider the duel (m,n), $\langle a\rangle$.

Case A. We define the strategies ξ and η of Players I and II.

Strategy of Player I: If Player II has not fired before, reach the point a_{mn}, fire at a^{ε}_{mn} and play optimally the duel $(m-1,n)$. If he has fired, play optimally the duel $(m,n-1)$.

Strategy of Player II: If Player I has not fired before, fire at $\langle a_{mn}\rangle$ and play optimally the duel $(m,n-1)$ or $(m-1,n-1)$. If he has fired, play optimally the duel $(m-1,n)$. If Player I has neither reached the point a_{mn} nor fired, do not fire either.

The numbers a_{mn} are determined from the equations

$$v_{mn} = -1 + (1 + v_{m,n-1})Q(a_{mn}) = 1 - (1 - v_{m-1,n})Q(a_{mn}) . \tag{16}$$

Solving this system we obtain

$$Q(a_{mn}) = \frac{2}{2 + v_{m,n-1} - v_{m-1,n}} , \tag{17}$$

where the numbers v_{mn} satisfy the recurrence equation

$$v_{mn} = \frac{v_{m,n-1} + v_{m-1,n}}{2 + v_{m,n-1} - v_{m-1,n}} , \tag{18}$$

$v_{\hat m 1}$, $v_{\hat n\hat n}$, $\hat m < m$, $\hat n < n$, being given.

We prove that if

$$n \le 6, \qquad m > n > 1 , \tag{19}$$

$$Q^2(a_{mn})v_{m-1,n-1} \le v_{mn} , \tag{20}$$

$$Q(a_{\hat m\hat n}) > Q(a_{\hat m,\hat n-1}) , \qquad Q(a_{\hat m\hat n}) > Q(a_{\hat m-1,\hat n}) \tag{21}$$

for all $\hat m > \hat n > 1$, $\hat m \le m$, $\hat n \le n$, then the strategies ξ and η are optimal in limit and v_{mn} given in (18) is the limit value of the game.

Proof. Suppose that Player II fires at $a' < a_{mn}$. We have

$$\begin{aligned} K(\xi; a', \hat\eta_0) &\ge -P(a') + Q(a')v_{m,n-1} - k(\hat\varepsilon) \\ &\ge -P(a_{mn}) + Q(a_{mn})v_{m,n-1} - k(\hat\varepsilon) = v_{mn} - k(\hat\varepsilon) \end{aligned}$$

if $a_{mn} < a_{m,n-1}$.

Suppose that Player II fires after $\langle a_{mn}\rangle + \alpha(\varepsilon)$. We obtain

$$K(\xi;\hat\eta) \ge P(a_{mn}) + Q(a_{mn})v_{m-1,n} - k(\hat\varepsilon) = v_{mn} - k(\hat\varepsilon) .$$

Then

$$K(\xi;\hat{\eta}) \geq v_{mn} - k(\hat{\varepsilon})$$

for any $\hat{\eta}$ if $k(\hat{\varepsilon})$ is chosen properly.

On the other hand, if Player I fires at $a' < a_{mn}$ we obtain

$$\begin{aligned} K(a',\hat{\xi}_0;\eta) &\leq P(a') + Q(a')v_{m-1,n} + k(\hat{\varepsilon}) \\ &\leq P(a_{mn}) + Q(a_{mn})v_{m-1,n} + k(\hat{\varepsilon}) = v_{mn} + k(\hat{\varepsilon}) \end{aligned}$$

if $a_{mn} < a_{m-1,n}$.

If Player I does not fire at $\langle a_{mn}\rangle$ or before we have

$$K(\hat{\xi};\eta) \leq -P(a_{mn}) + Q(a_{mn})v_{m,n-1} + k(\hat{\varepsilon}) = v_{mn} + k(\hat{\varepsilon})$$

if $a_{mn} < a_{m,n-1}$.

If Player I fires at $\langle a_{mn}\rangle$ then

$$K(\hat{\xi};\eta) \leq Q^2(a_{mn})v_{m-1,n-1} + k(\hat{\varepsilon}) \leq v_{mn} + k(\hat{\varepsilon})$$

by assumptions (20) and (21), since also

$$Q(a_{mn}) > Q(a_{m,n-1}) > Q(a_{m-1,n-1}).$$

Finally, suppose that Player I neither reaches the point a_{mn} nor fires. For such a strategy, say $\hat{\xi}$, we obtain

$$K(\hat{\xi};\eta) = 0 \leq v_{mn}$$

since from (16)

$$v_{mn} = 1 - (1 - v_{m-1,n})Q(a_{mn}) \geq 0$$

for $m > n$, $n \leq 6$, as can be proved using $v_{nn} \geq 0$ for $n \leq 6$ and $v_{m-1,n} \geq 0$ by an inductive argument with respect to m.

Moreover, the values v_{m1} can be determined for any m (see [19]) and if v_{mm} is determined for $m \leq 6$ (see [20], [21], [22]), then v_{mn} can be determined for all natural m, n satisfying conditions (19).

Thus the strategies ξ and η are optimal in limit.

Case B.

STRATEGY OF PLAYER I: If Player II has not fired before, fire at $\langle a_{mn}\rangle$ and play optimally the resulting duel $(m-1,n)$ or $(m-1,n-1)$. If he fired, play optimally the duel $(m,n-1)$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a^{\varepsilon}_{mn}\rangle$ and play optimally the duel $(m,n-1)$. If he has fired, play optimally the duel $(m-1,n)$. If Player I has neither reached the point a_{mn} nor fired, do not fire either.

These strategies are optimal if, besides (19) and (21), the condition

$$Q^2(a_{mn})v_{m-1,n-1} \geq v_{mn} \tag{22}$$

holds, and for those (m, n) the formulae (17) and (18) hold as well. The proof is omitted.

It is easy to see that the same strategies are optimal in limit for the duels (m, n), $\langle 1, a \wedge c, a \rangle$ and (m, n), $\langle 2, a, a \wedge c \rangle$.

Now we present tables of the values v_{mn} and $Q(a_{mn})$, $m > n$, $m \leq 25$, $n \leq 6$, computed on the basis of the paper [23], the obtained values v_{mm}, $m \leq 6$, and formulae (17) and (18). By asterisks we denote those (m, n) for which the inequality (20) holds.

9. Final remarks. The analysis presented in the paper is much more complicated than that in the corresponding classical duels when a player goes forward with constant speed even after firing all his shots. The strategies optimal in limit are completely different from those in the classical duels and their form is compound. However, the duels considered here can be analyzed and the strategies optimal in limit can be determined for any (m, n) using a computer. As is seen in the duels solved here and in [19]–[23],

Table 1

m	v_{m1}	$Q(a_{m1})$	v_{m2}	$Q(a_{m2})$	v_{m3}	$Q(a_{m3})$
2	0.41421	0.70711				
3	0.54692	0.77346	*0.28993	0.83387		
4	0.63060	0.81530	0.39328	0.85445	*0.23090	0.88345
5	0.68819	0.84410	0.47124	0.87149	0.31341	0.89272
6	0.73025	0.86512	0.53187	0.88535	0.38102	0.90153
7	0.76231	0.88115	0.58023	0.89668	0.43709	0.90942
8	0.78755	0.89378	0.61966	0.90608	0.48418	0.91635
9	0.80795	0.90398	0.65239	0.91395	0.52419	0.92242
10	0.82478	0.91239	0.67997	0.92064	0.55858	0.92774
11	0.83889	0.91945	0.70353	0.92638	0.58841	0.93242
12	0.85090	0.92545	0.72388	0.93137	0.61452	0.93656
13	0.86125	0.93062	0.74162	0.93573	0.63755	0.94024
14	0.87025	0.93512	0.75724	0.93957	0.65802	0.94354
15	0.87815	0.93908	0.77108	0.94299	0.67631	0.94649
16	0.88515	0.94258	0.78343	0.94604	0.69277	0.94917
17	0.89139	0.94569	0.79452	0.94879	0.70764	0.95159
18	0.89698	0.94849	0.80453	0.95127	0.72115	0.95379
19	0.90203	0.95101	0.81362	0.95352	0.73347	0.95581
20	0.90660	0.95330	0.82190	0.95557	0.74476	0.95766
21	0.91077	0.95539	0.82948	0.95746	0.75513	0.95936
22	0.91458	0.95729	0.83644	0.95918	0.76470	0.96094
23	0.91808	0.95904	0.84285	0.96078	0.77355	0.96239
24	0.92130	0.96065	0.84878	0.96225	0.78176	0.96375
25	0.92428	0.96214	0.85428	0.96362	0.78939	0.96501

to determine the optimal strategies for given (m, n) we only need the corresponding values $v^a_{m'n'}$, $\overset{1}{v}{}^a_{m'n'}$, $\overset{2}{v}{}^a_{m'n'}$ for $m' = m$, $n' = n-1$ and $m' = m-1$, $n' = n$. We can determine these strategies (if they exist) recursively and having the above values we need

(i) for Player I: to determine the direction in which this player should move and the place where he should fire if Player II has not fired before,

(ii) for Player II: to determine the places where Player I should fire (going back and forth).

It should also be established whether these shots are fired at fixed moments or at random in a short time interval. All these parameters can be tried by a computer and the proof of the limit optimality can be conducted along the lines developed in the papers of the author. For this purpose it may be convenient to present the limit values of subsequent duels (as functions of the variable a) numerically—in this case we need not consider different formulae for different a.

Noisy duels with retreat after firing all shots are considered by the author in [16]–[18].

For other noisy duels see [4], [10], [14], [26].

Table 2

m	v_{m4}	$Q(a_{m4})$	v_{m5}	$Q(a_{m5})$	v_{m6}	$Q(a_{m6})$
5	*0.19419	0.90923				
6	*0.26303	0.91457	*0.16809	0.92483		
7	0.32204	0.91994	*0.22755	0.92853	*0.14871	0.93578
8	0.37288	0.92501	0.27987	0.93226	*0.20110	0.93845
9	0.41699	0.92966	0.32608	0.93584	*0.24809	0.94119
10	0.45553	0.93389	0.36705	0.93921	0.29030	0.94386
11	0.48945	0.93770	0.40355	0.94233	0.32833	0.94640
12	0.51950	0.94115	0.43623	0.94520	0.36271	0.94881
13	0.54628	0.94426	0.46564	0.94785	0.39390	0.95106
14	0.57029	0.94709	0.49221	0.95028	0.42230	0.95315
15	0.59192	0.94966	0.51632	0.95251	0.44824	0.95510
16	0.61151	0.95200	0.53830	0.95457	0.47201	0.95691
17	0.62933	0.95414	0.55840	0.95647	0.49387	0.95860
18	0.64560	0.95610	0.57684	0.95822	0.51403	0.96017
19	0.66051	0.95791	0.59384	0.95985	0.53268	0.96163
20	0.67424	0.95958	0.60953	0.96135	0.54997	0.96300
21	0.68690	0.96112	0.62407	0.96276	0.56605	0.96427
22	0.69862	0.96256	0.63758	0.96407	0.58104	0.96547
23	0.70950	0.96389	0.65016	0.96529	0.59503	0.96659
24	0.71963	0.96513	0.66191	0.96643	0.60814	0.96764
25	0.72908	0.96630	0.67289	0.96750	0.62043	0.96864

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