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A NOISY DUEL UNDER ARBITRARY MOVING. V

1. Introduction. In the papers [18]–[22] of the author and in this paper an m-versus-n-bullets-noisy duel is considered in which duelists can move at will. The cases $m \le 25$, $n \le 6$, and $n = 1$ for any m are solved. Also an idea is given how to solve the duel for any (m, n) using the computer.

In this paper we consider the cases $n = 6$, $m = 1, 2, 3$.

Let us define a game which will be called the *game* (m, n). Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is supposed that $v_1 > v_2 \ge 0$. Player I has m bullets (or rockets), Player II has n bullets (rockets).

Assume that at time $t = 0$ the players are at distance 1 from each other and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability (the same for both players) that a player succeeds (destroys his opponent) if he fires at distance $1 - s$. It is assumed that the function $P(s)$ is increasing and continuous in $[0, 1]$, has a continuous second derivative in $(0, 1)$, $P(s) = 0$ for $s \le 0$ and $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The duel is *noisy*—the player hears the shot of his opponent.

Without loss of generality we also assume that Player II is motionless. Then $v_1 = 1$, $v_2 = 0$.

We suppose that between successive shots of the same players there has to pass a time $\hat{\varepsilon} > 0$.

We also assume that the reader knows the papers [18]–[21] and remembers the notations, assumptions and results given there.

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For definitions and notions in the theory of games of timing see [4], [23]. For results see [1], [2], [5], [7], [9]–[12], [14], [24].

2. Duel $(1,6)$, $\langle a\rangle$. In this section we solve the duel $(1,6)$ in the case when at the beginning the players are at distance $1-a$ from each other.

We define the strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(1,5)$, $\langle 2, a', a'\wedge c\rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ (at the beginning of the duel) and if Player I did not fire at that moment, play optimally the resulting duel $(1,4)$, $\langle 2, a, a\wedge c\rangle$.

The duels (m,n), $\langle 1, a\wedge c, a\rangle$ and (m,n), $\langle 2, a, a\wedge c\rangle$ are defined and discussed in [19], Section 5.

$\langle a\rangle$ denotes the earliest moment when Player I reaches the point a.

"Play optimally" means: apply a strategy optimal in limit (i.e. as $\hat{\varepsilon}\to 0$, see [19] for the precise definition).

We prove that if $a\le a_{16}$, where a_{16} is the root of the equation

$$(1)\qquad Q^6(a_{16})+Q^5(a_{16})+Q^4(a_{16})-Q(a_{16})-1=0,\quad Q(a_{16})\cong 0.913491,$$

$Q(s)=1-P(s)$, then the strategies ξ and η are optimal in limit and the limit value of the game $(1,6)$, $\langle a\rangle$ is

$$(2)\qquad v_{16}^{a}=-1+Q^4(a).$$

To prove this suppose that Player II fires at $a'\le a$ and then applies a strategy $\hat{\eta}_0$. For this strategy (call it $(a',\hat{\eta}_0)$) we have

$$K(\xi;a',\hat{\eta}_0)\ge -P(a')+Q(a')\overset{2}{v}{}_{15}^{a'}-k(\hat{\varepsilon}),$$

where $K(\cdot\,;\cdot)$ is the payoff function (the expected gain of Player I), $\overset{2}{v}{}_{15}^{a}$ is the limit value of the game $(1,5)$, $\langle 2, a, a\wedge c\rangle$ and $k(\hat{\varepsilon})\to 0$ as $\hat{\varepsilon}\to 0$.

Applying the formula for $\overset{2}{a}{}_{15}^{a'}$ (see [21]) we obtain

$$K(\xi;a',\hat{\eta}_0)\ge -1+Q^4(a')-k(\hat{\varepsilon})\ge -1+Q^4(a)-k(\hat{\varepsilon}).$$

Suppose that Player II fighting against the strategy ξ does not fire; call his strategy $\hat{\eta}$. Then

$$K(\xi;\hat{\eta})=0\ge -1+Q^4(a).$$

It follows that

$$K(\xi;\hat{\eta})\ge -1+Q^4(a)-k(\hat{\varepsilon})$$

for any strategy $\hat{\eta}$ of Player II.

Now we prove that if $a\le a_{16}$ then

$$K(\hat{\xi};\eta)\le -1+Q^4(a)+k(\hat{\varepsilon})$$

for any strategy $\hat{\xi}$ of Player I.

Assume that Player I does not fire at $\langle a\rangle$. For such a strategy $\hat{\xi}$

$$K(\hat{\xi};\eta) \leq -P(a) + Q(a)\overset{2a}{v}_{15} + k(\hat{\varepsilon}) = -1 + Q^4(a) + k(\hat{\varepsilon})$$

if $a \leq \hat{a}_{15}$, $Q(\hat{a}_{15}) \cong 0.902816$.

If Player I also fires at $\langle a\rangle$ we have

$$K(\hat{\xi};\eta) \leq -Q^2(a)(1 - Q^5(a)) + k(\hat{\varepsilon}) \leq -1 + Q^4(a) + k(\hat{\varepsilon}).$$

In the above inequality we took into account the fact that if both players fire at $\langle a\rangle$ and miss then Player I fires the remaining shots immediately since otherwise Player I can escape.

Thus we need

$$Q^7(a) - Q^4(a) - Q^2(a) + 1 \leq 0,$$

which after dividing by $Q(a) - 1$ leads to the inequality

$$Q^6(a) + Q^5(a) + Q^4(a) - Q(a) - 1 \geq 0,$$

which is satisfied for $a \leq a_{16}$.

Thus the strategies ξ and η are optimal in limit for $a \leq a_{16}$.

3. Duel $(1,6)$, $\langle 1, a\wedge c, a\rangle$. Suppose now that Player I can fire a shot from time $\langle a\rangle + c$ on and Player II can fire from $\langle a\rangle$ on (but sometimes not at $\langle a\rangle$, see [19]). In the sequel we denote by $\rangle t\langle$ the point where Player I is at time t. Moreover, for given a' set

$$a_1 = \rangle\langle a\rangle + c\langle, \qquad a_1' = \max(a', a_1).$$

We define the strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(1,5)$, $\langle 2, a_1', a_1'\wedge c_1\rangle$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(1,5)$, $\langle 2, a_1, a_1\wedge c_1\rangle$.

The value of the game is

$$\overset{1a}{v}_{16} = -1 + Q^4(a) \tag{3}$$

for $a \leq \hat{a}_{15}$, $Q(\hat{a}_{15}) \cong 0.902816$ (for the definition of $\hat{a}_{15}$ see [21]).

The optimality in limit of the strategies ξ and η for $a \leq \hat{a}_{15}$ can be easily established by comparing with the duel $(1,6)$, $\langle a\rangle$. The proof is omitted.

4. Duel $(1,6)$, $\langle 2, a, a\wedge c\rangle$. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(1,5)$, $\langle 2, a', a'\wedge c\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel $(1,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$. If he has fired, fire all shots as soon as possible.

The limit value of the game is

$$(3') \qquad \overset{2}{v}{}^a_{16} = -1 + Q^4(a),$$

thus it is the same as for the previous two duels but the set of the values for which it holds is different. Now the strategies ξ and η are optimal in limit and formula (3') holds for $a \le \hat{a}_{16}$, where

$$(4) \qquad Q^7(\hat{a}_{16}) - Q^4(\hat{a}_{16}) - 2Q(\hat{a}_{16}) + 2 = 0, \qquad Q(\hat{a}_{16}) \cong 0.921700.$$

The proof is omitted.

5. Results for the duels $(1,6)$

$$\overset{1}{v}{}^a_{16} = -1 + Q^4(a) \quad \text{for } Q(a) \ge Q(\hat{a}_{15}) \cong 0.902816,$$
$$v^a_{16} = -1 + Q^4(a) \quad \text{for } Q(a) \ge Q(a_{16}) \cong 0.913491,$$
$$\overset{2}{v}{}^a_{16} = -1 + Q^4(a) \quad \text{for } Q(a) \ge Q(\hat{a}_{16}) \cong 0.921700.$$

6. Duel $(2,6)$, $\langle a\rangle$

Case 1. We define the strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(2,5)$, $\langle 2, a', a' \wedge c\rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally the duel $(2,5)$, $\langle 2, a, a\wedge c\rangle$ or $(1,5)$, $\langle a_1\rangle$.

The above strategies are optimal in limit and

$$(5) \qquad v^a_{26} = \begin{cases} -1 + Q^2(a) & \text{if } a \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^3(a) & \text{if } a_{24} \le a \le a_{26}, \end{cases}$$

where the constants $v^{a_1}_{23}$ and a_{24} are defined in [19] and [20], respectively, $v^{a_1}_{23} \cong 0.013757$, $Q(a_{24}) \cong 0.986429$, the number a_{26} is the root of the equation

$$(6) \qquad Q^5(a_{26}) - (1 + v^{a_1}_{23})Q^3(a_{26}) - Q^2(a_{26}) + 1 = 0, \quad Q(a_{26}) \cong 0.953808.$$

To prove this suppose that Player II fires at $a' \le a$ and later applies a strategy $\hat{\eta}_0$. For this strategy (call it $(a', \hat{\eta}_0)$) we obtain

$$K(\xi; a', \eta_0) \ge -P(a') + Q(a')\overset{2}{v}{}^{a'}_{25} - k(\hat{\varepsilon})$$
$$= \begin{cases} -1 + Q^2(a') - k(\hat{\varepsilon}) & \text{if } a' \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^3(a') - k(\hat{\varepsilon}) & \text{if } a_{24} \le a' \le a_{26}. \end{cases}$$

Thus

$$K(\xi; a', \hat{\eta}_0) \geq v_{26}^a - k(\hat{\varepsilon})$$

for $a \leq a_{26}$ if v_{26}^a is given by (5).

Suppose that Player II fighting against ξ does not fire at all if Player I does not fire. For this strategy (call it $\hat{\eta}$)

$$K(\xi; \hat{\eta}) = 0 > v_{26}^a$$

if v_{26}^a is given by (5).

On the other hand, suppose that Player I also fires at $\langle a \rangle$. We obtain

$$K(\hat{\xi}; \eta) \leq Q^2(a) v_{15}^a + k(\hat{\varepsilon}) = -Q^2(a) + Q^5(a) + k(\hat{\varepsilon})$$

if $Q(a) \geq 0.889891$ (see [21]). Therefore we need

$$-Q^2(a) + Q^5(a) \leq -1 + Q^2(a)$$

or

$$S_1(Q(a)) = Q^5(a) - 2Q^2(a) + 1 \leq 0$$

for $a \leq a_{24}$. The function on the left hand side is decreasing in a in $[0, a_{24}]$ and $S_1(1) = 0$. Thus the inequality holds.

For $a_{24} \leq a \leq a_{26}$ we need the inequality

$$S_2(Q(a)) = Q^5(a) - (1 + v_{23}^{a_1}) Q^3(a) - Q^2(a) + 1 \leq 0\,.$$

This function is increasing in a and $S_2(Q(a_{26})) = 0$. Thus also here the inequality holds. This ends the proof in this case.

When Player I does not fire at $\langle a \rangle$ we have simply

$$K(\hat{\xi}; \eta) \leq -P(a) + Q(a) \overset{2}{v}{}_{25}^a + k(\hat{\varepsilon}) = v_{26}^a + k(\hat{\varepsilon})$$

if v_{26}^a is given by (5).

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: Fire at $\langle a \rangle$ and play optimally the duel $(1,6)$, $\langle 1, a \wedge c, a \rangle$ or $(1,5)$, $\langle a_1 \rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a \rangle$ and play optimally the duel $(2,5)$, $\langle 2, a, a \wedge c \rangle$ or $(1,5)$, $\langle a_1 \rangle$.

Now

$$v_{26}^a = Q^2(a) v_{15}^a = -Q^2(a) + Q^5(a) \tag{7}$$

for $a_{26} \leq a \leq a_{16}$.

To prove this suppose that Player II does not fire at $\langle a \rangle$. In this case

$$\begin{aligned} K(\xi; \hat{\eta}) &\geq P(a) + Q(a) \overset{1}{v}{}_{16}^a - k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^5(a) - k(\hat{\varepsilon}) \geq -Q^2(a) + Q^5(a) - k(\hat{\varepsilon}) \end{aligned}$$

for $a \leq a_{16}$.

If Player I does not fire at $\langle a\rangle$ we obtain

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}^{a}_{25} + k(\hat{\varepsilon})$$
$$= \begin{cases} -1 + (1 + v^{a_1}_{23})Q^3(a) & \text{if } a_{24} \le a \le \hat{a}_{25}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^5(a) & \text{if } \hat{a}_{25} \le a \le \check{a}_{14}, \end{cases}$$

$Q(\hat{a}_{25}) \cong 0.949182$, $Q(\check{a}_{14}) \cong 0.871757$.

Therefore we need

$$-1 + (1 + v^{a_1}_{23})Q^3(a) \le -Q^2(a) + Q^5(a)$$

for $a_{26} \le a \le \hat{a}_{25}$, which holds by the results of Case 1, and

$$-1 + 2Q(a) - 2Q^2(a) + Q^5(a) \le -Q^2(a) + Q^5(a)$$

for $\hat{a}_{25} \le a \le a_{16}$, which always holds.

7. Duel $(2,6)$, $\langle 1, a \wedge c, a\rangle$. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(2,5)$, $\langle 2, a'_1, a'_1 \wedge c_1\rangle$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(2,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$.

We remind that

$$a_1 = \rangle\langle a\rangle + c\langle, \quad a'_1 = \max(a', a_1)\,.$$

For the above duel the strategies ξ and η are optimal in limit and

$$\text{(8)} \qquad \overset{1}{v}{}^{a}_{26} = \begin{cases} -1 + Q^2(a) & \text{if } a \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^3(a) & \text{if } a_{24} \le a \le \hat{a}_{25}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^5(a) & \text{if } \hat{a}_{25} \le a \le \check{a}_{14}, \end{cases}$$

$Q(\hat{a}_{25}) \cong 0.949181$, $Q(\check{a}_{14}) = 0.871757$.

The proof of omitted.

8. Duel $(2,6)$, $\langle 2, a, a \wedge c\rangle$

Case 1. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(2,5)$, $\langle 2, a', a' \wedge c_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel $(2,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$ or $(1,5)$, $\langle a_2\rangle$, where $a_2 = \rangle\langle a\rangle + c + \hat{\varepsilon}\langle$. If Player I fired, play optimally the duel $(1,6)$, $\langle 1, a_1, a_1 \wedge c_1\rangle$.

Now

$$\begin{aligned}\overset{2}{v}{}^{a}_{26} &= -P(a) + Q(a)\overset{2}{v}{}^{a}_{24} \\ &= \begin{cases} -1 + Q^2(a) & \text{if } a \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^3(a) & \text{if } a_{24} \le a \le \hat{a}_{26}, \end{cases}\end{aligned}$$

where $\hat{a}_{26}$ is the only root of the equation

$$(9) \qquad Q^5(\hat{a}_{26}) - (1 + v^{a_1}_{23})Q^3(\hat{a}_{26}) - 2Q(\hat{a}_{26}) + 2 = 0, \quad Q(\hat{a}_{26}) \cong 0.957316.$$

The proofs of the optimality in limit of the strategies ξ and η and of the above formulae are omitted.

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(1,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the duel $(2,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$ or $(1,5)$, $\langle a_2\rangle$, where $a_2 = \rangle\langle a\rangle + c + \hat{\varepsilon}\langle$.

The limit value of the game is

$$v^{a}_{26} = 1 - 2Q(a) + Q^5(a)$$

for $\hat{a}_{26} \le a \le a_{16}$.

The proof is omitted.

9. Results for the duels (2,6)

$$\overset{1}{v}{}^{a}_{26} = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \ge Q(a_{24}) \cong 0.986429, \\ -1 + (1 + v^{a_1}_{23})Q^3(a) & \\ & \text{if } Q(a_{24}) \ge Q(a) \ge Q(\hat{a}_{25}) \cong 0.949181, \\ -1 + 2Q(a) - 2Q^2(a) + Q^5(a) & \\ & \text{if } Q(\hat{a}_{25}) \ge Q(a) \ge Q(\check{a}_{14}) \cong 0.871757, \end{cases}$$

$$v^{a}_{26} = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \ge Q(a_{24}), \\ -1 + (1 + v^{a_1}_{23})Q^3(a) & \\ & \text{if } Q(a_{24}) \ge Q(a) \ge Q(a_{26}) \cong 0.953808, \\ -Q^2(a) + Q^5(a) & \text{if } Q(a_{26}) \ge Q(a) \ge Q(a_{16}) \cong 0.913491, \end{cases}$$

$$\overset{2}{v}{}^{a}_{26} = \begin{cases} -1 + Q^2(a) & \text{if } Q(a) \ge Q(a_{24}), \\ -1 + (1 + v^{a_1}_{23})Q^3(a) & \\ & \text{if } Q(a_{24}) \ge Q(a) \ge Q(\hat{a}_{26}) \cong 0.957316, \\ 1 - 2Q(a) + Q^5(a) & \text{if } Q(\hat{a}_{26}) \ge Q(a) \ge Q(a_{16}). \end{cases}$$

10. Duel (3,6), $\langle a\rangle$

Case 1. As before, by ξ and η we denote the strategies which are next proved to be optimal in limit:

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(3,5)$, $\langle 2, a', a' \wedge c\rangle$.

STRATEGY OF PLAYER II: If Player I escapes, do not fire. If he comes nearer, do not fire till $\langle a_{36}\rangle$ and play optimally the duel $(3,5)$, $\langle 2, a_{36}, a_{36} \wedge c\rangle$ or $(2,5)$, $\langle\rangle a_{36}\langle +c\rangle$, where $c = \hat{\varepsilon}$.

In the considered case

$$v^a_{36} = 0 \tag{10}$$

for

$$Q(a) \geq Q(a_{36}) = \frac{1}{1 + P^2(a_{24})} \cong 0.999816\,. \tag{11}$$

Suppose then that Player II fires at $a' \leq a$. We obtain

$$\begin{aligned} K(\xi; a', \hat{\eta}_0) &\geq -P(a') + Q(a')\overset{2}{v}{}^{a'}_{35} - k(\hat{\varepsilon}) \\ &= -1 + (1 + P^2(a_{24}))Q(a') - k(\hat{\varepsilon}) \end{aligned}$$

if $a' \leq a_{35}$, $Q(a_{35}) \cong 0.980064$.

We have

$$-1 + (1 + P^2(a_{24}))Q(a') \geq -1 + (1 + P^2(a_{24}))Q(a) \geq 0 = v^a_{36}$$

if $a \leq a_{36}$.

On the other hand, if Player I does not reach the point a_{36} and does not fire then

$$K(\hat{\xi}; \eta) = 0 = v^a_{36}\,.$$

If Player I fires before a_{36} (at a') then

$$\begin{aligned} K(a', \hat{\xi}_0; \eta) &\leq P(a') + Q(a')\overset{1}{v}{}^{a'}_{26} + k(\hat{\varepsilon}) \\ &= 1 - 2Q(a') + Q^3(a') + k(\hat{\varepsilon}) \leq k(\hat{\varepsilon}) \end{aligned}$$

for $a' \leq a_{36}$.

If Player I fires at $\langle a_{36}\rangle$ then

$$K(\hat{\xi}; \eta) \leq Q^2(a_{36})v^{a_{36}}_{25} + k(\hat{\varepsilon}) = Q^2(a_{36})(-1 + Q(a_{36})) + k(\hat{\varepsilon}) \leq k(\hat{\varepsilon})\,.$$

If, finally, Player I does not fire before or at $\langle a_{36}\rangle$ but reaches the point a_{36} then

$$\begin{aligned} K(\hat{\xi}; \eta) &\leq -P(a_{36}) + Q(a_{36})\overset{2}{v}{}^{a_{36}}_{35} + k(\hat{\varepsilon}) \\ &= -P(a_{36}) + Q(a_{36})P^2(a_{24}) + k(\hat{\varepsilon}) = k(\hat{\varepsilon})\,. \end{aligned}$$

Thus ξ and η are optimal in limit for $a \leq a_{36}$.

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(3,5)$, $\langle 2, a', a' \wedge c\rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally the resulting duel $(3,5)$, $\langle 2,a,a\wedge c\rangle$ or $(2,5)$, $\langle a_1\rangle$.

Now

$$v_{36}^{a} = -P(a) + Q(a)\overset{2}{v}{}_{35}^{a} = -1 + (1+P^2(a_{24}))Q(a) \tag{12}$$

for $a_{36} \le a \le a_{36}^{(1)}$, where the number $a_{36}^{(1)}$ satisfies equation (13) below.

Assume that Player II fires at $a' \le a$. We have

$$K(\xi; a', \hat\eta_0) \ge -P(a') + Q(a')\overset{2}{v}{}_{35}^{a'} - k(\hat\varepsilon)$$
$$= -1 + (1+P^2(a_{24}))Q(a') - k(\hat\varepsilon) \ge -1 + (1+P^2(a_{24}))Q(a) - k(\hat\varepsilon)\,.$$

If Player II applying $\hat\eta$ against ξ does not fire at all then

$$K(\xi;\hat\eta) = 0 \ge -1 + (1+P^2(a_{24}))Q(a)$$

provided

$$Q(a) \le \frac{1}{1+P^2(a_{24})} = Q(a_{36})\,.$$

On the other hand, if Player I also fires at $\langle a\rangle$ then

$$K(\hat\xi;\eta) \le Q^2(a)v_{25}^{a} + k(\hat\varepsilon)$$
$$= -Q^2(a) + Q^3(a) + k(\hat\varepsilon) \le -1 + (1+P^2(a_{24}))Q(a) + k(\hat\varepsilon)$$

always in the interval $[a_{36}, a_{36}^{(1)}]$, where $a_{36}^{(1)}$ is the only root of the equation

$$Q^3(a_{36}^{(1)}) - Q^2(a_{36}^{(1)}) - (1+P^2(a_{24}))Q(a_{36}^{(1)}) + 1 = 0, \tag{13}$$
$$Q(a_{36}^{(1)}) \cong 0.990428\,.$$

Case 3. We define ξ and η.

STRATEGY OF PLAYER I: Fire at $\langle a\rangle$ and play optimally the resulting duel $(2,6)$, $\langle 1, a\wedge c, a\rangle$ or $(2,5)$, $\langle a_1\rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally the resulting duel $(3,5)$, $\langle 2,a,a\wedge c\rangle$ or $(2,5)$, $\langle a_1\rangle$.

We now prove that

$$v_{36}^{a} = Q^2(a)v_{25}^{a} = \begin{cases} -Q^2(a) + Q^3(a) & \text{if } a_{36}^{(1)} \le a \le a_{24}, \\ -Q^2(a) + (1+v_{23}^{a_1})Q^4(a) & \text{if } a_{24} \le a \le a_{36}^{(2)}, \end{cases} \tag{14}$$

where the number $a_{36}^{(2)}$ is the root of the equation

$$(1+v_{23}^{a_1})Q^4(a_{36}^{(2)}) - Q^2(a_{36}^{(2)}) - (1+P^2(a_{24}))Q(a_{36}^{(2)}) + 1 = 0 \tag{15}$$

in the interval (a_{24}, a_{35}), with $Q(a_{36}^{(2)}) \cong 0.986229$.

Suppose that Player I did not fire at $\langle a\rangle$. Then

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}^{a}_{35} + k(\hat{\varepsilon})$$
$$= \begin{cases} -1 + (1+P^2(a_{24}))Q(a) + k(\hat{\varepsilon}) & \text{if } a \le a_{35}, \\ -1 + (1+v^{a_1}_{34})Q^2(a) & \text{if } a_{35} \le a \le \check{a}_{35}, \end{cases}$$

$v^{a_1}_{34} \cong 0.020530$, $Q(\check{a}_{35}) = 0.948807$.

Thus we need the inequality

$$Q^3(a) - Q^2(a) - (1+P^2(a_{24}))Q(a) + 1 \ge 0$$

for $a^{(1)}_{36} \le a \le a_{24}$: in view of (13) it holds for those a.

For $a_{24} \le a \le a^{(2)}_{36}$ we need

$$S(Q(a)) = (1+v^{a_1}_{23})Q^4(a) - Q^2(a) - (1+P^2(a_{24}))Q(a) + 1 \ge 0,$$

which holds since the polynomial considered is increasing in Q for those a and $S(Q(a^{(2)}_{36})) = 0$.

Suppose then that Player II did not fire at $\langle a\rangle$. We have

$$K(\xi;\hat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}^{a}_{26} - k(\hat{\varepsilon})$$
$$= \begin{cases} 1 - 2Q(a) + Q^3(a) & \text{if } a \le a_{24}, \\ 1 - 2Q(a) + (1+v^{a_1}_{23})Q^4(a) & \text{if } a_{24} \le a \le \hat{a}_{25}. \end{cases}$$

Therefore we need

$$1 - 2Q(a) + Q^3(a) \ge -Q^2(a) + Q^3(a)$$

for $a^{(1)}_{36} \le a \le a_{24}$, which is always satisfied, and

$$1 - 2Q(a) + (1+v^{a_1}_{23})Q^4(a) \ge -Q^2(a) + (1+v^{a_1}_{23})Q^4(a)$$

for $a_{24} \le a \le a^{(2)}_{36}$ which also always holds. Thus the strategies ξ and η are optimal in limit.

Case 4. For given a denote by a^{ε} a random variable with an absolutely continuous probability distribution in $[\langle a\rangle, \langle a\rangle + \alpha(\varepsilon)]$, where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ (and as $\hat{\varepsilon} \to 0$, see [19]). We define the strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I: If Player I has not fired before, fire a shot at a^{ε} and play optimally the resulting duel $(2,6)$, $\langle 1, \rangle a^{\varepsilon}\langle \wedge c, \rangle a^{\varepsilon}\langle\rangle$. If he fired (say at a'), play optimally the resulting duel $(3,5)$, $\langle 2, a', a' \wedge c\rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and if Player I has not fired, play optimally the duel $(3,5)$, $\langle 2, a, a \wedge c\rangle$. If he has, play optimally the duel $(2,5)$, $\langle a_1\rangle$.

Now

$$v^{a}_{36} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{35} = -1 + (1+P^2(a_{24}))Q(a) \tag{16}$$

for $a_{36}^{(2)} \le a \le \hat{a}_{36}$, where

$$(17) \qquad (1+v_{23}^{a_1})Q^4(\hat{a}_{36}) - (3+P^2(a_{24}))Q(\hat{a}_{36}) + 2 = 0, \quad Q(\hat{a}_{36}) \cong 0.986016\,.$$

Suppose that Player I does not fire at $\langle a\rangle$. We obtain

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}_{35}^{a} + k(\hat{\varepsilon})\,.$$

If Player I fires at $\langle a\rangle$ then

$$K(\hat{\xi};\eta) \le Q^2(a)v_{25}^{a} + k(\hat{\varepsilon}) = -Q^2(a) + (1+v_{23}^{a_1})Q^4(a) + k(\hat{\varepsilon}) \le -1 + (1+P^2(a_{24}))Q(a) + k(\hat{\varepsilon})$$

for $a_{24} \le a \le a_{35}$. Therefore we need

$$S(Q(a)) = (1+v_{23}^{a_1})Q(a) - Q^2(a) - (1+P^2(a_{24}))Q(a) + 1 \le 0$$

for $a_{36}^{(2)} \le a \le \hat{a}_{36}$, which holds since $S(Q(a))$ is decreasing in a in this interval and $S(Q(a_{36}^{(2)})) = 0$. Thus Player II assures in limit the value $-1 + (1+P^2(a_{24}))Q(a)$.

On the other hand, if Player II fires at $\langle a\rangle$ then

$$K(\xi;\hat{\eta}) \ge -P(a) + Q(a)\overset{2}{v}{}_{35}^{a} - k(\hat{\varepsilon})\,.$$

If Player II does not fire before $\langle a\rangle + \alpha(\varepsilon)$ then

$$K(\xi;\hat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}_{26}^{a} - k(\hat{\varepsilon}) = 1 - 2Q(a) + (1+v_{23}^{a_1})Q^4(a) - k(\hat{\varepsilon}) \ge -1 + (1+P^2(a_{24}))Q(a) - k(\hat{\varepsilon})$$

for $a_{36}^{(2)} \le a \le \hat{a}_{36}$ provided

$$S(Q(a)) = (1+v_{23}^{a_1})Q^4(a) - (3+P^2(a_{24}))Q(a) + 2 \ge 0\,.$$

This function is decreasing in the considered interval and $S(Q(\hat{a}_{36})) = 0$ by (17). Thus the inequality holds.

From the above it follows that

$$K(\xi;\hat{\eta}) \ge -1 + (1+P^2(a_{24}))Q(a) - k(\hat{\varepsilon})$$

for properly chosen $k(\hat{\varepsilon})$, which proves that also Player I applying ξ assures in limit the value $-1 + (1+P^2(a_{24}))Q(a)$.

Case 5. We define ξ and η.

STRATEGY OF PLAYER I: If Player II has not fired before, escape, reach the point $\hat{a}_{36}$, fire at $\hat{a}_{36}^{\varepsilon}$ and play optimally the duel $(2,6)$, $\langle 1,\rangle\hat{a}_{36}^{\varepsilon}\langle\wedge c,\rangle\hat{a}_{36}^{\varepsilon}\langle\rangle$. If he fired (say at a'), play optimally the resulting duel $(3,5)$, $\langle 2, a', a' \wedge c\rangle$.

The random variable a_{36}^{ε} is defined similarly to a^{ε}.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and if Player I did not fire at that moment, play optimally the duel $(3,5)$, $\langle 2, a, a\wedge c\rangle$. If he fired, play optimally the duel $(2,5)$, $\langle a_1\rangle$.

In this case

$$v_{36}^{a} = \begin{cases} -1 + (1 + P^2(a_{24}))Q(a) & \text{if } \hat{a}_{36} \le a \le a_{35}, \\ -1 + (1 + v_{34}^{a_1})Q^2(a) & \text{if } a_{35} \le a \le a_{36}^{(3)}, \end{cases} \tag{18}$$

where

$$(1 + v_{23}^{a_1})Q^4(a_{36}^{(3)}) - (2 + v_{34}^{a_1})Q^2(a_{36}^{(3)}) + 1 = 0, \quad Q(a_{36}^{(3)}) \cong 0.956425. \tag{19}$$

To prove this suppose that Player I does not fire at $\langle a\rangle$. Then

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}_{35}^{a} + k(\hat{\varepsilon}).$$

If Player I fires at $\langle a\rangle$ we obtain

$$K(\hat{\xi};\eta) \le Q^2(a)v_{25}^{a} + k(\hat{\varepsilon}) = -Q^2(a) + (1 + v_{23}^{a_1})Q^4(a) + k(\hat{\varepsilon})$$

if $a_{24} \le a \le a_{25}$, $Q(a_{25}) \cong 0.943073$. Thus we need the inequality

$$-Q^2(a) + (1 + v_{23}^{a_1})Q^4(a) \le -1 + (1 + P^2(a_{24}))Q(a)$$

for $\hat{a}_{36} \le a \le a_{35}$, which holds by the proof in Case 4.

Moreover, we need

$$-Q^2(a) + (1 + v_{23}^{a_1})Q^4(a) \le -1 + (1 + v_{34}^{a_1})Q^2(a)$$

for $a_{35} \le a \le a_{36}^{(3)}$, which is satisfied for $a \le a_{36}^{(3)}$.

Thus for one side the proof is given.

For the other side, assume that Player II fires at a', $\hat{a}_{36} \le a' \le a$. We have

$$\begin{aligned} K(\xi; a', \hat{\eta}_0) &\ge -P(a') + Q(a')\overset{2}{v}{}_{35}^{a'} - k(\hat{\varepsilon}) \\ &= \begin{cases} -1 + (1 + P^2(a_{24}))Q(a') - k(\hat{\varepsilon}) & \text{if } \hat{a}_{36} \le a' \le a_{35}, \\ -1 + (1 + v_{34}^{a_1})Q^2(a') - k(\hat{\varepsilon}) & \text{if } a_{35} \le a' \le a_{36}^{(3)}, \end{cases} \end{aligned}$$

which gives

$$K(\xi; a', \hat{\eta}_0) \ge v_{36}^{a} - k(\hat{\varepsilon})$$

if v_{36}^{a} is given by (18).

Suppose now that Player I fighting against η fires after $\langle \hat{a}_{36}\rangle + \alpha(\varepsilon)$ or does not fire. Then

$$\begin{aligned} K(\xi;\hat{\eta}) &\ge P(\hat{a}_{36}) + Q(\hat{a}_{36})\overset{1}{v}{}_{26}^{\hat{a}_{36}}) - k(\hat{\varepsilon}) \\ &= 1 - 2Q(\hat{a}_{36}) + (1 + v_{23}^{a_1})Q^4(\hat{a}_{36}) - k(\hat{\varepsilon}) \\ &\ge -1 + (1 + P^2(a_{24}))Q(a) - k(\hat{\varepsilon}) \end{aligned}$$

for $\hat{a}_{36} \le a \le a_{35}$, since for $a = \hat{a}_{36}$ we have equality (see (19)).

Moreover,

$$1 - 2Q(\hat{a}_{36}) + (1 + v_{23}^{a_1})Q^4(\hat{a}_{36}) \geq -1 + (1 + v_{34}^{a_1})Q^2(a)$$

for $a_{35} \leq a \leq a_{36}^{(3)}$, because the right hand side of the above inequality is decreasing in a and for $a = a_{35}$ the inequality holds, since

$$-1 + (1 + P^2(a_{24}))Q(a) = -1 + (1 + v_{34}^{a_1})Q^2(a)$$

for this a.

This ends the proof of the optimality in limit of the strategies ξ and η.

Case 5′. Assume that instead of firing at $\hat{a}_{36}^{\varepsilon}$ if Player II has not fired before, Player I fires at $\langle \hat{a}_{36} \rangle$ and then plays optimally. This strategy (say ξ') is optimal in limit provided that, besides the inequalities proved above, we ensure that if Player II also fires at $\langle \hat{a}_{36} \rangle$ then

$$\begin{aligned}K(\xi; \hat{\eta}) &\geq Q^2(\hat{a}_{36})v_{25}^{\hat{a}_{36}} - k(\hat{\varepsilon}) \\ &= Q^2(\hat{a}_{36})(-1 + (1 + v_{23}^{a_1})Q^2(\hat{a}_{36})) - k(\hat{\varepsilon}) = -0.013998 - k(\hat{\varepsilon}) \\ &\geq -1 + (1 + (1 + P^2(a_{24}))Q(a) - k(\hat{\varepsilon}),\end{aligned}$$

i.e. provided

$$Q(a) \leq Q(\check{a}_{36}) \cong 0.985820, \tag{20}$$

since the inequality

$$-0.013998 \geq -1 + (1 + v_{34}^{a_1})Q^2(a)$$

for $a_{35} \leq a \leq a_{36}^{(3)}$ is satisfied too.

Thus for $\check{a}_{36} \leq a \leq a_{36}^{(3)}$ the strategy ξ' defined above is also optimal in limit.

Case 6. We define ξ and η.

STRATEGY OF PLAYER I: Fire at $\langle a \rangle$ and play optimally the duel $(2,6)$, $\langle 1, a \wedge c, a \rangle$ or $(2,5)$, $\langle a_1 \rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a \rangle$ and play optimally the duel $(3,5)$, $\langle 2, a, a \wedge c \rangle$ or $(2,5)$, $\langle a_1 \rangle$.

Now

$$v_{36}^a = Q^2(a)v_{25}^a = \begin{cases} -Q^2(a) + (1 + v_{23}^{a_1})Q^4(a) & \text{if } a_{36}^{(3)} \leq a \leq a_{25}, \\ -Q^4(a) + Q^6(a) & \text{if } a_{25} \leq a \leq a_{34}, \end{cases} \tag{21}$$

$Q(a_{34}) \cong 0.903576$ (see [20]), $Q(a_{25}) \cong 0.943073$ (see [21]).

Suppose that Player I does not fire at $\langle a \rangle$. Then

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}^{a}_{35} + k(\hat{\varepsilon})$$

$$= \begin{cases} -1 + (1 + v^{a_1}_{34})Q^2(a) + k(\hat{\varepsilon}) & \text{if } a^{(3)}_{36} \le a \le \check{a}_{35}, \\ -1 + 2Q(a) - 2Q^2(a) + (1 + v^{a_1}_{23})Q^4(a) + k(\hat{\varepsilon}) & \\ \qquad\qquad \text{if } \check{a}_{35} \le a \le \check{a}_{24}, & \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) - 2Q^4(a) + Q^6(a) + k(\hat{\varepsilon}) & \\ \qquad\qquad \text{if } \check{a}_{24} \le a \le a_{34}, & \end{cases}$$

$Q(\check{a}_{35}) \cong 0.948807$ (see [21]), $Q(\check{a}_{24}) \cong 0.933827$ (see [20]).

We consider several cases.

(i) $a^{(3)}_{36} \le a \le \check{a}_{35}$. In this case we need

$$-1 + (1 + v^{a_1}_{34})Q^2(a) \le -Q^2(a) + (1 + v^{a_1}_{23})Q^4(a),$$

which is satisfied by the results of Case 5 (see equation (19)).

(ii) $\check{a}_{35} \le a \le a_{25}$. In this case we need

$$-1 + 2Q(a) - 2Q^2(a) + (1 + v^{a_1}_{23})Q^4(a) \le -Q^2(a) + (1 + v^{a_1}_{23})Q^4(a),$$

which is always satisfied.

(iii) $a_{25} \le a \le \check{a}_{24}$. In this case we need

$$-1 + 2Q(a) - 2Q^2(a) + (1 + v^{a_1}_{23})Q^4(a) \le -Q^4(a) + Q^6(a)$$

or

$$S(Q(a)) = Q^6(a) - (2 + v^{a_1}_{23})Q^4(a) + 2Q^2(a) - 2Q(a) + 1 \ge 0.$$

This function is increasing in a in the interval considered and $S(Q(a_{25})) = S(0.943073) \cong 0.003241$. Thus the inequality holds.

(iv) $\check{a}_{24} \le a \le a_{34}$. In this case we need

$$-1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) - 2Q^4(a) + Q^6(a) \le -Q^4(a) + Q^6(a),$$

which is satisfied for any a.

On the other hand, if Player II does not fire at $\langle a \rangle$ we have

$$K(\xi;\hat{\eta}) \ge P(a) + Q(a)\overset{1}{v}{}^{a}_{26} - k(\hat{\varepsilon})$$

$$= \begin{cases} 1 - 2Q(a) + (1 + v^{a_1}_{23})Q^4(a) - k(\hat{\varepsilon}) & \text{if } a^{(3)}_{36} \le a \le \hat{a}_{25}, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^6(a) - k(\hat{\varepsilon}) & \\ \qquad\qquad \text{if } \hat{a}_{25} \le a \le a_{34}, & \end{cases}$$

$Q(\hat{a}_{25}) \cong 0.949181$ (see [21]), $Q(a_{34}) \cong 0.903576$ (see [20]).

We consider three cases.

(i) $a^{(3)}_{36} \le a \le \hat{a}_{25}$. We need the inequality

$$1 - 2Q(a) + (1 + v^{a_1}_{23})Q^4(a) \ge -Q^2(a) + (1 + v^{a_1}_{23})Q^4(a),$$

which is always satisfied.

(ii) $\hat{a}_{25} \le a \le a_{25}$. We need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^6(a) \ge -Q^2(a) + (1 + v_{23}^{a_1})Q^4(a)$$

or

$$S(Q(a)) = Q^6(a) - (1 + v_{23}^{a_1})Q^4(a) - 2Q^3(a) + 3Q^2(a) - 2Q(a) + 1 \ge 0\,.$$

This function is increasing in the interval considered and $S(Q(\hat{a}_{25})) \cong S(0.949181) \cong 0.002583 > 0$. Thus the inequality holds.

(iii) $a_{25} \le a \le a_{34}$. We need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^6(a) \ge -Q^4(a) + Q^6(a)\,,$$

which always holds.

Thus the strategies ξ and η are optimal in limit.

11. Duel $(3,6)$, $\langle 1, a \wedge c, a\rangle$. Since the paper would become very long we omit the proofs of the results given in this section.

Case 1: $a \le a_{36}$. The strategies optimal in limit are the same as in the duel $(3,6)$, $\langle a\rangle$ and the limit value of the game is the same.

Case 2: $a_{36} \le a \le a_{35}$. The strategies optimal in limit are:

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(3,5)$, $\langle 2, a_1', a_1' \wedge c_1\rangle$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(3,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$.

We remind that

$$a_1 = \rangle\langle a\rangle + c\langle, \quad a_1' = \max(a_1, a')\,.$$

The limit value of the game is

$$(22) \qquad \overset{1}{v}{}^{a}_{36} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{35} = -1 + (1 + P^2(a_{24}))Q(a)\,.$$

Case 3: $a_{35} \le a \le a_{34}$. The strategies optimal in limit are:

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at $\hat{a}^{\varepsilon}_{36}$ and play optimally the resulting duel $(2,6)$, $\langle 1, \rangle a^{\varepsilon}_{36}\langle \wedge c_1, \rangle a^{\varepsilon}_{36}\langle\rangle$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(3,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$.

Now

$$(23) \qquad \overset{1}{v}{}^{a}_{36} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{25}$$
$$= \begin{cases} -1 + (1 + v_{34}^{a_1})Q^2(a) & \text{if } a_{35} \le a \le \breve{a}_{35}, \\ -1 + 2Q(a) - 2Q^2(a) + (1 + v_{23}^{a_1})Q^4(a) & \text{if } \breve{a}_{35} \le a \le \breve{a}_{24}, \\ -1 + 2Q(a) - 2Q^2(a) + 2Q^3(a) - 2Q^4(a) + Q^6(a) & \text{if } \breve{a}_{24} \le a \le a_{34}, \end{cases}$$

$Q(\check{a}_{35}) \cong 0.948807$, $Q(\check{a}_{24}) \cong 0.933827$, $Q(a_{34}) \cong 0.903576$.

12. Duel $(3,6)$, $\langle 2, a, a \wedge c\rangle$. We also omit the proofs of the results obtained in this section.

Case 1: $a \le a_{36}$. The strategies optimal in limit are the same as in the duel $(3,6)$, $\langle a\rangle$ and the limit value of the game is the same.

Case 2: $a_{36} \le a \le a_{36}^{(4)}$, $a_{36}^{(4)}$ is the only root of the equation

$$(24) \qquad Q^3(a_{36}^{(4)}) - (3 + P^2(a_{24}))Q(a_{36}^{(4)}) + 2 = 0, \quad Q(a_{36}^{(4)}) \cong 0.992186.$$

The strategies optimal in limit are:

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(3,5)$, $\langle 2, a', a' \wedge c_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel $(3,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$ or $(2,5)$, $\langle a_2\rangle$, $a_2 = \rangle\langle a\rangle + c + \hat{\varepsilon}\langle$. If he has fired, play optimally the duel $(2,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

The limit value of the game is

$$(25) \qquad \overset{2}{v}{}^{a}_{36} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{25} = -1 + (1 + P^2(a_{24}))Q(a).$$

Case 3: $a_{36}^{(4)} \le a \le \hat{a}_{36}$. The strategies optimal in limit are:

STRATEGY OF PLAYER I: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(2,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the duel $(3,5)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$ or $(2,5)$, $\langle a_2\rangle$, $a_2 = \rangle\langle a\rangle + c + \hat{\varepsilon}\langle$. If he fired, play optimally the duel $(2,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

The limit value of the game is

$$(26) \qquad \overset{2}{v}{}^{a}_{36} = P(a) + Q(a)\overset{1}{v}{}^{a}_{26} = \begin{cases} 1 - 2Q(a) + Q^3(a) & \text{if } a_{36}^{(4)} \le a \le a_{24}, \\ 1 - 2Q(a) + (1 + v_{23}^{a_1})Q^4(a) & \text{if } a_{24} \le a \le \hat{a}_{36}. \end{cases}$$

Case 4: $\hat{a}_{36} \le a \le a_{36}^{(3)}$. The strategies optimal in limit are:

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at $\hat{a}_{36}^{\varepsilon}$ and play optimally the duel $(2,6)$, $\langle 1, \rangle\hat{a}_{36}^{\varepsilon}\langle \wedge c_1, \rangle\hat{a}_{36}^{\varepsilon}\langle\rangle$. If he fired (say at a'), play optimally the duel $(3,5)$, $\langle 2, a_1', a_1' \wedge c_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally afterwards. If he fired, play optimally the duel $(2,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

The limit value of the game is now

$$\text{(27)}\qquad \overset{2}{v}{}^{a}_{36} = -P(a) + Q(a)\overset{2}{v}{}^{a}_{35} = \begin{cases} -1+(1+P^2(a_{24}))Q(a) & \text{if } \hat{a}_{36} \le a \le a_{35}, \\ -1+(1+v^{a_1}_{34})Q^2(a) & \text{if } a_{35} \le a \le a^{(3)}_{36}. \end{cases}$$

Case 5: $a^{(3)}_{36} \le a \le a_{34}$. The strategies optimal in limit are:

STRATEGY OF PLAYER I: Fire before $\langle a\rangle + c$ and play optimally the resulting duel $(2,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally afterwards. If he has, play optimally the resulting duel $(2,6)$, $\langle 1, a_1 \wedge c_1, a_1\rangle$.

Now

$$\text{(28)}\qquad \overset{2}{v}{}^{a}_{36} = P(a) + Q(a)\overset{1}{v}{}^{a}_{26} = \begin{cases} 1-2Q(a)+(1+v^{a_1}_{23})Q^4(a) & \text{if } a^{(3)}_{36} \le a \le \hat{a}_{25}, \\ 1-2Q(a)+2Q^2(a)-2Q^3(a)+Q^6(a) & \text{if } \hat{a}_{25} \le a \le a_{34}. \end{cases}$$

13. Results for the duels (3, 6)

$$\overset{1}{v}{}^{a}_{36} = \begin{cases} 0 & \text{if } Q(a) \ge Q(a_{36}) \cong 0.999816, \\ -1+(1+P^2(a_{24}))Q(a) & \\ & \text{if } Q(a_{36}) \ge Q(a) \ge Q(a_{35}) \cong 0.980064, \\ -1+(1+v^{a_1}_{34})Q^2(a) & \text{if } Q(a_{35}) \ge Q(a) \ge Q(\breve{a}_{35}) \cong 0.948807, \\ -1+2Q(a)-2Q^2(a)+(1+v^{a_1}_{23})Q^4(a) & \\ & \text{if } Q(\breve{a}_{35}) \ge Q(a) \ge Q(\breve{a}_{24}) \cong 0.933827, \\ -1+2Q(a)-2Q^2(a)+2Q^3(a)-2Q^4(a)+Q^6(a) & \\ & \text{if } Q(\breve{a}_{24}) \ge Q(a) \ge Q(a_{34}) \cong 0.903576, \end{cases}$$

$$v^{a}_{36} = \begin{cases} 0 & \text{if } Q(a) \ge Q(a_{36}), \\ -1+(1+P^2(a_{24}))Q(a) & \\ & \text{if } Q(a_{36}) \ge Q(a) \ge Q(a^{(1)}_{36}) \cong 0.990428, \\ -Q^2(a)+Q^3(a) & \text{if } Q(a^{(1)}_{36}) \ge Q(a) \ge Q(a_{24}) \cong 0.986429, \\ -Q^2(a)+(1+v^{a_1}_{23})Q^4(a) & \\ & \text{if } Q(a_{24}) \ge Q(a) \ge Q(a^{(2)}_{36}) \cong 0.986229, \\ -1+(1+P^2(a_{24}))Q(a) & \\ & \text{if } Q(a^{(2)}_{36}) \ge Q(a) \ge Q(a_{35}), \\ -1+(1+v^{a_1}_{34})Q^2(a) & \text{if } Q(a_{35}) \ge Q(a) \ge Q(a^{(3)}_{36}) \cong 0.956425, \\ -Q^2(a)+(1+v^{a_1}_{23})Q^4(a) & \\ & \text{if } Q(a^{(3)}_{36}) \ge Q(a) \ge Q(a_{25}) = 0.943073, \\ -Q^4(a)+Q^6(a) & \text{if } Q(a_{25}) \ge Q(a) \ge Q(a_{34}), \end{cases}$$

$$\overset{2}{v}{}^{a}_{36} = \begin{cases} 0 & \text{if } Q(a) \geq Q(a_{36}), \\ -1 + (1 + P^2(a_{24}))Q(a) & \\ & \text{if } Q(a_{36}) \geq Q(a) \geq Q(a_{36}^{(4)}) \cong 0.992186, \\ 1 - 2Q(a) + Q^3(a) & \text{if } Q(a_{36}^{(4)}) \geq Q(a) \geq Q(a_{24}), \\ 1 - 2Q(a) + (1 + v_{23}^{a_1})Q^4(a) & \\ & \text{if } Q(a_{24}) \geq Q(a) \geq Q(\hat{a}_{36}) \cong 0.986016, \\ -1 + (1 + P^2(a_{24}))Q(a) & \\ & \text{if } Q(\hat{a}_{36}) \geq Q(a) \geq Q(a_{35}), \\ -1 + (1 + v_{34}^{a_1})Q^2(a) & \text{if } Q(a_{35}) \geq Q(a) \geq Q(a_{36}^{(3)}), \\ 1 - 2Q(a) + (1 + v_{23}^{a_1})Q^4(a) & \\ & \text{if } Q(a_{36}^{(3)}) \geq Q(a) \geq Q(\hat{a}_{25}) \cong 0.949181, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^6(a) & \\ & \text{if } Q(\hat{a}_{25}) \geq Q(a) \geq Q(a_{34}). \end{cases}$$

This ends the analysis of the duels $(m, 6)$, $m = 1, 2, 3$.

The duels $(m, 6)$, $4 \leq m \leq 25$ (and some others) are solved by the author in [22]. Noisy duels with retreat after firing all shots of the player are considered in [15]–[17]. For other noisy duels see [3], [9], [13], [25]

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