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A NOISY DUEL UNDER ARBITRARY MOVING. IV

1. Introduction. In the papers [17]–[21] of the author and in this paper an m-versus-n-bullets noisy duel is considered in which duelists can move at will. The cases $m \leq 25$, $n \leq 6$, and $n = 1$ for any m are solved. Also an idea is given how to determine the optimal (in limit) strategies for any (m, n) using the computer.

In this paper we solve the cases $n = 5$, $m \leq n$.

Let us define a game which will be called the *game* (m, n). Two Players I and II fight a duel. They can move as they want. The maximal speed of Player I is v_1, the maximal speed of Player II is v_2 and it is assumed that $v_1 > v_2 \geq 0$. Player I has m bullets (or rockets), Player II has n bullets (rockets).

Assume that at time $t = 0$ the players are at distance 1 from each other and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability (the same for both players) that a player succeeds (destroys his opponent) if he fires at distance $1 - s$. We assume that $P(s)$ is increasing and continuous in $[0, 1]$, has a continuous second derivative in $(0, 1)$, $P(s) = 0$ for $s \leq 0$, and $P(1) = 1$.

Player I gains 1 if only he succeeds, gains -1 if only Player II succeeds, and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The duel is *noisy*—the player hears the shot of his opponent.

Without loss of generality we can assume that Player II is motionless. Then $v_1 = 1$, $v_2 = 0$.

We suppose that between successive shots of the same player there has to pass a time $\hat{\varepsilon}$. We also assume that the reader knows the papers [17]–[19] and remembers the definitions, notations and assumptions given there.

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For definitions and notations in the theory of games of timing see [5], [22]. For results see [1]–[3], [6], [7], [10], [11], [13], [23].

2. Duel $(1,5)$, $\langle a\rangle$. In this section we solve the duel in which Player I has one bullet, Player II has five bullets and the game begins when the players are at distance a from each other.

We define the strategies ξ and η of Players I and II. We prove that for some a these strategies are optimal in limit (i.e. optimal as $\hat{\varepsilon} \to 0$, see [18] for the precise definition).

Strategy of Player I: Escape if Player II has not fired a shot yet. If he fired (say at a'), play optimally the resulting duel $(1,4)$, $\langle 2, a', a' \wedge c\rangle$.

Strategy of Player II: Fire at $\langle a\rangle$ (at the beginning of the duel) and if Player I did not fire at that moment, play optimally the resulting duel $(1,4)$, $\langle 2, a, a \wedge c\rangle$.

The duels (m,n), $\langle 1, a\wedge c, a\rangle$ and (m,n), $\langle 2, a, a \wedge c\rangle$ are defined in [18], Section 5.

$\langle \hat{a}\rangle$ denotes the earliest moment when Player I reaches the point $\hat{a}$.

"Play optimally" means: apply a strategy optimal in limit.

We prove that if $a \le a_{15}$, where a_{15} is the root of the equation

$$Q^5(a_{15}) + Q^4(a_{15}) + Q^3(a_{15}) - Q(a_{15}) - 1 = 0 \tag{1}$$

with $Q(a_{15}) \cong 0.889891$, $Q(s) = 1 - P(s)$, then the strategies ξ and η are optimal in limit and the limit value of the game $(1,5)$, $\langle a\rangle$ is

$$v_{15}^{a} = -1 + Q^3(a)\,. \tag{2}$$

Suppose then that Player II fires at $a' < a$ and later applies a strategy $\hat{\eta}_0$. For this strategy (call it $(a', \hat{\eta}_0)$; $\hat{\eta}_0$ may depend on a') we have

$$K(\xi; a', \hat{\eta}_0) \ge -P(a') + Q(a')\overset{2}{v}{}_{14}^{a'} - k(\hat{\varepsilon})\,,$$

where $\overset{2}{v}{}_{14}^{a}$ denotes the limit value of the game $(1,4)$, $\langle 2, a, a\wedge c\rangle$ and $k(\hat{\varepsilon}) \to 0$ as $\hat{\varepsilon} \to 0$. Taking into account that $\overset{2}{v}{}_{14}^{a} = -1 + Q^2(a)$ for $a < a_{12}$, $Q(a_{12}) \cong 0.853553$ we obtain

$$K(\xi; a', \hat{\eta}_0) \ge -1 + Q^3(a') - k(\hat{\varepsilon}) \ge -1 + Q^3(a) - k(\hat{\varepsilon})\,.$$

Suppose then that Player II playing against ξ does not fire; call this strategy $\hat{\eta}$. Then

$$K(\xi; \hat{\eta}) = 0 \ge -1 + Q^3(a)\,.$$

On the other hand, assume that Player I does not fire at $\langle a\rangle$; if we call this strategy $\hat{\xi}$ then

$$K(\hat{\xi}; \eta) \le -P(a) + Q(a)\overset{2}{v}{}_{14}^{a} + k(\hat{\varepsilon}) = -1 + Q^3(a) + k(\hat{\varepsilon}) \quad \text{for } a < a_{12}.$$

If Player I also fires at $\langle a\rangle$ we have

$$K(\hat{\xi};\eta) \le -Q^2(a)(1-Q^4(a)) + k(\hat{\varepsilon}) \le -1 + Q^3(a) + k(\hat{\varepsilon})$$

provided

$$Q^6(a) - Q^3(a) - Q^2(a) + 1 \le 0\,.$$

In the above bound on $K(\hat{\xi};\eta)$ we suppose that if both players have fired shots and survive then Player II fires all the remaining bullets immediately after those shots since otherwise Player I can escape.

Dividing the obtained polynomial by $Q(a) - 1$ shows that we need the inequality

$$Q^5(a) + Q^4(a) + Q^3(a) - Q(a) - 1 \ge 0\,,$$

which is satisfied for $a \le a_{15}$. This ends the proof of the assertion.

3. Duel $(1,5)$, $\langle 1, a\wedge c, a\rangle$. Suppose that Player I can fire a shot from time $\langle a\rangle + c$ on and Player II can fire a shot from a on (but sometimes not at $\langle a\rangle$, see [18]). Denote by $\rangle t\langle$ the coordinate of the point at which Player I was at time t and let $a_1 = \rangle\langle a\rangle + c\langle$, $a_1' = \max(a', a_1)$ for a given a'. We define the strategies ξ and η of Players I and II.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(1,4)$, $\langle 2, a_1', a_1'\wedge c_1\rangle$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the resulting duel (1,4), $\langle 2, a_1, a_1\wedge c_1\rangle$.

Now also

$$\overset{1}{v}{}^{a}_{15} = -1 + Q^3(a) \tag{3}$$

for $a \le \breve{a}_{14}$; $Q(\breve{a}_{14}) \cong 0.871757$ is defined in [19].

The proof that for these a the strategies ξ and η are optimal in limit is omitted.

4. Duel $(1,5)$, $\langle 2, a, a\wedge c\rangle$. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the resulting duel $(1,4)$, $\langle 2, a', a'\wedge c\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel $(1,4)$, $\langle 2, a_1, a_1\wedge c_1\rangle$. If he has fired, fire all shots as soon as possible.

The number

$$\overset{2}{v}{}^{a}_{15} = -1 + Q^3(a) \tag{4}$$

is the limit value of the game but only for $a \le \hat{a}_{15}$, where

$$Q^6(\hat{a}_{15}) - Q^3(\hat{a}_{15}) - 2Q(\hat{a}_{15}) + 2 = 0, \quad Q(\hat{a}_{15}) \cong 0.902816.$$

The proof that Player I, II assures in limit the value $-1 + Q^3(a)$ is the same as for the duel $(1,5)$, $\langle a \rangle$ with the only exception that now Player I can fire before Player II does. Thus assuming that Player I fires before $\langle a \rangle + c$ we obtain

$$K(\hat{\xi};\eta) \le P(a) - Q(a)(1 - Q^5(a)) + k(\hat{\varepsilon})$$
$$= 1 - 2Q(a) + Q^6(a) + k(\hat{\varepsilon}) \le -1 + Q^3(a) + k(\hat{\varepsilon}),$$

which requires the inequality

$$Q^6(a) - Q^3(a) - 2Q(a) + 2 \le 0.$$

This polynomial is zero for $Q = Q(\hat{a}_{15})$ and for $Q = 1$ and is negative for $Q(\hat{a}_{15}) < Q < 1$. Thus the inequality holds for $a \le \hat{a}_{15}$.

5. Results for the duels $(1,5)$

$$\overset{1}{v}{}^a_{15} = -1 + Q^3(a) \quad \text{for } Q(a) \ge Q(\check{a}_{14}) \cong 0.871757,$$
$$v^a_{15} = -1 + Q^3(a) \quad \text{for } Q(a) \ge Q(a_{15}) \cong 0.889891,$$
$$\overset{2}{v}{}^a_{15} = -1 + Q^3(a) \quad \text{for } Q(a) \ge Q(\hat{a}_{15}) \cong 0.902816.$$

6. Duel $(2,5)$, $\langle a \rangle$

Case 1. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the duel $(2,4)$, $\langle 2, a', a' \wedge c \rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a \rangle$ and play optimally the duel $(2,4)$, $\langle 2, a, a \wedge c \rangle$ or $(1,4)$, $\langle a_1 \rangle$.

We prove that the above ξ and η are optimal in limit and

$$v^a_{25} = \begin{cases} -1 + Q(a) & \text{for } a \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^2(a) & \text{for } a_{24} \le a \le a_{25}, \end{cases} \tag{5}$$

where $v^{a_1}_{23} \cong 0.013757$ (see [18]), $Q(a_{24}) \cong 0.986429$ (see [19]) and the number a_{25} satisfies the equation

$$Q^4(a_{25}) - (2 + v^{a_1}_{23})Q^2(a_{25}) + 1 = 0, \quad Q(a_{25}) \cong 0.943073. \tag{6}$$

Suppose that Player II, applying the strategy $(a', \hat{\eta}_0)$, fires at $a' \le a$. Then

$$K(\xi; a', \hat{\eta}_0) \ge -P(a') + Q(a')v^{a'}_{24} - k(\hat{\varepsilon})$$
$$= \begin{cases} -1 + Q(a') - k(\hat{\varepsilon}) & \text{if } a' \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^2(a') - k(\hat{\varepsilon}) & \text{if } a_{24} \le a' \le a_{25} \end{cases}$$

(see [19]). Both functions are decreasing in a', thus

$$K(\xi; a', \hat{\eta}_0) \geq \begin{cases} -1 + Q(a) - k(\hat{\varepsilon}) & \text{if } a \leq a_{24}, \\ -1 + (1 + v_{23}^{a_1})Q^2(a) - k(\hat{\varepsilon}) & \text{if } a_{24} \leq a \leq a_{25}, \end{cases}$$

and $K(\xi; a', \hat{\eta}_0) \geq v_{25}^a - k(\hat{\varepsilon})$ for $a \leq a_{25}$ if v_{25}^a is given by (5).

Suppose that Player II applying $\hat{\eta}$ against ξ does not fire. In this case

$$K(\xi; \hat{\eta}) = 0 \geq \overset{2}{v}{}_{25}^{a}$$

if v_{25}^a is given by (5), since

$$-1 + Q(a) \leq 0,$$

$$-1 + (1 + v_{23}^{a_1})Q^2(a) \leq -1 + (1 + v_{23}^{a_1})Q(a) \leq -1 + (1 + v_{23}^{a_1})Q(a_{24}) = 0$$

(see [19], (7)).

On the other hand, if Player I does not fire at $\langle a \rangle$ then

$$K(\hat{\xi}; \eta) \leq -P(a) + Q(a)\overset{2}{v}{}_{24}^{a} + k(\hat{\varepsilon}) = v_{25}^a + k(\hat{\varepsilon})$$

after taking into account the formulas for $\overset{2}{v}{}_{24}^{a}$ [19].

If Player I also fires at $\langle a \rangle$ we obtain

$$K(\hat{\xi}; \eta) \leq Q^2(a)v_{14}^a + k(\hat{\varepsilon}) = -Q^2(a) + Q^4(a) + k(\hat{\varepsilon})$$

if $a \leq a_{12}$, $Q(a_{12}) \cong 0.853553$. The first of the two cases considered in (5) requires the inequality

$$-Q^2(a) + Q^4(a) \leq -1 + Q(a),$$

or, after dividing by $Q - 1$,

$$Q^3(a) + Q^2(a) - 1 \geq 0,$$

which always holds for $a \leq a_{12}$. In the second case we need the inequality

$$Q^4(a) - (2 + v_{23}^{a_1})Q^2(a) + 1 \leq 0$$

satisfied for $a \leq a_{25}$ by (6). The assertion is proved.

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: Fire at $\langle a \rangle$ and play optimally the duel $(1,5)$, $\langle 1, a \wedge c, a \rangle$ or $(1,4)$, $\langle a_1 \rangle$.

STRATEGY OF PLAYER II: Fire at $\langle a \rangle$ and play optimally the duel $(2,4)$, $\langle 2, a, a \wedge c \rangle$ or $(1,4)$, $\langle a_1 \rangle$.

Now we prove that

$$v_{25}^a = Q^2(a)v_{14}^a = Q^4(a) - Q^2(a) \tag{7}$$

for $a_{25} \leq a < \breve{a}_{14}$, $Q(\breve{a}_{14}) \cong 0.871757$ (see [19]).

If Player II does not fire at $\langle a\rangle$ then

$$K(\xi;\hat{\eta}) \geq P(a) + Q(a)\overset{1}{v}{}^{a}_{15} - k(\hat{\varepsilon}) = 1 - 2Q(a) + Q^4(a) - k(\hat{\varepsilon}) \geq Q^4(a) - Q^2(a) - k(\hat{\varepsilon})$$

if $a \leq \breve{a}_{14}$.

On the other hand, if Player I does not fire at $\langle a\rangle$ then

$$K(\hat{\xi};\eta) \leq -P(a) + Q(a)\overset{2}{v}{}^{a}_{24} + k(\hat{\varepsilon}) = \begin{cases} -1 + (1 + v^{a_1}_{23})Q^2(a) & \text{if } a_{24} \leq a \leq \breve{a}_{24}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^4(a) & \text{if } \breve{a}_{24} \leq a \leq a_{12}, \end{cases}$$

$Q(\breve{a}_{24}) \cong 0.933827$ (see [19]).

In the first case we need

$$Q^4(a) - (2 + v^{a_1}_{23})Q^2(a) + 1 \geq 0$$

satisfied for $a \geq a_{25}$, by (6).

In the second case we need

$$-1 + 2Q(a) - 2Q^2(a) + Q^4(a) \leq Q^4(a) - Q^2(a),$$

which always holds. Thus the assertion is proved.

7. Duel $(2,5)$, $\langle 1, a \wedge c, a\rangle$

STRATEGY OF PLAYER I: Escape if Player I has not fired. If he fired (say at a'), play optimally the resulting duel $(2,4)$, $\langle 2, a'_1, a'_1 \wedge c_1\rangle$.

STRATEGY OF PLAYER II: Fire before $\langle a\rangle + c$ and play optimally the duel $(2,4)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$.

We recall that $a_1 = \rangle\langle a\rangle + c\langle$, $a'_1 = \max(a', a_1)$.

The limit value of the game is

$$\overset{1}{v}{}^{a}_{25} = \begin{cases} -1 + Q(a) & \text{if } a \leq a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^2(a) & \text{if } a_{24} \leq a \leq \breve{a}_{24}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^4(a) & \text{if } \breve{a}_{24} \leq a \leq a_{12}. \end{cases} \tag{8}$$

The proof is omitted.

8. Duel $(2,5)$, $\langle 2, a, a \wedge c\rangle$

Case 1. We define ξ and η.

STRATEGY OF PLAYER I: Escape if Player II has not fired. If he fired (say at a'), play optimally the duel $(2,4)$, $\langle 2, a', a' \wedge c_1\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the resulting duel. If he fired (say at a'), play optimally the duel $(1,5)$, $\langle 1, a'_1 \wedge c_1, a'_1\rangle$.

We now prove that

$$\overset{2}{v}{}^{a}_{25} = \begin{cases} -1 + Q(a) & \text{if } a \le a_{24}, \\ -1 + (1 + v^{a_1}_{23})Q^2(a) & \text{if } a_{24} \le a \le \hat{a}_{25}, \end{cases}$$

where the constant $\hat{a}_{25}$ is determined from the equation

(9) $Q^4(\hat{a}_{25}) - (1 + v^{a_1}_{23})Q^2(\hat{a}_{25}) - 2Q(\hat{a}_{25}) + 2 = 0, \quad Q(\hat{a}_{25}) \cong 0.949181\,.$

The proof that Player I assures in limit the value $\overset{2}{v}{}^{a}_{25}$ given above is the same as for the duel (2,5), $\langle a\rangle$. The same holds for Player II with the only exception when Player I fires before $\langle a\rangle + c$ (call such a strategy $\hat{\xi}$). Then

$$K(\hat{\xi};\eta) \le P(a) + Q(a)\overset{1}{v}{}^{a}_{15} + k(\hat{\varepsilon}) = 1 - 2Q(a) + Q^4(a) + k(\hat{\varepsilon})\,.$$

Consider two cases:

(i) $$1 - 2Q(a) + Q^4(a) \le -1 + Q(a) \quad \text{if} \quad a \le a_{24}\,.$$

This inequality can be presented in the form

$$(Q^3(a) + Q^2(a) + Q(a) - 2)(Q(a) - 1) \le 0$$

and is satisfied for $a \le a_{24}$.

(ii) $$1 - 2Q(a) + Q^4(a) \le -1 + (1 + v^{a_1}_{23})Q^2(a) \quad \text{if} \quad a_{24} \le a \le \hat{a}_{25}\,.$$

The polynomial

$$S(Q(a)) = Q^4(a) - (1 + v^{a_1}_{23})Q^2(a) - 2Q(a) + 2$$

is a decreasing function of Q and $S(Q(\hat{a}_{25})) = 0$ (see (9)). Thus the inequality holds for $a \le \hat{a}_{25}$.

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: Fire before $\langle a\rangle + c$ and play optimally the duel $(1,5)$, $\langle 1, a_1 \wedge c_1, a\rangle$, where $a_1 = \rangle\langle a\rangle + c\langle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a\rangle + c$ and play optimally the duel $(2,4)$, $\langle 2, a_1, a_1 \wedge c_1\rangle$ or $(1,4)$, $\langle a_2\rangle$, where a_2 is the point where Player I is at time $\langle a\rangle + c + \hat{\varepsilon}$.

The above strategies are optimal in limit and

(10) $\overset{2}{v}{}^{a}_{25} = P(a) + Q(a)\overset{1}{v}{}^{a}_{15} = P(a) + Q(a)(-1 + Q^3(a)) = 1 - 2Q(a) + Q^4(a)$

for $\hat{a}_{25} \le a \le \check{a}_{14}$.

Player I applying ξ assures in limit this value for the above a.

On the other hand, if Player I fires before $\langle a\rangle + c$ then

$$K(\hat{\xi};\eta) \le P(a) + Q(a)\overset{1}{v}{}^{a}_{15} + k(\hat{\varepsilon}) = \overset{2}{v}{}^{a}_{25} + k(\hat{\varepsilon})$$

if $\overset{2}{v}{}^{a}_{25}$ is given by (10).

If Player I fires at $\langle a\rangle + c$ we have

$$K(\hat{\xi};\eta) \le Q^2(a)v_{14}^{a} + k(\hat{\varepsilon}) = -Q^2(a) + Q^4(a) + k(\hat{\varepsilon})$$
$$= 1 - 2Q(a) + Q^4(a) + k(\hat{\varepsilon})$$

for $a \le a_{12}$.

Finally, if Player I does not fire before or at $\langle a\rangle + c$ we obtain

$$K(\hat{\xi};\eta) \le -P(a) + Q(a)\overset{2}{v}{}_{24}^{a} + k(\hat{\varepsilon})$$
$$= \begin{cases} -1 + (1 + v_{23}^{a_1})Q^2(a) + k(\hat{\varepsilon}) & \text{if } a_{24} \le a \le \check{a}_{24}, \\ -1 + 2Q(a) - 2Q^2(a) + Q^4(a) + k(\hat{\varepsilon}) & \text{if } \check{a}_{24} \le a \le a_{12}. \end{cases}$$

The inequality

$$-1 + 2Q(a) - 2Q^2(a) + Q^4(a) \le 1 - 2Q(a) + Q^4(a)$$

always holds.

Consider

$$-1 + (1 + v_{23}^{a_1})Q^2(a) \le 1 - 2Q(a) + Q^4(a).$$

From (9) one finds that this inequality holds for $a \ge \hat{a}_{25}$, which ends the proof of the assertion.

9. Results for the duels $(2,5)$

$$\overset{1}{v}{}_{25}^{a} = \begin{cases} -1 + Q(a) \quad \text{if } Q(a) \ge Q(a_{24}) \cong 0.986429, \\ -1 + (1 + v_{23}^{a_1})Q^2(a) \\ \qquad\qquad \text{if } Q(a_{24}) \ge Q(a) \ge Q(\check{a}_{24}) \cong 0.933827, \\ -1 + 2Q(a) - 2Q^2(a) + Q^4(a) \\ \qquad\qquad \text{if } Q(\check{a}_{24}) \ge Q(a) \ge Q(a_{12}) \cong 0.853553, \end{cases}$$

$$v_{25}^{a} = \begin{cases} -1 + Q(a) \quad \text{if } Q(a) \ge Q(a_{24}), \\ -1 + (1 + v_{23}^{a_1})Q^2(a) \\ \qquad\qquad \text{if } Q(a_{24}) \ge Q(a) \ge Q(a_{25}) \cong 0.943073, \\ -Q^2(a) + Q^4(a) \\ \qquad\qquad \text{if } Q(a_{25}) \ge Q(a) \ge Q(\check{a}_{14}) \cong 0.871757, \end{cases}$$

$$\overset{2}{v}{}_{25}^{a} = \begin{cases} -1 + Q(a) \quad \text{if } Q(a) \ge Q(a_{24}), \\ -1 + (1 + v_{23}^{a_1})Q^2(a) \\ \qquad\qquad \text{if } Q(a_{24}) \ge Q(a) \ge Q(\hat{a}_{25}) \cong 0.949181, \\ 1 - 2Q(a) + Q^4(a) \\ \qquad\qquad \text{if } Q(\hat{a}_{25}) \ge Q(a) \ge Q(\check{a}_{14}). \end{cases}$$

10. Duel $(3,5)$, $\langle a\rangle$

Case 1. Let a^ε_{mn} denote a random moment, $\langle a_{mn}\rangle \le a^\varepsilon_{mn} \le \langle a_{mn}\rangle + \alpha(\varepsilon)$, with an absolutely continuous distribution in the above interval, where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. We define the strategies ξ and η of Players I and II.

Strategy of Player I: Reach the point a_{24} and if Player II has not fired before, fire a shot at a^ε_{24} and play optimally the duel $(2,5)$, $\langle 1, \rangle a^\varepsilon_{24}\langle \wedge c, \rangle a^\varepsilon_{24}\langle\rangle$. If Player II fired (say at a'), play optimally the duel $(3,4)$, $\langle 2, a', a' \wedge c\rangle$.

Strategy of Player II: If Player I has not fired before, fire a shot at $\langle a_{35}\rangle$ and play optimally the resulting duel $(3,4)$. If he fired (say at a'), play optimally the duel $(2,5)$, $\langle 1, a' \wedge c, a'\rangle$. If Player I has not reached the point a_{35}, do not fire.

We have $Q(a_{24}) \cong 0.986429$ (see [19]). The number a_{35} is determined from the equations

$$v^a_{35} = P(a_{24}) + Q(a_{24})\overset{1}{v}{}^{a_{24}}_{25} = -P(a_{35}) + Q(a_{35})\overset{2}{v}{}^{a_{35}}_{34} \overset{\text{df}}{=} v^{a_1}_{35}\,.$$

Since

$$P(a_{24}) + Q(a_{24})\overset{1}{v}{}^{a_{24}}_{25} = P^2(a_{24})\,,$$
$$\overset{2}{v}{}^{a_{35}}_{34} = v^{a_1}_{34} \cong 0.020530 \quad \text{for } a_{35} < a_{34}\,,$$

we have

$$Q(a_{35}) = \frac{1 + P^2(a_{24})}{1 + v^{a_1}_{34}} \cong 0.980064 \tag{11}$$

and $Q(a_{35}) > Q(a_{34}) = 0.903576$, as was assumed. Moreover,

$$v^a_{35} = P^2(a_{24}) = 0.000184\,. \tag{12}$$

To prove that the strategies ξ and η are optimal in limit and v^a_{35} is given by (12) for $a < a_{24}$, assume that Player II fires at $a' < a_{24}$ and then plays according to $\hat\eta_0$. Denote this strategy by $(a', \hat\eta_0)$; then

$$\begin{aligned} K(\xi; a', \hat\eta_0) &\ge -P(a') + Q(a')v^{a_1}_{34} - k(\hat\varepsilon) \ge -P(a_{24}) + Q(a_{24})v^{a_1}_{34} - k(\hat\varepsilon) \\ &\ge -P(a_{35}) + Q(a_{35})v^{a_1}_{34} - k(\hat\varepsilon) = v^{a_1}_{35} - k(\hat\varepsilon)\,. \end{aligned}$$

If Player II does not fire before $\langle a_{24}\rangle + \alpha(\varepsilon)$ we obtain

$$K(\xi; \hat\eta) \ge P(a_{24}) + Q(a_{24})\overset{1}{v}{}^{a_{24}}_{25} - k(\hat\varepsilon) = v^{a_1}_{35} - k(\hat\varepsilon)\,.$$

Then

$$K(\xi; \hat\eta) \ge v^{a_1}_{35} - k(\hat\varepsilon) \quad \text{for any } \hat\eta$$

if the function $k(\hat\varepsilon)$ is chosen properly.

On the other hand, if $a' < a_{35}$ then

$$K(a',\hat{\xi}_0;\eta) \le P(a') + Q(a')\overset{1}{v}{}_{25}^{a'} + k(\hat{\varepsilon})$$
$$= \begin{cases} 1 - 2Q(a') + Q^2(a') + k(\hat{\varepsilon}) & \text{if } a' \le a_{24}, \\ 1 - 2Q(a') + (1 + v_{23}^{a_1})Q^3(a') + k(\hat{\varepsilon}) & \text{if } a_{24} \le a' \le a_{35}. \end{cases}$$

The first function is increasing and the second is decreasing in a'. Therefore

$$K(a',\hat{\xi}_0;\eta) \le 1 - 2Q(a_{24}) + Q^2(a_{24}) + k(\hat{\varepsilon}) = v_{35}^{a_1} + k(\hat{\varepsilon}).$$

If Player I fires at $\langle a_{35}\rangle$ then

$$K(\hat{\xi};\eta) \le Q^2(a_{35})v_{24}^{a_{35}} + k(\hat{\varepsilon})$$
$$= Q^2(a_{35})(-1 + (1 + v_{23}^{a_1})Q(a_{35})) + k(\hat{\varepsilon}) < k(\hat{\varepsilon}) < v_{35}^{a_1} + k(\hat{\varepsilon}).$$

If Player I does not fire before or at $\langle a_{35}\rangle$ but reaches the point a_{35} then

$$K(\hat{\xi};\eta) \le -P(a_{35}) + Q(a_{35})v_{34}^{a_1} + k(\hat{\varepsilon}) = v_{35}^{a_1} + k(\hat{\varepsilon}).$$

If Player I neither fires nor reaches a_{35} then

$$K(\hat{\xi};\eta) = 0 < v_{35}^{a_1}.$$

Thus the strategies ξ and η are optimal in limit and $v_{35}^{a_1}$ is the limit value of the game.

Case 2. We define ξ and η.

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at a_{24}^{ε} and play optimally the resulting duel $(2,5)$, $\langle 1,\rangle a_{24}^{\varepsilon}\langle\wedge c,\rangle a_{24}^{\varepsilon}\langle\rangle$. If he fired (say at a'), play optimally the duel $(3,4)$, $\langle 2, a', a' \wedge c\rangle$.

STRATEGY OF PLAYER II: If Player I has not fired before and reached the point a_{35}, fire at $\langle a_{35}\rangle$ and play optimally the resulting duel $(3,4)$. If he fired (say at a'), play optimally the duel $(2,5)$, $\langle 1, a' \wedge c, a'\rangle$. If Player I has not reached a_{35}, do not fire.

Now also

$$v_{35}^{a} = P^2(a_{24}) = v_{35}^{a_1}.$$

We prove that the above strategies are optimal in limit for $a_{24} \le a \le a_{35}$.

Suppose that Player II fires at a', $\rangle\langle a_{24}\rangle + \alpha(\varepsilon)\langle \le a' < a \le a_{35}$. We have

$$K(\xi; a', \hat{\eta}_0) \ge -P(a') + Q(a')v_{34}^{a_1} - k(\hat{\varepsilon})$$
$$\ge -P(a) + Q(a)v_{34}^{a_1} - k(\hat{\varepsilon}) \ge P^2(a_{24}) - k(\hat{\varepsilon})$$

provided

$$Q(a) \ge \frac{1 + P^2(a_{24})}{1 + v_{34}^{a_1}} = Q(a_{35}),$$

which is satisfied.

If Player II intends to fire at $a' > a_{24}$ or does not fire at all we obtain

$$\begin{aligned} K(\xi;\hat\eta) &\geq P(a_{24}) + Q(a_{24}) v_{25}^{1a_{24}} - k(\hat\varepsilon) \\ &= 1 - Q(a_{24}) + Q(a_{24})(-1 + (1 + v_{23}^{a_1})Q^2(a_{24})) - k(\hat\varepsilon) \\ &= 1 - 2Q(a_{24}) + (1 + v_{23}^{a_1})Q^3(a_{24}) - k(\hat\varepsilon) = v_{35}^{a_1} - k(\hat\varepsilon) \end{aligned}$$

by the equations obtained in the proof of the previous case.

If Player I fires at $a' < a_{35}$ then

$$\begin{aligned} K(a', \hat\xi_0; \eta) &\leq P(a') + Q(a') v_{25}^{1a'} + k(\hat\varepsilon) \\ &= \begin{cases} 1 - 2Q(a') + Q^2(a') + k(\hat\varepsilon) & \text{if } a' \leq a_{24}, \\ 1 - 2Q(a') + (1 + v_{23}^{a_1})Q^3(a') + k(\hat\varepsilon) & \text{if } a_{24} \leq a \leq a_{35}. \end{cases} \end{aligned}$$

Both functions are not greater than $P^2(a_{24}) + k(\hat\varepsilon)$.

If Player I did not fire before or at $\langle a_{35}\rangle$ but did reach this point we have

$$K(\hat\xi;\eta) \leq -P(a_{35}) + Q(a_{35}) v_{34}^{a_1} + k(\hat\varepsilon) = P^2(a_{24}) + k(\hat\varepsilon).$$

If Player I fires at $\langle a_{35}\rangle$ then

$$K(\hat\xi;\eta) \leq Q^2(a_{35}) v_{24}^{a_{35}} + k(\hat\varepsilon) < k(\hat\varepsilon) < P^2(a_{24}) + k(\hat\varepsilon),$$

as shown in the previous case.

If Player I neither reaches a_{35} nor fires then

$$K(\hat\xi;\eta) = 0 < P^2(a_{24}).$$

Thus this case is also solved.

Case 3. We define ξ and η.

STRATEGY OF PLAYER I: If Player II has not fired before, escape, fire at a_{24}^{ξ} and play optimally the duel $(2,5)$, $\langle 1, \rangle a_{24}^{\xi} \langle \wedge c, \rangle a_{24}^{\xi} \langle\rangle$. If he fired (say at a'), play optimally the duel $(3,4)$.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally the duel $(3,4)$ or $(2,4)$, $\langle a_1\rangle$.

Let $\hat a_{35}$ be the number satisfying the equation

$$(13) \qquad (1 + v_{23}^{a_1})Q^3(\hat a_{35}) - Q^2(\hat a_{35}) - (1 + v_{34}^{a_1})Q(\hat a_{35}) + 1 = 0, \quad Q(\hat a_{35}) \cong 0.935980.$$

We now prove that the value of the game is

$$(14) \qquad v_{35}^{a} = -P(a) + Q(a) v_{34}^{a_1}$$

if $a_{35} \leq a \leq \hat a_{35}$.

To prove this assume that Player II fires at a', $\rangle\langle a_{24}\rangle + \alpha(\varepsilon)\langle \leq a' \leq a \leq a_{35}$. Then

$$K(\xi; a', \hat\eta_0) \geq -P(a') + Q(a') v_{34}^{a_1} - k(\hat\varepsilon) \geq -P(a) + Q(a) v_{34}^{a_1} - k(\hat\varepsilon).$$

If Player I intends to fire at $a' > a_{24}$ or not to fire at all then

$$K(\xi;\hat\eta) \ge P(a_{24}) + Q(a_{24})\overset{1}{v}{}_{25}^{a_{24}} - k(\hat\varepsilon) = P^2(a_{24}) - k(\hat\varepsilon) \ge -1 + (1+v_{34}^{a_1})Q(a) - k(\hat\varepsilon)$$

provided

$$Q(a) \le \frac{1+P^2(a_{24})}{1+v_{34}^{a_1}} = Q(a_{35}),$$

which is satisfied.

On the other hand, if Player I also fires at $\langle a\rangle$ then

$$K(\hat\xi;\eta) \le Q^2(a)v_{24}^{a} + k(\hat\varepsilon) = -Q^2(a) + (1+v_{23}^{a_1})Q^3(a) + k(\hat\varepsilon) \le -1 + (1+v_{34}^{a_1})Q(a) + k(\hat\varepsilon)$$

provided

$$(1+v_{23}^{a_1})Q^3(a) - Q^2(a) - (1+v_{34}^{a_1})Q(a) + 1 \le 0\,.$$

Since the function on the left hand side is increasing for $a_{35} \le a \le \hat a_{35}$ and the number $\hat a_{35}$ is its root, the inequality holds for $a_{35} \le a \le \hat a_{35}$. This ends the proof of the assertion.

Case 4. We define ξ and η.

STRATEGY OF PLAYER I: Fire at $\langle a\rangle$ and play optimally afterwards.

STRATEGY OF PLAYER II: Fire at $\langle a\rangle$ and play optimally afterwards.

Now

$$v_{35}^{a} = Q^2(a)v_{24}^{a} = \begin{cases} -Q^2(a) + (1+v_{23}^{a_1})Q^3(a) & \text{if } \hat a_{35} \le a \le \hat a_{24},\\ -Q^4(a) + Q^5(a) & \text{if } \hat a_{24} \le a \le a_{34}, \end{cases} \tag{15}$$

$Q(\hat a_{24}) \cong 0.918836$, $Q(a_{34}) = 0.903576$ (see [19]).

When Player II does not fire at $\langle a\rangle$ we have

$$K(\xi;\hat\eta) \ge P(a) + Q(a)\overset{1}{v}{}_{24}^{a} - k(\hat\varepsilon) = \begin{cases} 1 - 2Q(a) + (1+v_{23}^{a_1})Q^3(a) - k(\hat\varepsilon) & \text{if } a_{24} \le a \le \check a_{24},\\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) - k(\hat\varepsilon) & \\ \qquad\qquad \text{if } \check a_{24} \le a \le a_{12}, & \end{cases}$$

$Q(a_{24}) \cong 0.986429$, $Q(\check a_{24}) \cong 0.933827$, $Q(a_{12}) \cong 0.853553$.

When $\hat a_{35} \le a \le \check a_{24}$ we need

$$1 - 2Q(a) + (1+v_{23}^{a_1})Q^3(a) \ge -Q^2(a) + (1+v_{23}^{a_1})Q^3(a),$$

which always holds.

When $\check a_{24} \le a \le \hat a_{24}$ we need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) \ge -Q^2(a) + (1+v_{23}^{a_1})Q^3(a)$$

or

$$S(Q(a)) = Q^5(a) - (3 + v_{23}^{a_1})Q^3(a) + 3Q^2(a) - 2Q(a) + 1 \geq 0\,.$$

The function S is decreasing in Q for $Q(\check{a}_{24}) \geq Q \geq Q(\hat{a}_{24})$ and $S(Q(\check{a}_{34})) = 0.004379 > 0$. Thus the inequality holds for $\check{a}_{24} \leq a \leq \hat{a}_{24}$.

Finally, when $\hat{a}_{24} \leq a < a_{34}$ we need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) \geq -Q^4(a) + Q^5(a)\,,$$

which is satisfied for any a.

Therefore Player I applying ξ assures in limit the value v_{35}^a given in (15).

To prove that so does Player II applying η, assume that Player I does not fire at $\langle a\rangle$. In this case

$$K(\hat{\xi};\eta) \leq -P(a) + Q(a)\overset{2}{v}{}_{34}^{a} + k(\hat{\varepsilon})$$

for $a \leq a_{34}$. Then if $\hat{a}_{35} \leq a \leq \hat{a}_{24}$ we need

$$-1 + (1 + v_{34}^{a_1})Q(a) \leq -Q^2(a) + (1 + v_{23}^{a_1})Q^3(a)\,,$$

which is satisfied for $a \geq \hat{a}_{35}$ by (13).

If $\hat{a}_{24} \leq a \leq a_{34}$ we need

$$S(Q(a)) = Q^5(a) - Q^4(a) + (1 + v_{34}^{a_1})Q(a) + 1 \geq 0\,.$$

S is decreasing in Q and $S(Q(\hat{a}_{24})) = 0.004449 > 0$. Thus the inequality holds.

This ends the analysis of Case 4.

11. Duel $(3,5)$, $\langle 1, a \wedge c, a\rangle$

Case 1: $a \leq a_{24}$.

Case 2: $a_{24} \leq a \leq a_{35}$.

For these two cases the strategies optimal in limit are the same as for the duel $(3,5)$, $\langle a\rangle$ (and the limit values of the game are the same).

Case 3: $a_{35} \leq a \leq a_{34}$. In this case the strategies optimal in limit are the same as for the duel $(3,5)$, $\langle a\rangle$ but the set of values of a for which these strategies are optimal in limit is different: there we have $a_{35} \leq a \leq \hat{a}_{35}$, and here $a_{35} \leq a \leq a_{34}$.

12. Duel $(3,5)$, $\langle 2, a, a \wedge c\rangle$

Case 1: $a \leq a_{24}$.

Case 2: $a_{24} \leq a \leq a_{35}$.

Also here the strategies optimal in limit are the same as for the duel $(3,5)$, $\langle a\rangle$ (and the limit values of the game are the same).

Case 3. We define ξ and η.

STRATEGY OF PLAYER I: If Player I has not fired before, escape, fire at a_{24}^{ε} and play optimally the duel $(2,5)$, $\langle 1, \rangle a_{24}^{\varepsilon} \langle \wedge c, \rangle a_{24}^{\varepsilon} \langle \rangle$. If he fired (say at a'), play optimally the duel $(3,4)$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a \rangle + c$ and play the duel $(3,4)$ or the duel $(2,4)$, $\langle a_1 \rangle$, where $a_1 = \rangle \langle a \rangle + \hat{\varepsilon} \langle$. If he has fired, play optimally the duel $(2,5)$, $\langle 1, a_1 \wedge c_1, a_1 \rangle$.

Moreover,

$$\overset{2}{v}{}_{35}^{a} = -P(a) + Q(a) v_{34}^{a_1} \tag{16}$$

for $a_{35} \le a \le \breve{a}_{35}$, where the number $\breve{a}_{35}$ satisfies the equation

$$(1 + v_{23}^{a_1}) Q^3(\breve{a}_{35}) - (3 + v_{34}^{a_1}) Q(\breve{a}_{35}) + 2 = 0, \quad Q(\breve{a}_{35}) \cong 0.948807. \tag{17}$$

It is easy to see, comparing with the duel $(3,5)$, $\langle a \rangle$, that Player I always assures in limit the value $\overset{2}{v}{}_{35}^{a}$ given by (16) if $a_{35} \le a \le \breve{a}_{35}$.

On the other hand, comparing with the same duel, we find that Player II assures in limit the value $\overset{2}{v}{}_{35}^{a}$ for $a_{35} \le a \le \hat{a}_{35}$ if Player I fires at $\langle a \rangle + c$ or later or does not fire. Therefore assume that Player I fires before $\langle a \rangle + c$ (call this strategy $(a', \hat{\xi}_0)$). Then

$$\begin{aligned} K(a', \hat{\xi}_0; \eta) &\le P(a) + Q(a) \overset{1}{v}{}_{25}^{a} + k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + (1 + v_{23}^{a_1}) Q^3(a) + k(\hat{\varepsilon}) \end{aligned}$$

if $a_{24} \le a \le \breve{a}_{24}$, $Q(a_{24}) \cong 0.986429$, $Q(\breve{a}_{24}) \cong 0.933827$.

Comparing with (16) shows that we need the inequality

$$S(Q(a)) = (1 + v_{23}^{a_1}) Q^3(a) - (3 + v_{34}^{a_1}) Q(a) + 2 \le 0.$$

The above function is increasing in a in the interval $a_{35} \le a \le \breve{a}_{35}$ and $S(Q(\breve{a}_{35})) = 0$. Thus the inequality holds.

Case 4. We define ξ and η.

STRATEGY OF PLAYER I: Fire before $\langle a \rangle + c$ and play optimally the resulting duel $(2,5)$, $\langle 1, a_1 \wedge c_1, a_1 \rangle$, where $a_1 = \rangle \langle a \rangle + c \langle$.

STRATEGY OF PLAYER II: If Player I has not fired before, fire at $\langle a \rangle + c$ and play optimally the obtained duel. If he has fired, play optimally the duel $(2,5)$, $\langle 1, a_1 \wedge c_1, a_1 \rangle$.

Now

$$\begin{aligned} \overset{2}{v}{}_{35}^{a} &= P(a) + Q(a) \overset{1}{v}{}_{25}^{a} \\ &= \begin{cases} 1 - 2Q(a) + (1 + v_{23}^{a_1}) Q^3(a) & \text{if } \breve{a}_{35} \le a \le \breve{a}_{24}, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) & \text{if } \breve{a}_{24} \le a \le a_{34}. \end{cases} \end{aligned}$$

It is easy to see that Player I always assures in limit the above values.

Suppose then that Player I fires before $\langle a\rangle + c$. We have

$$K(\hat{\xi};\eta) \le P(a) + Q(a)\overset{1}{v}{}^{a}_{25} + k(\hat{\varepsilon}) = \overset{2}{v}{}^{a}_{35} + k(\hat{\varepsilon}),$$

as desired.

If Player I fires at $\langle a\rangle + c$ we obtain

$$K(\hat{\xi};\eta) \le Q^2(a)v^{a}_{24} + k(\hat{\varepsilon})$$
$$= \begin{cases} -Q^2(a) + (1 + v^{a_1}_{23})Q^3(a) + k(\hat{\varepsilon}) \\ \qquad\qquad\qquad \text{if } a_{24} \le a \le \hat{a}_{24},\ Q(\hat{a}_{24}) \cong 0.918836, \\ -Q^4(a) + Q^5(a) + k(\hat{\varepsilon}) \quad \text{if } \hat{a}_{24} \le a \le a_{12},\ Q(a_{12}) \cong 0.853553. \end{cases}$$

Then for $\check{a}_{35} \le a \le \check{a}_{24}$ we need

$$-Q^2(a) + (1 + v^{a_1}_{23})Q^3(a) \le 1 - 2Q(a) + (1 + v^{a_1}_{23})Q^3(a),$$

which always holds.

For $\check{a}_{24} \le a \le \hat{a}_{24}$ we need

$$-Q^2(a) + (1 + v^{a_1}_{23})Q^3(a) \le 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a),$$

i.e.

$$S(Q(a)) = Q^5(a) - (3 + v^{a_1}_{23})Q^3(a) + 3Q^2(a) - 2Q(a) + 1 \ge 0,$$

which is the same as in the duel $(3,5)$, $\langle a\rangle$, Case 4.

For $\hat{a}_{24} \le a \le a_{12}$ we need

$$1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) \ge -Q^4(a) + Q^5(a),$$

which always holds.

Suppose then, finally, that Player I fires neither before nor at $\langle a\rangle + c$. In this case we have

$$K(\hat{\xi};\eta) \le P(a) + Q(a)v^{a_1}_{34} + k(\hat{\varepsilon})$$

for $a \le a_{34}$, $Q(a_{34}) \cong 0.903576$. Then for $\check{a}_{35} \le a \le \check{a}_{24}$ we obtain

$$-1 + (1 + v^{a_1}_{34})Q(a) \le 1 - 2Q(a) + (1 + v^{a_1}_{23})Q^3(a).$$

This inequality is opposite to that at the end of Case 3 and the function $S(Q(a))$ defined there is monotonic for $a_{35} \le a \le \check{a}_{24}$. Thus the inequality holds.

If $\check{a}_{24} \le a \le a_{34}$ we need

$$-1 + (1 + v^{a_1}_{34})Q(a) \le 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a)$$

or

$$S(Q(a)) = Q^5(a) - 2Q^3(a) + 2Q^2(a) - (3 + v^{a_1}_{34})Q(a) + 2 \ge 0.$$

This function is increasing (in a) in the given interval and $S(Q(\check{a}_{24})) = -0.7152 < 0$. Thus the inequality holds also in this case.

13. Results for the duels $(3,5)$

$$\overset{1}{v}{}^{a}_{35} = \begin{cases} P^2(a_{24}) & \text{if } Q(a) \geq Q(a_{35}) \cong 0.980064, \\ -1 + (1 + v^{a_1}_{34})Q(a) & \\ & \text{if } Q(a_{35}) \geq Q(a) \geq Q(a_{34}) \cong 0.903576, \end{cases}$$

$$v^{a}_{35} = \begin{cases} P^2(a_{24}) \cong 0.000184 & \text{if } Q(a) \geq Q(a_{35}), \\ -1 + (1 + v^{a_1}_{34})Q(a) & \text{if } Q(a_{35}) \geq Q(a) \geq Q(\hat{a}_{35}) \cong 0.935980, \\ -Q^2(a) + (1 + v^{a_1}_{23})Q^3(a) & \\ & \text{if } Q(\hat{a}_{35}) \geq Q(a) \geq Q(\hat{a}_{24}) \cong 0.918836, \\ -Q^4(a) + Q^5(a) & \text{if } Q(\hat{a}_{24}) \geq Q(a) \geq Q(a_{34}), \end{cases}$$

$$\overset{2}{v}{}^{a}_{35} = \begin{cases} P^2(a_{24}) & \text{if } Q(a) \geq Q(a_{35}), \\ -1 + (1 + v^{a_1}_{34})Q(a) & \\ & \text{if } Q(a_{35}) \geq Q(a) \geq Q(\check{a}_{35}) \cong 0.948807, \\ 1 - 2Q(a) + (1 + v^{a_1}_{23})Q^3(a) & \\ & \text{if } Q(\check{a}_{35}) \geq Q(a) \geq Q(\check{a}_{24}) \cong 0.933827, \\ 1 - 2Q(a) + 2Q^2(a) - 2Q^3(a) + Q^5(a) & \\ & \text{if } Q(\check{a}_{24}) \geq Q(a) \geq Q(a_{34}). \end{cases}$$

14. Duel $(4,5)$. Consider the duel $(4,5)$, $\langle a \rangle$. We define ξ and η.

Strategy of Player I: If Player II has not fired before, reach the point a_{45}, fire at a^{ε}_{45} and play optimally the duel $(3,5)$, $\langle 1, \rangle a^{\varepsilon}_{45} \langle \wedge c, \rangle a^{\varepsilon}_{45} \langle \rangle$. If he has fired, play optimally the duel $(4,4)$.

Strategy of Player II: If Player I has not fired before, fire at $\langle a_{45} \rangle$ and play optimally the duel $(4,4)$ or $(3,4)$. If he fired (say at a'), play optimally the duel $(3,5)$, $\langle 1, a' \wedge c, a' \rangle$. If Player I has not reached the point a_{45}, do not fire.

Assume that the numbers v^{a}_{45} and a_{45} are related as follows:

$$v^{a}_{45} = P(a_{45}) + Q(a_{45}) \overset{1}{v}{}^{a_{45}}_{35} = -P(a_{45}) + Q(a_{45}) v_{44} \overset{\text{df}}{=} v^{a_1}_{45} . \tag{18}$$

If $0.980064 \geq Q(a_{45}) \geq 0.903576$ we obtain

$$P(a_{45}) + Q(a_{45}) \overset{1}{v}{}^{a_{45}}_{35} = 1 - 2Q(a_{45}) + (1 + v^{a_1}_{34}) Q^2(a_{45}) ,$$

which leads to the equation

$$(1 + v^{a_1}_{34}) Q^2(a_{45}) - (3 + v_{44}) Q(a_{45}) + 2 = 0, \quad Q(a_{45}) \cong 0.919295 . \tag{19}$$

We prove that for $a \leq a_{45}$, a_{45} being the root of equation (19), the strategies ξ and η are optimal in limit and

$$v^{a_1}_{45} = -1 + (1 + v_{44}) Q(a_{45}) \cong 0.023863 \tag{20}$$

is the limit value of the game.

Suppose that Player II fires at $a' < a_{45}$. We have

$$K(\xi; a', \hat{\eta}_0) \geq -P(a') + Q(a')v_{44} - k(\hat{\varepsilon})$$
$$\geq -P(a_{45}) + Q(a_{45})v_{44} - k(\hat{\varepsilon}) = v_{45}^{a_1} - k(\hat{\varepsilon}).$$

Suppose that Player II fires after $\langle a_{45}\rangle + \alpha(\varepsilon)$ or does not fire at all. Then

$$K(\xi; \hat{\eta}) \geq P(a_{45}) + Q(a_{45}) \overset{1}{v}{}_{35}^{a_{45}} - k(\hat{\varepsilon}) = v_{45}^{a_1} - k(\hat{\varepsilon}).$$

On the other hand, if Player I fires before he reaches a_{45}, $a' < a_{45}$, then

$$K(a', \hat{\xi}_0; \eta) \leq P(a') + Q(a') \overset{1}{v}{}_{35}^{a'} + k(\hat{\varepsilon})$$
$$= \begin{cases} 1 - (1 - P^2(a_{24}))Q(a') + k(\hat{\varepsilon}) & \text{if } a' \leq a_{35}, \\ 1 - 2Q(a') + (1 + v_{34}^{a_1})Q^2(a') + k(\hat{\varepsilon}) & \text{if } a_{35} \leq a' \leq a_{34} \end{cases}$$

(see Section 13).

The first function on the right hand side is increasing in a'. The second has its minimum at

$$Q(a') = \frac{1}{1 + v_{34}^{a_1}} \cong 0.979883.$$

Moreover,

$$1 - (1 - P^2(a_{24}))Q(a_{35}) \cong 0.020117 < v_{45}^{a_1},$$
$$1 - 2Q(a_{45}) + (1 + v_{34}^{a_1})Q^2(a_{45}) \cong -1 + (1 + v_{44})Q(a_{45}) = v_{45}^{a_1}$$

by (19) and (20). Thus

$$K(a', \hat{\xi}_0; \eta) \leq v_{45}^{a_1} + k(\hat{\varepsilon}).$$

The rest of the proof is simple. If Player I does not fire before or at $\langle a_{45}\rangle$ then

$$K(\hat{\xi}; \eta) \leq -P(a_{45}) + Q(a_{45})v_{44} + k(\hat{\varepsilon}) = v_{45}^{a_1} + k(\hat{\varepsilon}).$$

If Player I fires at $\langle a_{45}\rangle$ then

$$K(\hat{\xi}; \eta) \leq Q^2(a_{45})v_{34}^{a_1} + k(\hat{\varepsilon}) \cong 0.017350 + k(\hat{\varepsilon}) < v_{45}^{a_1} + k(\hat{\varepsilon}).$$

If Player I neither reaches a_{45} nor fires then

$$K(\hat{\xi}; \eta) = 0 < v_{45}^{a_1}.$$

Finally, notice that if $a \leq a_{45}$ then the same strategies are optimal in limit in the duels $(4,5)$, $\langle 1, a \wedge c, a\rangle$ and $(4,5)$, $\langle 2, a, a \wedge c\rangle$ as well.

15. Duel $(5,5)$. We define ξ and η.

STRATEGY OF PLAYER I: If Player I has not fired before, reach the point a_{55}, fire at a_{55}^{ε} and play optimally the duel $(4,5)$. If he has fired, play optimally the duel $(5,4)$.

Strategy of Player II: If Player I has not fired before, fire at $\langle a_{55} \rangle$ and play optimally the duel $(5,4)$ or $(4,4)$. If he has fired, play optimally the obtained duel $(4,5)$. If he has not reached the point a_{55}, do not fire.

The number a_{55} is determined from the equations

$$v_{55} = P(a_{55}) + Q(a_{55})\overset{1}{v}{}_{45}^{a_{55}} = -P(a_{55}) + Q(a_{55})v_{54}\,.$$

Since $v_{54} \cong 0.194191$ (see [21]) we obtain

$$Q(a_{55}) = \frac{2}{2 + v_{54} - v_{45}^{a_1}} \cong 0.921520\,, \tag{21}$$

which gives

$$v_{55} = -1 + (1 + v_{54})Q(a_{55}) \cong 0.100470\,. \tag{22}$$

The proof that the strategies ξ and η are optimal in limit for $a \le a_{55}$ is omitted.

This ends the analysis of the duel $(m,5)$, $m \le 5$.

The duels $(m,5)$, $5 < m \le 25$ (and some others) are solved by the author in [21].

Noisy duels with retreat after the shots are considered by the author in [14]–[16].

For other noisy duels see [4], [8], [12], [24].

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