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CONNECTEDNESS OF ROW AND COLUMN DESIGNS

Abstract. This paper is concerned with the investigation of connected row and column designs. It is known that a connected row and column design is row-connected and column-connected. For certain classes of row and column designs it is shown that row-connectedness and/or column-connectedness implies connectedness.

1. Introduction. The connectedness of row and column designs has been studied by Shah and Khatri [7], Raghavarao and Federer [6], Eccleston and Russell [3], Sia [8], Baksalary and Kala [1]. It has been shown in [6] that if a design is connected, then it is row-connected and column-connected. Therefore, it is interesting to provide conditions under which a row and column design is connected. Baksalary and Kala [1] give such a condition, but for ordinary designs only. The main aim of the present paper is to examine the connectedness of designs with unequal row sizes and unequal column sizes.

2. Preliminaries. Suppose v treatments are applied to n experimental units arranged in b_1 rows and b_2 columns. Let $\mathbf{N}_1 = [n^1_{ij}]$ be the $v \times b_1$ *row incidence matrix* of the design, n^1_{ij} being the number of units which receive the ith treatment in the jth row, let $\mathbf{N}_2 = [n^2_{ih}]$ be the $v \times b_2$ *column incidence matrix*, n^2_{ih} being the number of units which receive the ith treatment in the hth column, and let $\mathbf{N}_3 = [n^3_{jh}]$ be the $b_1 \times b_2$ *row-column incidence matrix*, n^3_{jh} being the number of units which appear in the jth row and hth column. It is known that $\mathbf{N}_1\mathbf{1} = \mathbf{N}_2\mathbf{1} = \mathbf{r}$, $\mathbf{N}_1'\mathbf{1} = \mathbf{N}_3\mathbf{1} = \mathbf{k}_1$, $\mathbf{N}_2'\mathbf{1} = \mathbf{N}_3'\mathbf{1} = \mathbf{k}_2$, where $\mathbf{r}$ is the vector of replications, $\mathbf{k}_1$ the vector of

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row sizes, $\mathbf{k}_2$ the vector of column sizes, and $\mathbf{1}$ is the vector of ones, of appropriate dimension.

The properties of a row and column design can be considered by examining the matrices

$$\begin{aligned} &\mathbf{C}_1 = \mathbf{R} - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_1', \\ &\mathbf{C}_2 = \mathbf{R} - \mathbf{N}_2\mathbf{K}_2^{-1}\mathbf{N}_2', \\ (1)\qquad &\mathbf{C}_3 = \mathbf{K}_2 - \mathbf{N}_3'\mathbf{K}_1^{-1}\mathbf{N}_3, \\ &\mathbf{C}_0 = \mathbf{R} - \mathbf{r}\mathbf{r}'/n, \end{aligned}$$

$$(2)\qquad \mathbf{C} = \mathbf{C}_1 - (\mathbf{N}_2 - \mathbf{N}_1\mathbf{K}_1^{-1}\mathbf{N}_3)\mathbf{C}_3^-(\mathbf{N}_2' - \mathbf{N}_3'\mathbf{K}_1^{-1}\mathbf{N}_1'),$$

where

$$\mathbf{R} = \mathrm{diag}[r_1, \ldots, r_v], \quad \mathbf{K}_1 = \mathrm{diag}[k_{11}, \ldots, k_{1b_1}], \quad \mathbf{K}_2 = \mathrm{diag}[k_{21}, \ldots, k_{2b_2}],$$

and $\mathbf{C}_3^-$ denotes the generalized inverse of the matrix $\mathbf{C}_3$. An equivalent formula for $\mathbf{C}$ can be obtained by replacing $\mathbf{N}_1$ by $\mathbf{N}_2$, $\mathbf{N}_2$ by $\mathbf{N}_1$, $\mathbf{N}_3$ by $\mathbf{N}_3'$, $\mathbf{K}_1$ by $\mathbf{K}_2$ and $\mathbf{C}_3$ by $\mathbf{C}_4 = \mathbf{K}_1 - \mathbf{N}_3\mathbf{K}_2^{-1}\mathbf{N}_3'$. Let λ_{1i} be an eigenvalue of the matrix $\mathbf{C}_1$ with respect to $\mathbf{R}$. Let λ_{2i} be an eigenvalue of $\mathbf{C}_2$ with respect to $\mathbf{R}$ and let λ_i be an eigenvalue of $\mathbf{C}$ with respect to $\mathbf{R}$. It is known that all eigenvalues of $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}$ with respect to $\mathbf{R}$ belong to the interval $[0, 1]$ (see e.g. [5]). The design is said to be *row-connected* if the rank of $\mathbf{C}_1$ is $v - 1$, $r(\mathbf{C}_1) = v - 1$; *column-connected* if $r(\mathbf{C}_2) = v - 1$; and *connected* if $r(\mathbf{C}) = v - 1$. A row and column design is said to be *ordinary* if the row-column incidence matrix satisfies $\mathbf{N}_3 = \mathbf{1}\mathbf{1}'$. In this case $\mathbf{k}_1 = b_2\mathbf{1}$ and $\mathbf{k}_2 = b_1\mathbf{1}$. If $\mathbf{N}_3$ can be expressed as

$$\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$$

then the matrix $\mathbf{C}_3$, defined in (1), takes the form $\mathbf{C}_3 = \mathbf{K}_2 - \mathbf{k}_2\mathbf{k}_2'/n$, and it can be easily checked that $\mathbf{K}_2^{-1}$ is its generalized inverse. In consequence the matrix $\mathbf{C}$, given in (2), reduces to the form

$$(3)\qquad \mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_0 .$$

3. Results. Consider a row and column design with incidence matrices $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ satisfying for $i = 1$ or $i = 2$ the condition

$$(4)\qquad \mathbf{N}_i = \mathbf{N}_j\mathbf{K}_j^{-1}\mathbf{N}_{3i}$$

where $i \neq j$, $j = 1, 2$ and $\mathbf{N}_{31} = \mathbf{N}_3'$ and $\mathbf{N}_{32} = \mathbf{N}_3$. Eccleston and Russell [3] show that this design is connected if and only if $r(\mathbf{C}_1) = v - 1$ and $r(\mathbf{C}_i^*) = b_i - 1$, where $\mathbf{C}_i^* = \mathbf{K}_i - \mathbf{N}_i'\mathbf{R}^{-1}\mathbf{N}_i$, $i = 1, 2$.

THEOREM 1. *A row and column design with incidence matrices satisfying* (4) *for $i = 1$ or $i = 2$ is connected if and only if it is column-connected or row-connected respectively.*

Proof. By (2) and (4) we have $\mathbf{C} = \mathbf{C}_2$ for $i = 1$ and $\mathbf{C} = \mathbf{C}_1$ for $i = 2$, which completes the proof.

Now we consider a row and column design with row-column incidence matrix $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$ (see [4]). Let the incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ satisfy, for $i = 1$ or $i = 2$, the relation

$$\mathbf{N}_i'\mathbf{R}^{-1}\mathbf{N}_j = \mathbf{k}_i\mathbf{k}_j'/n\,, \quad j \neq i\,,\ j = 1,2. \tag{5}$$

In the special case when $\mathbf{N}_3 = \mathbf{1}\mathbf{1}'$ and $\mathbf{r} = r\mathbf{1}$ the condition (5) describes the class of row and column designs considered in [2].

THEOREM 2. *If a row and column design with row-column incidence matrix $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$ satisfies (5) for $i = 1$ or $i = 2$, then the design is connected if and only if it is row-connected and column-connected.*

Proof. Since $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$, the matrix $\mathbf{C}$ has the form (3). From (5) for $i = 1$ or $i = 2$, it is easily seen that the matrices $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2$, $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_0$ and $\mathbf{C}_2\mathbf{R}^{-1}\mathbf{C}_0$ are symmetric. Hence $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_0$ have a common set of eigenvectors with respect to the matrix $\mathbf{R}$. Moreover, it can be verified that each of these matrices has eigenvalue zero with respect to $\mathbf{R}$, and that this eigenvalue corresponds to the same eigenvector for all three matrices. The other eigenvalue of $\mathbf{C}_0$ with respect to $\mathbf{R}$ is 1 and it appears with multiplicity $v - 1$. Thus, in view of (3) the eigenvalues of $\mathbf{C}$ with respect to $\mathbf{R}$ have the form $\lambda_h = \lambda_{1h} + \lambda_{2h} - 1$, $h = 1,\ldots,v-1$, where λ_{1h} and λ_{2h} are the eigenvalues of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively with respect to $\mathbf{R}$. It can be easily seen that $(\mathbf{C}_0 - \mathbf{C}_1)\mathbf{R}^{-1}(\mathbf{C}_0 - \mathbf{C}_2) = \mathbf{0}$. It follows that for each $h = 1,\ldots,v-1$, either λ_{1h} or λ_{2h} (or both) is equal to 1. This completes the proof.

If a row or column incidence matrix is of the form $\mathbf{N}_i = \mathbf{r}\mathbf{k}_i'/n$, $i = 1,2$, then (5) is satisfied and hence from Theorem 2 we have

COROLLARY 1. *A row and column design with row-column incidence matrix $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$ and with $\mathbf{N}_i = \mathbf{r}\mathbf{k}_i'/n$ $(i = 1,2)$ is connected if and only if it is column-connected or row-connected, respectively.*

Consider now row and column designs for which $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$ and $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2$ is a symmetric matrix.

THEOREM 3. *If for a row and column design with row-column incidence matrix $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$ the matrix $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2$ is symmetric then the design is connected if and only if it is row-connected and column-connected.*

Proof. Since $\mathbf{C}$ is of the form (3) and $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2$, $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_0$ and $\mathbf{C}_2\mathbf{R}^{-1}\mathbf{C}_0$ are symmetric, the eigenvalues of $\mathbf{C}$ with respect to $\mathbf{R}$ are $\lambda_h = \lambda_{1h} + \lambda_{2h} - 1$, $h = 1,\ldots,v-1$. Hence, for each h, $\lambda_{1h} + \lambda_{2h} > 1$. This holds if and only if $\lambda_{1h} > 0$ and $\lambda_{2h} > 0$.

If the non-zero eigenvalues of the matrix $\mathbf{C}_1$ with respect to $\mathbf{R}$ are all equal to λ_1, then the design is said to be *row-balanced*. If the non-zero eigenvalues of $\mathbf{C}_2$ with respect to $\mathbf{R}$ are all equal to λ_2, then the design is said to be *column-balanced*. Finally, if the non-zero eigenvalues of $\mathbf{C}$ with respect to $\mathbf{R}$ are all equal to λ, then the design is said to be *balanced*. It can be shown that a row-connected row and column design is row-balanced if and only if its matrix $\mathbf{C}_1$ is of the form $\mathbf{C}_1 = \lambda_1(\mathbf{R} - \mathbf{r}\mathbf{r}'/n)$. Similarly, a column-connected row and column design is column-balanced if and only if $\mathbf{C}_2$ is of the form $\mathbf{C}_2 = \lambda_2(\mathbf{R} - \mathbf{r}\mathbf{r}'/n)$. Hence, if $\mathbf{C}_1$ and $\mathbf{C}_2$ are of the form given above, then $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2$ is also symmetric. In addition, it has been shown in [6] that if a row and column design is connected, then it is row-connected and column-connected. Using these notions we can formulate the following corollary:

COROLLARY 2. *If a row and column design with row-column incidence matrix* $\mathbf{N}_3 = \mathbf{k}_1\mathbf{k}_2'/n$ *is row-balanced and column-balanced then it is connected if and only if it is row-connected and column-connected.*

4. Examples. First we discuss the design with incidence matrices

$$\mathbf{N}_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \qquad \mathbf{N}_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

$$\mathbf{N}_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

For this design $\mathbf{N}_2 = \mathbf{N}_1\mathbf{N}_3/3$. Hence the condition (4) of Theorem 1 is satisfied. Moreover, since $r(\mathbf{C}_1) = 2$, the design is row-connected, and hence connected by Theorem 1.

Consider the row and column design with

$$\mathbf{N}_3 = \mathbf{N}_1' = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

These incidence matrices satisfy the conditions of Corollary 1. The column incidence matrix $\mathbf{N}_2$ may be of the form

$$\mathbf{N}_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

Since $r(\mathbf{C}_2) = 2$, the design is column-connected and hence connected.

Now consider the row and column design with incidence matrices

$$\mathbf{N}_1 = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 1 & 4 \end{pmatrix}, \qquad \mathbf{N}_2 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \qquad \mathbf{N}_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}.$$

This design is non-ordinary and thus neither the criterion of Sia [8] nor that of Baksalary and Kala [1] may be used to decide whether the design is connected or not. Observe, however, that this design satisfies the necessary conditions for connectedness, i.e. it is row-connected and column-connected. Since

$$\mathbf{C}_1 = \frac{1}{5}\begin{pmatrix} 14 & -7 & -7 \\ -7 & 16 & -9 \\ -7 & -9 & 16 \end{pmatrix}, \quad \mathbf{C}_2 = \frac{1}{3}\begin{pmatrix} 10 & -5 & -5 \\ -5 & 9 & -4 \\ -5 & -4 & 9 \end{pmatrix},$$

and, in consequence, $\mathbf{C}_1\mathbf{R}^{-1}\mathbf{C}_2$ is a symmetric matrix, Theorem 3 of the present paper is applicable.

References

[1] J. K. Baksalary and R. Kala, *On connectedness of ordinary two-way elimination of heterogeneity designs*, Biom. J. 22 (1980), 105–109.

[2] J. Eccleston and J. Kiefer, *Relationships of optimality for individual factors of a design*, J. Statist. Planning and Inference 5 (1981), 213–219.

[3] J. Eccleston and K. Russell, *Connectedness and orthogonality in multi-factor designs*, Biometrika 62 (1975), 341–345.

[4] S. Pal, *On designs with one-way and two-way elimination of heterogeneity*, Calcutta Statist. Assoc. Bull. 26 (1977), 79–103.

[5] J. Przybyłowski and R. Walkowiak, *The $r^{-\delta}$-balance in row-column designs with special respect to the ordinary designs*, in: XI Methodological Colloquium on Agro-Biometry in Kielce, 1981, 264–283 (in Polish).

[6] D. Raghavarao and W. T. Federer, *On connectedness in two-way elimination of heterogeneity designs*, Ann. Statist. 3 (1975), 730–735.

[7] K.R. Shah and C. G. Khatri, *Connectedness in row-column designs*, Comm. Statist. 2 (1973), 571–573.

[8] L.L. Sia, *Some properties of connectedness in two-way designs*, Comm. Statist. -Theor. Math. A6 (1977), 1165–1169.

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