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CONNECTEDNESS OF ROW AND COLUMN DESIGNS

Abstract. This paper is concerned with the investigation of connected row and column designs. It is known that a connected row and column design is row-connected and column-connected. For certain classes of row and column designs it is shown that row-connectedness and/or column-connectedness implies connectedness.

1. Introduction. The connectedness of row and column designs has been studied by Shah and Khatri [7], Raghava Rao and Federer [6], Eccleston and Russell [3], Sia [8], Baksalary and Kala [1]. It has been shown in [6] that if a design is connected, then it is row-connected and column-connected. Therefore, it is interesting to provide conditions under which a row and column design is connected. Baksalary and Kala [1] give such a condition, but for ordinary designs only. The main aim of the present paper is to examine the connectedness of designs with unequal row sizes and unequal column sizes.

2. Preliminaries. Suppose $v$ treatments are applied to $n$ experimental units arranged in $b_1$ rows and $b_2$ columns. Let $N_1 = [n_{ij}^1]$ be the $v \times b_1$ row incidence matrix of the design, $n_{ij}^1$ being the number of units which receive the $i$th treatment in the $j$th row, let $N_2 = [n_{ih}^2]$ be the $v \times b_2$ column incidence matrix, $n_{ih}^2$ being the number of units which receive the $i$th treatment in the $h$th column, and let $N_3 = [n_{jh}^3]$ be the $b_1 \times b_2$ row-column incidence matrix, $n_{jh}^3$ being the number of units which appear in the $j$th row and $h$th column. It is known that $N_1 1 = N_2 1 = r$, $N_1' 1 = N_3 1 = k_1$, $N_2' 1 = N_3' 1 = k_2$, where $r$ is the vector of replications, $k_1$ the vector of

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row sizes, $k_2$ the vector of column sizes, and $1$ is the vector of ones, of appropriate dimension.

The properties of a row and column design can be considered by examining the matrices

\[
\begin{align*}
C_1 &= R - N_1 K_1^{-1} N_1', \\
C_2 &= R - N_2 K_2^{-1} N_2', \\
C_3 &= K_2 - N_3 K_1^{-1} N_3, \\
C_0 &= R - r r'/n, \\
C &= C_1 - (N_2 - N_1 K_1^{-1} N_3) C_3 (N_2' - N_3' K_1^{-1} N_1'),
\end{align*}
\]

where

\[
R = \text{diag}[r_1, \ldots, r_v], \quad K_1 = \text{diag}[k_{11}, \ldots, k_{1b_1}], \quad K_2 = \text{diag}[k_{21}, \ldots, k_{2b_2}],
\]

and $C_3^{-}$ denotes the generalized inverse of the matrix $C_3$. An equivalent formula for $C$ can be obtained by replacing $N_1$ by $N_2$, $N_2$ by $N_1$, $N_3$ by $N_3'$, $K_1$ by $K_2$ and $C_3$ by $C_4 = K_1 - N_3 K_2^{-1} N_3'$. Let $\lambda_i$ be an eigenvalue of the matrix $C_1$ with respect to $R$. Let $\lambda_i'$ be an eigenvalue of $C_2$ with respect to $R$ and let $\lambda_i$ be an eigenvalue of $C$ with respect to $R$. It is known that all eigenvalues of $C_1$, $C_2$, $C$ with respect to $R$ belong to the interval $[0,1]$ (see e.g. [5]). The design is said to be row-connected if the rank of $C_1$ is $v - 1$, $r(C_1) = v - 1$; column-connected if $r(C_2) = v - 1$; and connected if $r(C) = v - 1$. A row and column design is said to be ordinary if the row-column incidence matrix satisfies $N_3 = 11'$. In this case $k_1 = b_21$ and $k_2 = b_11$. If $N_3$ can be expressed as

\[
N_3 = k_1 k_2' / n
\]

then the matrix $C_3$, defined in (1), takes the form $C_3 = K_2 - k_2 k_2' / n$, and it can be easily checked that $K_2^{-1}$ is its generalized inverse. In consequence the matrix $C$, given in (2), reduces to the form

\[
C = C_1 + C_2 - C_0.
\]

3. Results. Consider a row and column design with incidence matrices $N_1$, $N_2$ and $N_3$ satisfying for $i = 1$ or $i = 2$ the condition

\[
N_i = N_j K_j^{-1} N_{3i}
\]

where $i \neq j$, $j = 1, 2$ and $N_{31} = N_3'$ and $N_{32} = N_3$. Eccleston and Russell [3] show that this design is connected if and only if $r(C_1) = v - 1$ and $r(C_2') = b_1 - 1$, where $C_2' = K_i - N_i' R^{-1} N_i$, $i = 1, 2$.

**Theorem 1.** A row and column design with incidence matrices satisfying (4) for $i = 1$ or $i = 2$ is connected if and only if it is column-connected or row-connected respectively.
Proof. By (2) and (4) we have $C = C_2$ for $i = 1$ and $C = C_1$ for $i = 2$, which completes the proof.

Now we consider a row and column design with row-column incidence matrix $N_3 = k_1 k_2 / n$ (see [4]). Let the incidence matrices $N_1$ and $N_2$ satisfy, for $i = 1$ or $i = 2$, the relation

$$N_i' R^{-1} N_j = k_i k_j' / n, \quad j \neq i, \quad j, 1, 2.$$  

(5)

In the special case when $N_3 = 11'$ and $r = r1$ the condition (5) describes the class of row and column designs considered in [2].

**Theorem 2.** If a row and column design with row-column incidence matrix $N_3 = k_1 k_2 / n$ satisfies (5) for $i = 1$ or $i = 2$, then the design is connected if and only if it is row-connected and column-connected.

**Proof.** Since $N_3 = k_1 k_2 / n$, the matrix $C$ has the form (3). From (5) for $i = 1$ or $i = 2$, it is easily seen that the matrices $C_1 R^{-1} C_2$, $C_1 R^{-1} C_0$ and $C_2 R^{-1} C_0$ are symmetric. Hence $C_1$, $C_2$ and $C_0$ have a common set of eigenvectors with respect to the matrix $R$. Moreover, it can be verified that each of these matrices has eigenvalue zero with respect to $R$, and that this eigenvalue corresponds to the same eigenvector for all three matrices. The other eigenvalue of $C_0$ with respect to $R$ is 1 and it appears with multiplicity $v - 1$. Thus, in view of (3) the eigenvalues of $C$ with respect to $R$ have the form $\lambda_h = \lambda_{1h} + \lambda_{2h} - 1, \quad h = 1, \ldots, v - 1$, where $\lambda_{1h}$ and $\lambda_{2h}$ are the eigenvalues of $C_1$ and $C_2$ respectively with respect to $R$. It can be easily seen that $(C_0 - C_1) R^{-1} (C_0 - C_2) = 0$. It follows that for each $h = 1, \ldots, v - 1$, either $\lambda_{1h}$ or $\lambda_{2h}$ (or both) is equal to 1. This completes the proof.

If a row or column incidence matrix is of the form $N_i = rk_i' / n$, $i = 1, 2$, then (5) is satisfied and hence from Theorem 2 we have

**Corollary 1.** A row and column design with row-column incidence matrix $N_3 = k_1 k_2 / n$ and with $N_i = rk_i' / n$ ($i = 1, 2$) is connected if and only if it is column-connected or row-connected, respectively.

Consider now row and column designs for which $N_3 = k_1 k_2 / n$ and $C_1 R^{-1} C_2$ is a symmetric matrix.

**Theorem 3.** If for a row and column design with row-column incidence matrix $N_3 = k_1 k_2 / n$ the matrix $C_1 R^{-1} C_2$ is symmetric then the design is connected if and only if it is row-connected and column-connected.

**Proof.** Since $C$ is of the form (3) and $C_1 R^{-1} C_2$, $C_1 R^{-1} C_0$ and $C_2 R^{-1} C_0$ are symmetric, the eigenvalues of $C$ with respect to $R$ are $\lambda_h = \lambda_{1h} + \lambda_{2h} - 1, \quad h = 1, \ldots, v - 1$. Hence, for each $h$, $\lambda_{1h} + \lambda_{2h} > 1$. This holds if and only if $\lambda_{1h} > 0$ and $\lambda_{2h} > 0$. 


If the non-zero eigenvalues of the matrix $C_1$ with respect to $R$ are all equal to $\lambda_1$, then the design is said to be row-balanced. If the non-zero eigenvalues of $C_2$ with respect to $R$ are all equal to $\lambda_2$, then the design is said to be column-balanced. Finally, if the non-zero eigenvalues of $C$ with respect to $R$ are all equal to $\lambda$, then the design is said to be balanced. It can be shown that a row-connected row and column design is row-balanced if and only if its matrix $C_1$ is of the form $C_1 = \lambda_1(R - rr'}/n)$. Similarly, a column-connected row and column design is column-balanced if and only if $C_2$ is of the form $C_2 = \lambda_2(R - rr'}/n)$. Hence, if $C_1$ and $C_2$ are of the form given above, then $C_1R^{-1}C_2$ is also symmetric. In addition, it has been shown in [6] that if a row and column design is connected, then it is row-connected and column-connected. Using these notions we can formulate the following corollary:

**Corollary 2.** If a row and column design with row-column incidence matrix $N_3 = k_1k_2'/n$ is row-balanced and column-balanced then it is connected if and only if it is row-connected and column-connected.

4. Examples. First we discuss the design with incidence matrices

$$N_1 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

For this design $N_2 = N_1N_3/3$. Hence the condition (4) of Theorem 1 is satisfied. Moreover, since $r(C_1) = 2$, the design is row-connected, and hence connected by Theorem 1.

Consider the row and column design with

$$N_3 = N_1 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

These incidence matrices satisfy the conditions of Corollary 1. The column incidence matrix $N_2$ may be of the form

$$N_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

Since $r(C_2) = 2$, the design is column-connected and hence connected.
Now consider the row and column design with incidence matrices

\[ N_1 = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \end{pmatrix}. \]

This design is non-ordinary and thus neither the criterion of Sia [8] nor that of Baksalary and Kala [1] may be used to decide whether the design is connected or not. Observe, however, that this design satisfies the necessary conditions for connectedness, i.e. it is row-connected and column-connected. Since

\[ C_1 = \frac{1}{5} \begin{pmatrix} 14 & -7 & -7 \\ -7 & 16 & -9 \\ -7 & -9 & 16 \end{pmatrix}, \quad C_2 = \frac{1}{3} \begin{pmatrix} 10 & -5 & -5 \\ -5 & 9 & -4 \\ -5 & -4 & 9 \end{pmatrix}, \]

and, in consequence, \( C_1 R^{-1} C_2 \) is a symmetric matrix, Theorem 3 of the present paper is applicable.

References


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