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STATISTICAL PROPERTIES OF THE EJGIELIES MODEL OF A COGGED BIT

Abstract. We study the motion of cogged bits in the rotary drilling of hard rock with a high rotational speed. When a cogged bit rotates in a drilling fluid, it experiences random checks to its motion. For this reason we consider a probabilistic model in which the motion of the bit is treated as a realization of some dynamical system with a multiplicative perturbation. Our aim is to give the conditions which should be satisfied by the perturbation for the asymptotical stability of the model investigated.

1. The aim of the present investigations was to examine statistical properties of the random trajectory of the dynamical system describing the motion of a drilling tool. As in the paper of Lasota and Rusek [7] the theoretical considerations are based on the R. M. Ejgielies model of a cogged bit. He proposed replacing the whole tool with a cogged wheel turning over a flat base (Fig. 1).

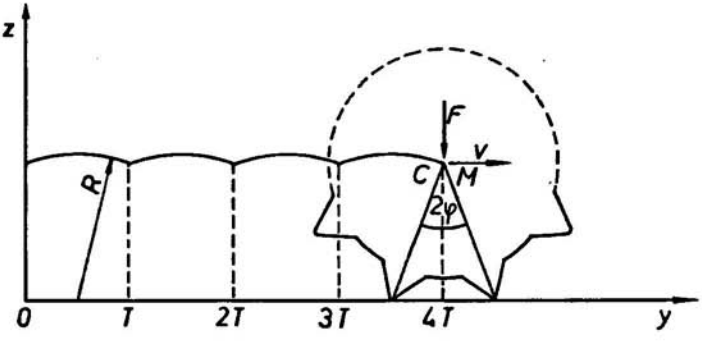

Fig. 1. The Ejgielies model of a cogged bit

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We assume that the cogged wheel has diameter $2R$, mass M and turns with linear velocity v. The central angle between the cutting edges of neighbouring cogs is 2φ. The cogged wheel is pressed against the base with force F. A significant role in further considerations is played by the quantity $\lambda = v^2M/(FR)$ which is the Froude number for the system considered. The curve described by the arcs of circles of radii R and centres at distances of $T = 2R\sin\varphi$ is called the *basic curve* and denoted by $\overline{z} = p(y)$. For $\lambda > 1$, in the time between two successive contacts of the cogs with the base, the wheel centre C moves above the basic curve according to the equation

$$d^2z/dy^2 = -F/(Mv^2). \tag{1}$$

At the initial point $y = y_0$ $(0 \le y_0 < T)$ the solution $z(y)$ of equation (1) satisfies the conditions

$$z(y_0) = p(y_0), \quad z'(y_0) = p'(y_0).$$

Let us denote by y_n the successive points of the impact of the tool against the bottom of the bore-hole. The points y_n are called *nodal points* of the trajectory (see Fig. 2). In the intervals (y_i, y_{i+1}) the trajectory $z = z(y)$ is a solution of equation (1) and at the points y_i it is tangential to the basic curve $\overline{z} = p(y)$.

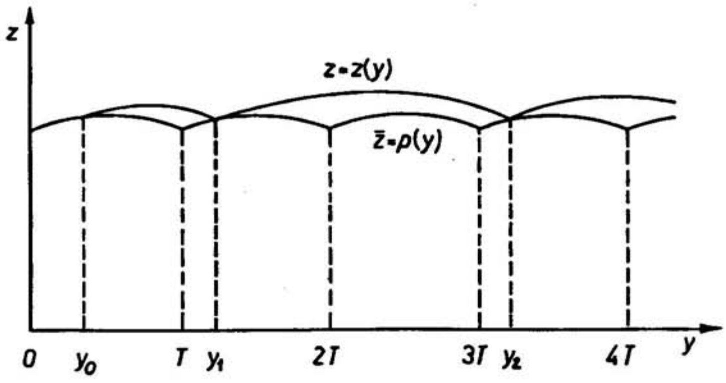

Fig. 2. Determination of the sequence of nodal points $\{y_n\}$

In [7] it was shown that all the properties of the Ejgielies model that are interesting from the technical point of view may be described by the sequence

$$s_n = y_n/T \pmod 1. \tag{2}$$

This sequence may be expressed by the recurrence formula

$$s_{n+1} = T_\lambda(s_n), \tag{3}$$

where

$$T_\lambda(s) = s + r_\lambda(s) - I(s + r_\lambda(s)), \quad 0 \le s \le 1. \tag{4}$$

In this formula

$$r_\lambda(s) = \alpha q(s) - (\alpha^2 q(s)^2 + 2\alpha q(s)(s - I(s)) - \alpha q(s)(1 + q(s)))^{1/2},$$

$$\alpha = \frac{\lambda}{\lambda - 1}, \quad q(s) = 1 + I\left(\frac{1 - 2(s - I(s))}{\alpha - 1}\right),$$

and $I(s)$ denotes the integer part of the number s ($I(s)$ equals 0 for $s < 1$ and $I(s)$ is the largest integer less than or equal to s for $s \ge 1$). A. Lasota and P. Rusek showed that for a small Froude number ($\lambda < 2$) the motion of the system is periodic and stable. For $\lambda > 2$ the system does not have stable trajectories; however, in this case there exists an absolutely continuous invariant measure. From the theoretical point of view a basic drawback in the theory formulated by A. Lasota and P. Rusek is the lack of proof of the uniqueness and ergodicity of the invariant measure. This makes it impossible to check whether the mean values they found associated with formula (3) are uniquely determined. A proof of the ergodicity of the system considered would present a considerable difficulty. From the practical point of view it may also be objected that the model presented by A. Lasota and P. Rusek does not take into account random perturbations. The cogged bit rotating in the drilling fluid environment and hitting against a base covered with drill borings has a motion subject to random deceleration. This usually leads to a reduction in the real values of s_n relative to the values calculated from formula (2). Hence it would appear to be useful to replace (3) by the sequence $s_{n+1} = T_\lambda(s_n)\xi_n$, where ξ_n is a random variable with values in the interval $[0, 1]$.

It turns out that taking this more realistic model simultaneously allows certain theoretical difficulties to be eliminated. We shall show that the model considered is asymptotically stable, which is a much stronger property than ergodicity.

Dynamical systems with stochastic perturbations may often be viewed as special cases of Markov processes; there is a large applied literature concerning their stability properties [5].

In [2] and [3] we gave sufficient conditions for the asymptotical stability of a Markov operator governing the evolution of densities corresponding to a dynamical system with multiplicative perturbations. However, the transformation T_λ given by (4) does not satisfy those conditions.

2. Consider a stochastically perturbed discrete time dynamical system of the form

$$x_{n+1} = T_\lambda(x_n)\xi_n \quad \text{for } n = 0, 1, \ldots, \tag{5}$$

where T_λ is the transformation given by (4) and ξ_n is a random variable with values in $[0,1]$.

We assume that the random variables ξ_n are independent and all identically distributed with density g, i.e., for all n and a Borel set B,

$$\Pr(\xi_n \in B) = \int_B g(x)\,dx\,.$$

In addition, we assume that the initial condition x_0 is independent of the sequence of perturbations $\{\xi_n\}$.

Let D be the set of all nonnegative functions $f \in L^1([0,1])$ such that $\|f\| = \int_0^1 f(x)\,dx = 1$.

Our goal is to study the asymptotic behaviour of the sequence $\{x_n\}$. Since the ξ_n are random, our strategy is to study the sequence of distributions of x_n. Denote by f_n the density of the distribution of x_n. In order to calculate f_{n+1} from f_n denote by h an arbitrary bounded measurable function defined on $[0,1]$. The mean value $\mathrm{E}(h(x_{n+1}))$ of $h(x_{n+1})$ is evidently given by

$$\mathrm{E}(h(x_{n+1})) = \int_0^1 h(x) f_{n+1}(x)\,dx\,.$$

Since $x_{n+1} = T_\lambda(x_n)\xi_n$ and the random variables x_n and ξ_n are independent, we also have

$$\mathrm{E}(h(x_{n+1})) = \int_0^1 \int_0^1 h(T_\lambda(y)z) f_n(y) g(z)\,dy\,dz\,. \tag{6}$$

Furthermore, because the adjoint operator $P^*_{T_\lambda}$ to the Frobenius–Perron operator P_{T_λ} is given by $P^*_{T_\lambda} f = f \circ T_\lambda$ (cf. [6]), setting $h_z(y) = h(yz)$ we obtain

$$\begin{aligned} \int_0^1 h(T_\lambda(y)z) f_n(y)\,dy &= \int_0^1 h_z(T_\lambda(y)) f_n(y)\,dy \\ &= \int_0^1 P_{T_\lambda} f_n(y) h_z(y)\,dy = \int_0^1 P_{T_\lambda} f_n(y) h(yz)\,dy\,. \end{aligned} \tag{7}$$

Using (6) and (7) it is easy to calculate that

$$\begin{aligned} \mathrm{E}(h(x_{n+1})) &= \int_0^1 \int_0^1 P_{T_\lambda} f_n(y) h(yz) g(z)\,dy\,dz \\ &= \int_0^1 P_{T_\lambda} f_n(y) \int_0^y h(z) g(z/y) y^{-1}\,dz\,dy \end{aligned}$$

$$= \int_0^1 h(z) \int_z^1 P_{T_\lambda} f_n(y) g(z/y) y^{-1}\, dy\, dz\,.$$

Since h is arbitrary, we obtain the following relation between f_{n+1} and f_n:

(8) $$f_{n+1}(x) = \int_x^1 P_{T_\lambda} f_n(y) g(x/y) y^{-1}\, dy\,.$$

Thus, given an arbitrary initial density f_0, the evolution of densities corresponding to the system (5) is described by the sequence of iterates $\{P^n f_0\}$, where

(9) $$Pf(x) = \int_x^1 P_{T_\lambda} f(y) g(x/y) y^{-1}\, dy\,, \quad f \in L^1([0,1])\,.$$

It is easy to prove that $P : L^1([0,1]) \to L^1([0,1])$ is a Markov operator (i.e. $Pf \geq 0$ and $\|Pf\| = \|f\|$ for $f \in L^1([0,1])$ and $f \geq 0$).

3. We say that a Markov operator P is *asymptotically stable* if there exists a unique $f_* \in D$ such that $Pf_* = f_*$ and

$$\lim_{n\to+\infty} \|P^n f - f_*\| = 0 \quad \text{for every } f \in D\,.$$

Our first step in the study of asymptotic stability of the Markov operator P is to show that P is weakly constrictive. By definition, an operator P is *weakly constrictive* if there exists a weakly precompact set $\mathcal{F} \subset L^1([0,1])$ such that

(10) $$\lim_{n\to+\infty} \varrho(P^n f, \mathcal{F}) = 0 \quad \text{for } f \in D\,,$$

where $\varrho(f, \mathcal{F})$ denotes the distance, in $L^1([0,1])$-norm, between the element f and the set $\mathcal{F}$. An answer to the problem considered is given by the following

THEOREM 1. *Assume that the density g of the random variables ξ_n satisfies the condition*

(11) $$g(x) \leq Kx^r \quad \textit{for } x \in (0,1)\,,$$

where K and r are positive constants. Then the Markov operator P defined by (9) *is weakly constrictive.*

In the proof of Theorem 1 we shall use the following

LEMMA 1. *Let $f \in L^1([0,1])$ be given by the equality $f(x) = x^r \omega(x)$, $x \in [0,1]$, where ω is a nonnegative, nonincreasing function and r is a nonnegative constant. Then*

$$f(x) \leq \|f\|(r+1)/x \quad \textit{for } x \in (0,1)\,.$$

The proof of this lemma may be found in [2]. ∎

Proof of Theorem 1. For an arbitrary function $h \in L^1([0,1])$, denote by $\mathcal{F}_h$ the set of densities f satisfying $f(x) \le h(x)$ for $x \in (0,1)$. Evidently, $\mathcal{F}_h$ is a weakly precompact set. We are going to find an h such that condition (10) holds for $\mathcal{F} = \mathcal{F}_h$.

We consider two cases: the first where the Froude number $\lambda > 2$ and the second where $1 < \lambda \le 2$.

In the case where $\lambda > 2$, the transformation T_λ given by (4) may be written in the form

$$T_\lambda(s) = \begin{cases} s + \alpha k(\lambda) - k(\lambda) - \sqrt{\alpha^2 k(\lambda)^2 + 2\alpha k(\lambda) s - \alpha k(\lambda)(1 + k(\lambda))} \\ \qquad\qquad \text{for } 0 \le s \le \dfrac{\lambda - k(\lambda)}{2(\lambda - 1)}, \\ s + \alpha p - p - \sqrt{\alpha^2 p^2 + 2\alpha p s - \alpha p(1+p)} \\ \qquad \text{for } \dfrac{\lambda - p - 1}{2(\lambda - 1)} < s \le \dfrac{\lambda - p}{2(\lambda - 1)},\ p = 2, 3, \ldots, k(\lambda) - 1, \\ s + \alpha - 1 - \sqrt{\alpha^2 + 2\alpha s - 2\alpha} \quad \text{for } s > \frac{\lambda - 2}{2(\lambda - 1)}, \end{cases}$$

where

$$k(\lambda) = \begin{cases} \lambda - 1 & \text{when } \lambda \text{ is an integer,} \\ I(\lambda) & \text{when } \lambda \text{ is not an integer.} \end{cases}$$

Set

$$T_{\lambda,p} = T_\lambda\,|_U, \quad U = \left[\frac{\lambda - p - 1}{2(\lambda - 1)}, \frac{\lambda - p}{2(\lambda - 1)}\right], \quad p = 2, 3, \ldots, k(\lambda) - 1,$$

$$T_{\lambda,k(\lambda)} = T_\lambda\,|_V, \quad V = \left[0, \frac{\lambda - k(\lambda)}{2(\lambda - 1)}\right],$$

where $T_\lambda\,|_W$ denotes the restriction of T_λ to the interval W.

The Frobenius–Perron operator P_{T_λ} corresponding to the transformation T_λ may be expressed by the formula (cf. [6])

$$\begin{aligned} P_{T_\lambda} f(y) &= f(T^{-1}_{\lambda,k(\lambda)}(y)) \frac{1_{T_{\lambda,k(\lambda)}(V)}(y)}{|T'_{\lambda,k(\lambda)}(T^{-1}_{\lambda,k(\lambda)}(y))|} \\ &\quad + \sum_{p=2}^{k(\lambda)-1} f(T^{-1}_{\lambda,p}(y)) \frac{1}{|T'_{\lambda,p}(T^{-1}_{\lambda,p}(y))|} 1_{T_{\lambda,p}(U)}(y) \\ &\quad + f(1 + y - \sqrt{2\alpha y}) \left|1 - \sqrt{\frac{\alpha}{2y}}\right| 1_{(0,\alpha/2]}(y), \end{aligned} \tag{12}$$

where 1_A is the characteristic function of the set A.

Fix an arbitrary initial density f and set $f_n = P^n f$ for $n = 0, 1, \ldots$ Using (8) and (12) we may derive an explicit formula for f_{n+1}:

$$
\begin{aligned}
(13)\quad f_{n+1}(x) = & \int_x^1 f_n(T^{-1}_{\lambda,k(\lambda)}(y)) \frac{1_{T_{\lambda,k(\lambda)}(V)}(y)}{|T'_{\lambda,k(\lambda)}(T^{-1}_{\lambda,k(\lambda)}(y))|} g\left(\frac{x}{y}\right) \frac{dy}{y} \\
& + \sum_{p=2}^{k(\lambda)-1} \int_x^1 f_n(T^{-1}_{\lambda,p}(y)) \frac{g(x/y)y^{-1}}{|T'_{\lambda,p}(T^{-1}_{\lambda,p}(y))|} 1_{T_{\lambda,p}(U)}(y)\, dy \\
& + \int_x^1 f_n(1 + y - \sqrt{2\alpha y}) \left|1 - \sqrt{\frac{\alpha}{2y}}\right| 1_{(0,\alpha/2]}(y) g\left(\frac{x}{y}\right) \frac{dy}{y}.
\end{aligned}
$$

Using the inequality $g(x) \leq Kx^r$ we obtain

$$
f_{n+1}(x) = \int_x^1 P_{T_\lambda} f_n(y) g\left(\frac{x}{y}\right) \frac{dy}{y} \leq Kx^r \int_x^1 P_{T_\lambda} f_n(y) \frac{dy}{y^{r+1}}.
$$

Putting

$$
\omega(x) = \int_x^1 P_{T_\lambda} f_n(y) \frac{dy}{y^{r+1}}
$$

and applying Lemma 1 we obtain

$$
(14)\qquad f_n(x) \leq K/x \quad \text{for } n = 1, 2, \ldots,\ x \in (0,1).
$$

Now we are going to estimate the first term in the sum (13). Since T_λ is a piecewise C^2, decreasing function and

$$
a_{\lambda,k(\lambda)} = T_{\lambda,k(\lambda)}\left(\frac{\lambda - k(\lambda)}{2(\lambda - 1)}\right) > 0,
$$

we have

$$
\begin{aligned}
(15)\qquad A_1 = & \int_x^1 f_n(T^{-1}_{\lambda,k(\lambda)}(y)) \frac{1_{T_{\lambda,k(\lambda)}(V)}(y)}{|T'_{\lambda,k(\lambda)}(T^{-1}_{\lambda,k(\lambda)}(y))|} g\left(\frac{x}{y}\right) \frac{dy}{y} \\
& \leq Kx^r \int_x^1 f_n(T^{-1}_{\lambda,k(\lambda)}(y)) \frac{1_{[a_{\lambda,k(\lambda)}, T_{\lambda,k(\lambda)}(0))}(y)}{|T'_{\lambda,k(\lambda)}(T^{-1}_{\lambda,k(\lambda)}(y))| y^{r+1}}\, dy \\
& \leq \frac{Kx^r}{a^{r+1}_{\lambda,k(\lambda)}} \|f_n\| \leq \frac{K}{a^{r+1}_{\lambda,k(\lambda)}}.
\end{aligned}
$$

Since

$$
|T'_{\lambda,p}(s)| = \frac{\alpha p}{\sqrt{\alpha^2 p^2 + 2\alpha p s - \alpha p(1+p)}} - 1 > \frac{2\alpha}{\sqrt{4\alpha^2 - 2\alpha}} - 1
$$

and

$$T_{\lambda,p}^{-1}(y) > \frac{\lambda - p}{2(\lambda - 1)},$$

and (14) holds, we may estimate the second term in the sum (13) as follows:

$$\text{(16)} \qquad A_2 = \sum_{p=2}^{k(\lambda)-1} \int_x^1 f_n(T_{\lambda,p}^{-1}(y)) \frac{1_{T_{\lambda,p}(U)}(y)}{|T'_{\lambda,p}(T_{\lambda,p}^{-1}(y))|} g\left(\frac{x}{y}\right) \frac{dy}{y}$$

$$\leq \frac{2K^2x^2(\lambda-1)\sqrt{4\alpha^2-2\alpha}(\lambda-2)}{(\lambda-p)(2\alpha-\sqrt{4\alpha^2-2\alpha})} \int_x^1 \frac{dy}{y^{r+1}}.$$

Setting

$$d = \frac{2K^2(\lambda-1)(\lambda-2)\sqrt{4\alpha^2-2\alpha}}{(\lambda-p)(2\alpha-\sqrt{4\alpha^2-2\alpha})r}$$

we have $A_2 \leq d$. Furthermore, by the inequality

$$1 + y - \sqrt{2\alpha y} \geq \frac{\lambda-2}{2(\lambda-1)} \quad \text{for } y \in (0, \alpha/2]$$

the last term in (13) may be estimated as follows:

$$\text{(17)} \quad A_3 = \int_x^1 f_n(1+y-\sqrt{2\alpha y}) \frac{\sqrt{\alpha/2}-\sqrt{y}}{\sqrt{y}} 1_{(0,\alpha/2]}(y) g\left(\frac{x}{y}\right) \frac{dy}{y}$$

$$\leq Kx^r \int_x^1 \frac{K}{1+y-\sqrt{2\alpha y}} \frac{\sqrt{\alpha/2}-\sqrt{y}}{\sqrt{y}} 1_{(0,\alpha/2]}(y) \frac{dy}{y^{r+1}}$$

$$\leq \frac{2K^2x^r(\lambda-1)\sqrt{\alpha/2}}{\lambda-2} \int_x^1 \frac{dy}{y^{r+3/2}}$$

$$\leq \frac{K^2\sqrt{2\lambda(\lambda-1)}}{(\lambda-2)(r+1/2)} \frac{1}{\sqrt{x}}.$$

Combining these inequalities and (15), (16) with equality (13) we immediately obtain

$$f_{n+1}(x) \leq \frac{K}{a_{\lambda,k(\lambda)}^{r+1}} + d + \frac{K^2\sqrt{2\lambda(\lambda-1)}}{(\lambda-2)(r+1/2)} \frac{1}{\sqrt{x}}.$$

Defining

$$h(x) = \frac{K}{a_{\lambda,k(\lambda)}^{r+1}} + d + \frac{K^2\sqrt{2\lambda(\lambda-1)}}{(\lambda-2)(r+1/2)} \frac{1}{\sqrt{x}}$$

we finally obtain $P^n f \in \mathcal{F}_h$ for $n = 2, 3, \ldots$ Thus we have proved that for $\lambda > 2$ the operator P is weakly constrictive.

Now we consider the case where $1 < \lambda \le 2$. Then

$$T_\lambda(s) = s + \alpha - 1 - \sqrt{\alpha^2 + 2\alpha s - 2\alpha}, \quad s \in [0,1].$$

It is easy to verify that the Frobenius–Perron operator P_{T_λ} corresponding to T_λ is given by

$$P_{T_\lambda} f(y) = f(1 + y - \sqrt{2\alpha y})\frac{\sqrt{\alpha/2} - \sqrt{y}}{\sqrt{y}} 1_{T_\lambda(0,1)}(y).$$

As a consequence,

$$f_{n+1}(x) = \int_x^1 f_n(1 + y - \sqrt{2\alpha y})\frac{\sqrt{\alpha/2} - \sqrt{y}}{\sqrt{y}} 1_{(0,\alpha-1-\sqrt{\alpha^2-2\alpha})}(y) g\left(\frac{x}{y}\right)\frac{dy}{y}$$

$$\le K x^r \int_x^1 f_n(1 + y - \sqrt{2\alpha y})\frac{\sqrt{\alpha/2} - \sqrt{y}}{\sqrt{y}} 1_{(0,\alpha-1-\sqrt{\alpha^2-2\alpha})}(y)\frac{dy}{y^{r+1}}.$$

Choose $c_\lambda \in (0, \alpha - 1 - \sqrt{\alpha^2 - 2\alpha})$. Then

$$\begin{aligned}
f_{n+1}(x) &\le K x^r \int_{[x,1]\cap(0,c_\lambda)} f_n(1 + y - \sqrt{2\alpha y})\frac{\sqrt{\alpha/2} - \sqrt{y}}{\sqrt{y}}\frac{dy}{y^{r+1}} \\
&\quad + K x^r \int_{[x,1]\cap[c_\lambda,1]} f_n(1 + y - \sqrt{2\alpha y})\frac{\sqrt{\alpha/2} - \sqrt{y}}{\sqrt{y}}\frac{dy}{y^{r+1}} \\
&\le K x^r \int_{[x,1]\cap(0,c_\lambda)} \frac{K}{1 + y - \sqrt{2\alpha y}}\frac{\sqrt{\alpha/2} - \sqrt{y}}{y^{r+3/2}}\,dy \\
&\quad + \frac{K x^r}{c_\lambda^{r+1}} \int_{[x,1]\cap[c_\lambda,1]} f_n(1 + y - \sqrt{2\alpha y})\frac{\sqrt{\alpha/2} - \sqrt{y}}{\sqrt{y}}\,dy \\
&\le \frac{K^2 x^r \sqrt{\alpha/2}}{1 + c_\lambda - \sqrt{2\alpha c_\lambda}} \int_x^1 \frac{dy}{y^{r+3/2}} + \frac{K}{c_\lambda^{r+1}} \\
&\le \frac{K^2 \sqrt{\alpha/2}}{(1 + c_\lambda - \sqrt{2\alpha c_\lambda})(r + 1/2)} \cdot \frac{1}{\sqrt{x}} + \frac{K}{c_\lambda^{r+1}}.
\end{aligned}$$

Setting

$$h(x) = \frac{K^2 \sqrt{\alpha/2}}{(1 + c_\lambda - \sqrt{2\alpha c_\lambda})(r + 1/2)} \cdot \frac{1}{\sqrt{x}} + \frac{K}{c_\lambda^{r+1}}$$

and proceeding as in the case where $\lambda > 2$ we obtain the weak constrictiveness of the operator P. ∎

Using Theorem 1 we may prove the following result concerning the asymptotic stability of the Markov operator P defined by (9).

Denote by m the first moment of the density g, i.e. $m = \int_0^1 xg(x)\,dx$, and set

$$d_\lambda = \min\{T_\lambda(y):\ 0 \le y \le m\}.$$

THEOREM 2. *If the density g satisfies condition* (11) *and there is a nonnegative constant ε_λ such that $\varepsilon_\lambda < d_\lambda$ and*

$$g(x) > 0 \quad \text{for } x \ge \varepsilon_\lambda, \tag{18}$$

then the Markov operator P defined by (9) *is asymptotically stable.*

To prove this theorem we use the following

LEMMA 2. *Let P be a weakly constrictive Markov operator. Assume that there is a set $A \subset [0,1]$ of nonzero measure, $\mu(A) > 0$, with the property that for every $f \in D$ there is an integer $n_1(f)$ such that*

$$P^n f(x) > 0 \tag{19}$$

for almost all $x \in A$ and all $n \ge n_1(f)$. Then P is asymptotically stable.

The proof of this lemma may be found in [4]. ■

Proof of Theorem 2. Since P is weakly constrictive by Theorem 1, we need only show that P satisfies the remaining assumptions of Lemma 2. Define

$$\mathrm{E}(f) = \int_0^1 xf(x)\,dx, \quad f \in D.$$

From the properties of Frobenius–Perron operators it follows that (cf. [6])

$$\int_0^1 P_{T_\lambda} f(y) g\left(\frac{x}{y}\right)\frac{1}{y}1_{[x,1]}(y)\,dy = \int_0^1 f(y) g\left(\frac{x}{T_\lambda(y)}\right)\frac{1}{T_\lambda(y)}1_{[x,1]}(T_\lambda(y))\,dy. \tag{20}$$

Then we may estimate $\mathrm{E}(Pf)$ as follows:

$$\mathrm{E}(Pf) = \int_0^1 xPf(x)\,dx = \int_0^1 x \int_0^1 f(y) g\left(\frac{x}{T_\lambda(y)}\right)\frac{1}{T_\lambda(y)}1_{[x,1]}(T_\lambda(y))\,dy\,dx$$

$$= \int_0^1 f(y)T_\lambda(y)\int_0^1 zg(z)\,dz\,dy$$

$$= m\int_0^1 f(y)T_\lambda(y)\,dy \le m \quad \text{for } f \in D.$$

As a consequence, $\mathrm{E}(P^n f) \le m$ for all $n = 1, 2, \ldots$ and $f \in D$. Thus

$$(21)\qquad \int_0^b P^n f(x)\,dx = 1 - \int_b^1 P^n f(x)\,dx = 1 - b^{-1}\int_b^1 bP^n f(x)\,dx$$
$$\ge 1 - b^{-1}\int_b^1 xP^n f(x)\,dx \ge 1 - b^{-1}\mathrm{E}(P^n f)$$
$$\ge 1 - m/b > 0$$

for arbitrary $b > m$ and $n = 1, 2, \ldots$ Using (20) we may write

$$P^n f(x) = \int_0^1 P^{n-1} f(y) g\left(\frac{x}{T_\lambda(y)}\right)\frac{1}{T_\lambda(y)} 1_{[x,1]}(T_\lambda(y))\,dy.$$

Hence

$$(22)\qquad P^n f(x) \ge \int_0^b P^{n-1} f(y) g\left(\frac{x}{T_\lambda(y)}\right)\frac{1}{T_\lambda(y)} 1_{[x,1]}(T_\lambda(y))\,dy.$$

From the properties of T_λ and the inequality $\varepsilon_\lambda < d_\lambda$ it follows that there exists a positive constant $b_0 > m$ such that

$$\varepsilon_\lambda < \min\{T_\lambda(y) : \ 0 \le y \le b_0\}.$$

We set

$$u_\lambda = \min\{T_\lambda(y) : \ 0 \le y \le b_0\}$$

and define $A = [\varepsilon_\lambda, u_\lambda]$. Using inequality (19) it is easy to verify that

$$g\left(\frac{x}{T_\lambda(y)}\right)\frac{1}{T_\lambda(y)} 1_{[x,1]}(T_\lambda(y)) > 0$$

for $x \in A$ and $0 \le y \le b_0$. From this and inequality (21) we conclude that for all $x \in A$

$$\int_0^{b_0} P^{n-1} f(y) g\left(\frac{x}{T_\lambda(y)}\right)\frac{1}{T_\lambda(y)} 1_{[x,1]}(T_\lambda(y))\,dy > 0 \quad \text{for } n = 2, 3, \ldots$$

As a consequence, applying inequality (22) we finally obtain $P^n f(x) > 0$ for all $x \in A$ and $n = 2, 3, \ldots$ Thus, the proof of the theorem is complete. ∎

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