W. MYDLARCZYK (Wroclaw)

GALERKIN METHODS FOR MULTIDIMENSIONAL NONLINEAR VOLterra TYPE EQUATIONS

1. Introduction. Our aim in this paper is to construct some approximate solution of the nonlinear Volterra type equation

\( z(t, x) = \int_0^t \int_D m(t, x, s, y)k(t - s, x - y)G(z(s, y)) \, dy \, ds = f(t, x) , \)

where \( D \subseteq \mathbb{R}^q \) is a bounded domain with Lipschitz continuous boundary, \( G(z)(s, y) = g(z(s, y), s, y) \) and \( t \in [0, T] , 0 < T < \infty \). We assume that \( m \) is a regular function and \( k \) is in a suitable Nikol’skiǐ space.

After proving the unique solvability of (1.1) in the Nikol’skiǐ or \( L^p \)-spaces, \( 1 \leq p \leq \infty \), we discretize our problem by the Galerkin method and we seek approximate solutions in finite-dimensional spaces.

We apply a fixed point theorem due to Leray and Schauder ([4], p. 189) to prove existence theorems for the approximate problem. The estimates of approximations are given in the \( L^p \)-norm, \( 1 \leq p \leq \infty \).

The results presented in this paper are a generalization of those obtained in our previous paper [8], where similar questions for a one-dimensional equation were considered.

2. Basic notations and assumptions. For \( \mathcal{U} \subseteq \mathbb{R}^N \) and \( \delta \in \mathbb{R}^N \) we set

\( \tilde{\mathcal{U}} = \{ z \in \mathbb{R}^N : z = x - y \text{ for some } x, y \in \mathcal{U} \} , \quad \mathcal{U}_\delta = \{ z \in \mathcal{U} : z + \delta \in \mathcal{U} \} \).

The euclidean norm of a vector \( \delta \) is denoted by \( |\delta| \). We put

\( \Delta_\delta f(u) = f(u + \delta) - f(u) \).

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Let $q$ be a fixed integer. We set
\[ \Omega = [0,T] \times D, \quad \Omega(a) = [0,a] \times D, \quad \Omega(a,b) = [a,b] \times D \quad \text{for } a,b \in \mathbb{R}. \]

Throughout this paper $C$ with or without a subscript always denotes a constant. We permit it to change its value from paragraph to paragraph.

Given an operator $T : L^p(\mathcal{U}) \to L^p(\mathcal{U})$ we let $\|T\|_{p,p}$ denote its operator norm.

3. Some auxiliary definitions and theorems. For the convenience of the reader we give in this section some definitions and theorems used in the sequel.

By $C^\beta(\mathcal{U})$, $0 < \beta < 1$, we denote the Hölder space of functions defined on $\mathcal{U}$.

The Nikol’skii space $N^\alpha_p(\mathcal{U})$, $0 < \alpha < 1$, $1 \leq p \leq \infty$, is the set of all functions $\varphi \in L^p(\mathcal{U})$ satisfying the condition
\[ |\varphi|_{p,\alpha} = \sup_{\delta \neq 0} |\delta|^{-\alpha} \|\Delta_\delta \varphi\|_{p,\mathcal{U}} < \infty. \]

It is known [9] that $N^\alpha_p(\mathcal{U})$ equipped with the norm
\[ \|\varphi\|_{p,\alpha} = \|\varphi\|_p + |\varphi|_{p,\alpha} \]
is a Banach space.

**Lemma 3.1.** Let $k \in N^\alpha_1([0,1] \times \tilde{D})$, $0 < \alpha < 1$, and put $k(u) = 0$ for $u \in [-1,0) \times \tilde{D}$. Then $k \in N^\alpha_1(\tilde{D})$.

**Proof.** For details, see [5], [6].

In the study of the integral operators appearing in the equation (1.1) we need the following lemma.

**Lemma 3.2.** Let nonnegative functions $z, f \in L^1(\Omega(d))$ and $k \in L^1([0,d] \times \tilde{D})$ for some $0 < d < \infty$ satisfy the inequality
\[ z(u) \leq \int_{\Omega(t)} k(u-v)z(v) \, dv + f(u) \quad (3.1) \]
for $u = (t,x), u \in \Omega(d)$. Then $\|z\|_1 \leq C\|f\|_1$, where $C$ depends on $k$ only.

**Proof.** Define
\[ Z(T) = \int_{\Omega(T)} z(v) \, dv, \quad I(T) = \int_A k(v) \, dv, \quad \text{where } A = [0,T] \times \tilde{D}, \]
\[ F(T) = \int_{\Omega(T)} f(v) \, dv. \]
Setting \( u = (t, x) \) and \( v = (s, y) \) and changing the order of integration we get

\[
\int_{\Omega(T)} \left( \int_{\Omega(t)} k(u - v)z(v) \, dv \right) du
\]

\[= \int_0^T \int_D z(s, y) \left( \int_0^T \int_D k(t - s, x - y) \, dx \right) dt \, dy \, ds \]

\[\leq \int_0^T \int_D z(s, y) I(T - s) \, dy \, ds . \]

Let us integrate both sides of the inequality (3.1) on \( \Omega(T) \). Since \( Z \) and \( I \) are nondecreasing, in view of (3.2), we get

\[Z(T) \leq I(T)Z(T_1) + I(T - T_1)Z(T) + F(T),\]

for every \( 0 \leq T_1 \leq T \leq d \). Therefore selecting \( \eta > 0 \) so that \( I(t) \leq 1/2 \) for \( 0 \leq t \leq \eta \) and applying once more the monotonicity argument we can write

\[Z(T) \leq 2(I(d)Z(T_1) + F(d))\]

for every \( 0 \leq T_1 \leq T \leq d \) with \( T - T_1 \leq \eta \).

Hence beginning with \( T_1 = 0 \) and \( T = \eta \) we get the recurrent estimates of \( Z(\eta) \), \( Z(2\eta) \) and so on. After a finite number of steps we obtain the required estimate of \( Z(d) = \|z\|_1 \).

Let \([X]\) denote the Banach space of bounded linear operators \( L : X \rightarrow X \), where \( X \) is a Banach space. Then a set \( K \subseteq [X] \) is collectively compact provided that the set \( KB = \{Lz : L \in K, \|z\|_X \leq 1\} \) is relatively compact.

The essential results concerning the integral operators which are considered in this paper follow from the properties of functions belonging to suitable Nikol'skii spaces.

Let \( \mathcal{U} \subseteq \mathbb{R}^N \) be an arbitrary bounded domain with Lipschitz continuous boundary. The proof of the following lemma can be found in [9].

**Lemma 3.3.** Let \( 0 < \alpha < 1 \). Then the imbedding \( N_1^\alpha(\mathcal{U}) \subseteq L^1(\mathcal{U}) \) is compact.

In the next sections we use the following fixed point theorem due to Leray and Schauder ([4], p. 189).

**Theorem 3.1.** Let \( X \) be a Banach space and let \( T : X \rightarrow X \) be a continuous and compact operator. Assume that for any fixed point \( z_\lambda \) of the operator \( \lambda T \), \( \lambda \in [0, 1] \), we have an a priori estimate \( \|z_\lambda\|_X \leq C \), where \( C \) is independent of \( \lambda \). Then there exists \( z \in X \) such that \( z = T(z) \).
4. The exact problem. We suppose from now on that \( g : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) is a function which satisfies the following conditions:

\[
|g(t_1, u) - g(t_2, u)| \leq M|t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R}, \tag{4.1}
\]

\[
|g(0, u)| \leq M \tag{4.2}
\]

for almost every \( u \in \Omega \).

We assume that \( m \) and \( k \) appearing in (1.1) satisfy

\[
k \in N^\alpha_1([0, 1] \times \bar{D}), \tag{4.3}
\]

\[
m \in C(\bar{\Omega} \times \bar{\Omega}), \quad \|m\|_\infty \leq M, \tag{4.4}
\]

\[
|m(u_1, v) - m(u_2, v)| \leq M|u_1 - u_2|^\alpha \quad \text{for } u_1, u_2 \in \Omega. \tag{4.5}
\]

In (4.1)–(4.5), \( M \) is a constant and \( \alpha \in (0, 1) \).

By Lemma 3.1 the function \( k \) extended by 0 to the whole set \( \tilde{\Omega} \) belongs to \( N^\alpha_1(\tilde{\Omega}) \). Therefore it will be convenient to replace (4.3) by the condition

\[
k \in N^\alpha_1(\tilde{\Omega}) \quad \text{and} \quad k(0) = 0 \quad \text{for } u \in [-1, 0] \times \bar{D}. \tag{4.6}
\]

We denote by \( \|k\|_1 \) and \( \|k\|_{1, \alpha} \) the suitable norms of the kernel \( k \) over the domain \( \Omega \).

In view of (4.6), the Volterra type operator

\[
Kz(u) = \int_{\Omega(t)} m(u, v)k(u - v)z(v) \, dv,
\]

where \( u = (t, x), \ t \in [0, 1] \) and \( x \in D \), may be written in the form of the Fredholm operator

\[
Kz(u) = \int_{\Omega} m(u - v)k(u - v)z(v) \, dv \quad \text{for } u \in \Omega.
\]

Define the operator

\[
G(z)(u) = g(z(u), u) \quad \text{for } z \in L^1(\Omega), \ u \in \Omega.
\]

Thus

\[
KG(z)(u) = \int_{\Omega} m(u, v)k(u - v)G(z)(v) \, dv \quad \text{for } u \in \Omega.
\]

For an arbitrary \( \psi \in L^\infty(\Omega), \|\psi\|_\infty \leq M \), we define the operator

\[
L(\psi)z = \psi z \quad \text{for } z \in L^1(\Omega).
\]

The family of all these operators will be denoted by \( \mathcal{L}_M \).

Let \( z', z'' \in L^1(\Omega) \). Taking

\[
\psi(u) = \begin{cases} 
\frac{g(z'(u), u) - g(z''(u), u)}{z'(u) - z''(u)} & \text{if } z'(u) \neq z''(u), \\
0 & \text{if } z'(u) = z''(u),
\end{cases}
\]
we note that \( \psi \in L^\infty(\Omega), \|\psi\|_\infty \leq M \) and

\[
(4.7) \quad G(z') - G(z'') = L(\psi)(z' - z'').
\]

**Lemma 4.1.** The operators

(a) \( K : L^1(\Omega) \to N^a(\Omega) \), \quad (b) \( K : L^\infty(\Omega) \to C^a(\overline{\Omega}) \)

are bounded.

**Proof.** For simplicity we prove part (a) only. Case (b) can be treated in a similar way. For \( z \in L^1(\Omega) \) we have

\[
\Delta_\delta K z(u) = I_1 + I_2,
\]

where

\[
(4.8) \quad I_1 = \int_\Omega m(u + \delta, v)[k(u + \delta - v) - k(u - v)]z(v) \, dv,
\]

\[
(4.9) \quad I_2 = \int_\Omega [m(u + \delta, v) - m(u, v)]k(u - v)z(v) \, dv
\]

for \( u \in \Omega_\delta, \delta \in \mathbb{R}^{n+1} \).

Changing the order of integration we obtain

\[
(4.10) \quad \int_{\Omega_\delta} |I_1| \, du \leq \int_{\Omega} |z(v)| \left( \int_{\Omega} |k(u + \delta - v) - k(u - v)| \, du \right) dv \leq M |k|_{1,1,\alpha} \|z\|_{1,\alpha} |\delta|^{\alpha}.
\]

Applying (4.5) to \( m(u + \delta, v) - m(u, v) \) and changing the order of integration we get

\[
(4.11) \quad \int_{\Omega_\delta} |I_2| \, du \leq M |k|_{1,1,\alpha} \|z\|_{1,\alpha} |\delta|^{\alpha}.
\]

Thus

\[
(4.12) \quad |Kz|_{1,\alpha} \leq M |k|_{1,1,\alpha} \|z\|_{1,\alpha}.
\]

By the inequality

\[
\int_{\Omega} |Kz(u)| \, du \leq \int_{\Omega} |z(v)| \left( \int_{\Omega} |m(u, v)k(u - v)| \, du \right) dv
\]

it follows that

\[
(4.13) \quad \|Kz\|_{1,\alpha} \leq M |k|_{1,1,\alpha} \|z\|_{1,\alpha} \leq M |k|_{1,\alpha} \|z\|_{1,\alpha}.
\]

Now the inequalities (4.12) and (4.13) give

\[
(4.14) \quad \|Kz\|_{1,\alpha} \leq 2M |k|_{1,\alpha} \|z\|_{1,\alpha},
\]

which completes the proof.
As a consequence of Lemma 4.1 we get

**Corollary 4.1.** The operators $K : L^1(\Omega) \to L^1(\Omega)$ and $K : L^\infty(\Omega) \to L^\infty(\Omega)$ are compact.

Using the Riesz–Thorin interpolation theorem we obtain

**Corollary 4.2.** We have $K : L^p(\Omega) \to L^p(\Omega)$, $1 \leq p \leq \infty$, with $\|K\|_{p,p} \leq M\|k\|_1$.

**Lemma 4.2.** Let $0 < d \leq 1$. Then

(a) for any $z_1, z_2 \in L^p(\Omega(d))$, $1 \leq p \leq \infty$,  

$$\|KG(z_1) - KG(z_2)\|_{p,\Omega(d)} \leq M^2 \int_A |k(v)| dv \|z_1 - z_2\|_{p,\Omega(d)},$$

where $A = [0, d] \times \tilde{D}$,

(b) $KG(z) \in N^*_p(\Omega(d))$ for any $z \in L^1(\Omega(d))$,

(c) $KG(z) \in C^\alpha(\Omega(d))$ for any $z \in L^\infty(\Omega(d))$.

**Proof.** First we note that for any $z \in L^p(\Omega(d))$, in view of (4.1), we have

(4.15)  

$$|G(z)(u)| \leq M|z(u)| + |G(0)(u)|$$

for almost every $u \in \Omega(d)$. Therefore by (4.2) and Corollary 4.2 the operator $KG$ is well defined on $L^p(\Omega(d))$, $1 \leq p \leq \infty$.

(a) By (4.7) we get  

$$\|KG(z_1) - KG(z_2)\|_p \leq \|K\|_{p,p}\|L(\psi)\|_{p,p}\|z_1 - z_2\|_p$$

for $z_1, z_2 \in L^p(\Omega(d))$. It is clear that $\|L(\psi)\|_{p,p} \leq M$. Therefore our assertion follows by Corollary 4.2.

(b) and (c). In view of (4.15) our assertions follow by Lemma 4.1.

**Lemma 4.3.** The operators $KG : L^\infty(\Omega) \to L^\infty(\Omega)$ and $KG : L^1(\Omega) \to L^1(\Omega)$ are continuous and compact.

**Proof.** By (4.7) and (4.15) it follows that $G$ considered both on $L^1(\Omega)$ and on $L^\infty(\Omega)$ is continuous and bounded. Therefore our assertion follows from Corollary 4.1.

We are now ready to consider the equation (1.1):

**Theorem 4.1.** Let $f \in L^p(\Omega)$, $1 \leq p \leq \infty$. Then the equation (1.1) has a unique solution $z_0 \in L^p(\Omega)$. If $f \in C(\Omega)$, then $z_0 \in C(\Omega)$.

**Proof.** Choose a sufficiently small $d > 0$ so that $2M^2 \int_A |k(v)| dv < 1$, where $A = [0, d] \times \tilde{D}$. By Lemma 4.2(a) the operator $KG$ considered both on $L^1(\Omega(d))$ and on $L^\infty(\Omega(d))$ is a contraction. Hence there exists a unique solution $z_0$ of (1.1) defined on $\Omega(d)$, if $f$ belongs to $L^1(\Omega)$ or to $L^\infty(\Omega)$. 
Consider the following complete metric spaces:
\[ X_1 = \{ z \in L^1(\Omega(2d)) : z = z_0 \text{ on } \Omega(d) \} \quad \text{in the first case}, \]
\[ X_2 = \{ z \in L^\infty(\Omega(2d)) : z = z_0 \text{ on } \Omega(d) \} \quad \text{in the second case}. \]

For any \( z_1 \) and \( z_2 \) either both in \( X_1 \) or both in \( X_2 \) we have
\begin{align*}
(4.16) \quad KG(z_1)(u) - KG(z_2)(u) &= 0 \quad \text{for } u \in \Omega(d), \\
(4.17) \quad |KG(z_1)(u) - KG(z_2)(u)| &\leq M^2 \int_{\Omega(d,2d)} |k(u-v)||z_1(v) - z_2(v)| \, dv \quad \text{for } u \in \Omega(d,2d).
\end{align*}
Combining (4.16), (4.17) with (4.6) we get
\[ \|KG(z_1) - KG(z_2)\|_p \leq M^2 \int_{\Omega(d)} |k(v)| \, dv \|z_1 - z_2\|_p, \]
where \( p = 1 \) and \( \infty \) in the first and second case, respectively. All the norms are taken over \( \Omega(2d) \).

Therefore \( KG \) is a contraction on \( X_1 \) and on \( X_2 \). As a consequence there exists a unique extension of \( z_0 \) to a solution of (1.1) on \( \Omega(2d) \). Repeating this procedure we are led to a unique solution of our equation defined on the whole \( \Omega \), when \( f \) belongs to \( L^1(\Omega) \) or to \( L^\infty(\Omega) \).

We now turn to the case \( 1 < p < \infty \). Let \( f \in L^p(\Omega) \subseteq L^1(\Omega) \). Then (1.1) has a unique solution \( z_0 \in L^1(\Omega) \). According to (4.7) we have
\[ KG(z_0) - KG(0) = L(\psi_0)z_0 \]
for some \( \psi_0 \in L^\infty(\Omega) \), \( \|\psi_0\|_\infty \leq M \). Therefore taking \( f_0 = f + KG(0) \) we can write
\[ z_0 - KL(\psi_0)z_0 = f_0. \]
According to Lemma 4.2(c), \( KG(0) \in C^\alpha(\Omega) \). Hence \( f_0 \in L^p(\Omega) \).

Now we consider the operator \( I - KL(\psi_0) \). Define
\[ g(t, u) = t\psi_0(u) \quad \text{for } t \in \mathbb{R}, \ u \in \Omega. \]
Since \( g \) satisfies (4.1) and (4.2) the homogeneous equation \( z - KL(\psi_0)z = 0 \) has no nontrivial solution. Now Lemma 4.3 allows us to use the Fredholm alternative to prove the existence of the bounded linear operators \((I - KL_0)^{-1} : L^\infty(\Omega) \to L^\infty(\Omega) \) and \((I - KL_0)^{-1} : L^1(\Omega) \to L^1(\Omega) \), where \( L_0 = L(\psi_0) \). By the Riesz–Thorin interpolation theorem \((I - KL_0)^{-1} \) is bounded on \( L^p(\Omega) \) for all \( 1 \leq p \leq \infty \). Since \( z_0 = (I - KL_0)^{-1}f_0 \), we get \( z_0 \in L^p(\Omega) \).

It remains to consider the case \( f \in C(\Omega) \). Since then \( z_0 \in L^\infty(\Omega) \), by Lemma 4.2(c) it follows that \( KG(z_0) \in C^\alpha(\Omega) \). Therefore by (1.1) we get \( z_0 \in C(\Omega) \).
Corollary 4.3. Let $f \in N_1^\beta(\Omega), \ 0 < \beta < 1$. Then the solution $z_0$ of (1.1) belongs to $N_1^\gamma(\Omega)$, where $\gamma = \min(\alpha, \beta)$. If $f \in C^\beta(\Omega)$, then $z_0 \in C^\gamma(\overline{\Omega})$.

Proof. Since $z_0 = KG(z_0) + f$ our assertion follows from Theorem 4.1 and the obvious imbeddings
\[ C^\alpha(\overline{\Omega}) \subset C^\beta(\overline{\Omega}) \subset C^\gamma(\overline{\Omega}) \subset C(\overline{\Omega}), \quad N_1^\alpha(\Omega) \subset N_1^\beta(\Omega) \subset N_1^\gamma(\Omega) \subset L^1(\Omega). \]

5. The approximate problem. In this section we construct a certain approximate solution of (1.1) in finite-dimensional function spaces.

Consider a sequence $\{X_n\}, \ n \in \mathbb{N}$ of finite-dimensional subspaces of $L^\infty(\Omega)$ and a sequence $\{P_n\}, \ n \in \mathbb{N}$, of projections $P_n : L^1(\Omega) \rightarrow X_n$ which satisfy the following conditions:

\begin{align*}
(5.1) & \quad P_n \circ P_n = P_n, \quad n \in \mathbb{N}, \\
(5.2) & \quad \|P_n\|_{1,1} \leq C \quad \text{for every } n \in \mathbb{N}, \\
(5.3) & \quad \|P_n\|_{\infty,\infty} \leq C \quad \text{for every } n \in \mathbb{N}, \\
(5.4) & \quad \|\varphi - P_n \varphi\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ for every } \varphi \in C(\overline{\Omega}),
\end{align*}

where the constant $C$ in (5.2) and (5.3) is independent of $n$.

Since $C(\overline{\Omega})$ is a dense subspace of $L^1(\Omega)$, the Banach–Steinhaus theorem yields

Remark 5.1. $\|\varphi - P_n \varphi\|_1 \rightarrow 0$ as $n \rightarrow \infty$ for every $\varphi \in L^1(\Omega)$.

By the Riesz–Thorin interpolation theorem we immediately obtain

Remark 5.2. $P_n : L^p(\Omega) \rightarrow L^p(\Omega)$ with $\|P_n\|_{p,p} \leq C$ for $1 \leq p \leq \infty$, where $C$ is the same constant as in (5.2) and (5.3).

Our approximate problem is formulated as follows: find $z_n \in X_n$ which satisfies

\[ z_n - P_n KG(z_n) = P_n f. \]

For the study of (5.5) it will be convenient to introduce auxiliary functions $y_n$ defined by

\[ y_n = f + KG(z_n). \]

It is easy to verify that $y_n$ satisfies the equations

\begin{align*}
(5.6) & \quad z_n = P_n y_n, \\
(5.7) & \quad y_n - KG(P_n y_n) = f.
\end{align*}

It should also be noted that if $y_n$ is a solution of (5.7) and $z_n = P_n y_n$, then $z_n$ satisfies (5.5).

Before proving the existence and uniqueness theorem for (5.5) and (5.7) we prove a lemma essential for our further considerations.
For any \( n \in \mathbb{N} \) and for any function \( \varphi \in L^\infty(\Omega) \) with \( \|\varphi\|_\infty \leq M \) we denote by \( L(n, \varphi) \) the linear operator defined on \( L^1(\Omega) \) by the formula
\[
L(n, \varphi)z = \varphi P_n z \quad \text{for} \ z \in L^1(\Omega).
\]
The family of all these operators will be denoted by \( \mathcal{L}_M(\mathbb{N}) \).

**Lemma 5.1.** There exist positive constants \( C_1, C_2 \) and \( N_0 \) such that
\[
C_1 \|z\|_p \leq \|(I - KL(n, \psi))z\|_p \leq C_2 \|z\|_p
\]
for any \( z \in L^p(\Omega) \), \( 1 \leq p \leq \infty \), and any \( L(n, \psi) \in \mathcal{L}_M(\mathbb{N}) \) with \( n \geq N_0 \).

**Proof.** The right inequality in (5.9) follows immediately from Corollary 4.2 and the estimate (5.2).

Since
\[
P_n z = KL(n, \psi)z + (P_n z - KL(n, \psi)z)
\]
we get
\[
|P_n z(u)| \leq M \int_\Omega |k(u - v)||P_n z(v)| \, dv + |P_n z(u) - KL(n, \psi)z(u)|
\]
for \( u \in \Omega \). Now applying Lemma 3.2 we conclude that
\[
P_n z(\cdot) \rightarrow C \|P_n z - KL(n, \psi)z\|_1,
\]
where the constant \( C \) depends on \( k \) only.

First we consider the case \( p = 1 \) or \( \infty \). Of course, it suffices to study the case \( \|z\|_p = 1 \). To get a contradiction assume that there exists a sequence \( z_m \in L^1(\Omega) \) with \( \|z_m\|_p = 1 \) and a sequence of operators \( L_m = L(n_m, \psi_m) \) with \( n_m \rightarrow \infty \) as \( m \rightarrow \infty \) such that
\[
\|(I - KL_m)z_m\|_p \rightarrow 0 \quad \text{as} \ m \rightarrow \infty.
\]
Since, by Lemma 4.1, \( K \) is compact, and by (5.2), the \( L_m \) are uniformly bounded, the set \( \{KL_mz_m : m \in \mathbb{N}\} \) is relatively compact in \( L^1(\Omega) \). Without loss of generality we can assume that \( KL_mz_m \rightarrow z \) in \( L^1(\Omega) \) as \( m \rightarrow \infty \) for some \( z \). It then follows by (5.12) that
\[
z_m \rightarrow z \quad \text{as} \ m \rightarrow \infty, \quad \|z\|_p = 1.
\]
Since \( KL_m, \ m \in \mathbb{N}, \) are uniformly bounded we get
\[
\|KL_m(z_m - z)\|_p \rightarrow 0 \quad \text{as} \ m \rightarrow \infty.
\]
Combining (5.12), (5.13) and (5.14) we obtain
\[
\|z - KL_mz\|_p \rightarrow 0 \quad \text{as} \ m \rightarrow \infty.
\]
Noting that \( P_{n_m} z - KL(n_m, \psi_m)z = (z - KL(n_m, \psi_m)z) - (z - P_{n_m} z) \) we get, in view of (5.11),
\[
(1/C)\|P_{n_m} z\|_1 \leq \|z - KL(n_m, \psi_m)z\|_1 + \|z - P_{n_m} z\|_1.
\]
By (5.15) and Remark 5.1 the right-hand side of this inequality tends to 0 as $m \to \infty$, while the left-hand side tends to $(1/C)\|z\|_1 = 1/C$. The derived contradiction completes the proof in the case $p = 1$ or $\infty$.

From the obtained estimates it follows immediately that the homogeneous equation

\[(5.16) \quad z - KL(n, \psi)z = 0\]

has no nontrivial solutions in $L^1(\Omega)$, if $n > N_0$.

In view of Lemma 4.1 and the conditions (5.2) and (5.3) the operators $KL(n, \psi)$ considered both on $L^1(\Omega)$ and on $L^\infty(\Omega)$ are compact. Therefore we can apply the Fredholm alternative to prove the existence of the inverse operators $(I - KL(n, \psi))^{-1}$ for $n > N_0$ both on $L^1(\Omega)$ and on $L^\infty(\Omega)$. Taking $C_1$ so that

\[
\|(I - KL(n, \psi))^{-1}\|_{p, p} \leq 1/C_1 \quad \text{for } p = 1 \text{ and } \infty
\]

from the Riesz–Thorin interpolation theorem we get

\[
\|(I - KL(n, \psi))^{-1}\|_{p, p} \leq 1/C_1 \quad \text{for } 1 < p < \infty.
\]

This completes the proof.

The results obtained at the end of the proof of Lemma 5.1 are collected in

**Corollary 5.1.** For $n > N_0$ and $\psi \in L^\infty(\Omega)$ with $\|\psi\|_\infty \leq M$ there exists the inverse operator $(I - KL(n, \psi))^{-1}: L^p(\Omega) \to L^p(\Omega)$, $1 \leq p \leq \infty$, with $\|(I - KL(n, \psi))^{-1}\|_{p, p} \leq 1/C_1$, where $C_1$ is the same constant as in Lemma 5.1.

Now we estimate the errors of approximations to the solution $z_0$ of (1.1) by solutions of (5.5) and (5.7). The results are collected in

**Theorem 5.1.** Let $f \in L^p(\Omega)$, $1 \leq p \leq \infty$. Then for sufficiently large $n$

(a) the problem (5.5) has a unique solution $z_n \in X_n$ for which

\[(5.17) \quad C_1 \|z_0 - P_nz_0\|_p \leq \|z_0 - z_n\|_p \leq C_2 \|z_0 - P_nz_0\|_p,
\]

(b) the problem (5.7) has a unique solution $y_n$ for which

\[(5.18) \quad \|z_0 - y_n\|_p \leq C_3 \|z_0 - P_nz_0\|_p,
\]

where the constants $C_1, C_2$ and $C_3$ are independent of $n$.

**Proof.** Similarly to (4.7) for any $z$ we can find $\psi \in L^\infty(\Omega)$ with $\|\psi\|_\infty \leq M$ such that

\[g(P_nz(u), u) - g(0, u) = \psi(u)P_nz(u).
\]

Therefore for $z \in L^1(\Omega)$ we get

\[(5.19) \quad KG(P_nz) - KG(0) = KL(n, \psi)z.
\]
Now applying Lemma 5.1 we obtain the estimate
\[(5.20) \quad C_1 \|z\|_1 - \|KG(0)\|_1 \leq \|(I - \lambda KGP_n)z\|_1 \leq C_2 \|z\|_1 + \|KG(0)\|_1\]
for \(\lambda \in [0,1], z \in L^1(\Omega)\) and \(n > N_0\).

Note that by (5.20) any solution \(z_\lambda\) of the equation
\[z_\lambda - \lambda KGP_n z_\lambda = f,\]
where \(\lambda \in [0,1]\) and \(n > N_0\), can be estimated as follows:
\[\|z_\lambda\|_1 \leq (1/C_1)(\|f\|_1 + \|KG(0)\|_1).\]
By Lemma 4.2(a) and (5.2) it follows that \(\lambda KGP_n, \lambda \in [0,1]\), are continuous and compact as the operators on \(L^1(\Omega)\). Therefore using Theorem 3.1 we conclude that (5.7) has a solution \(y_n\). Then \(P_n y_n\) satisfies (5.5).

We are going to demonstrate the required regularity of the solutions of (5.7) in the case of \(L^p(\Omega) \subseteq L^1(\Omega), 1 < p < \infty\). Adding \(KG(0)\) to both sides of (5.7) and proceeding similarly to (5.19) we find \(\psi_1 \in L^\infty(\Omega)\) with \(\|\psi_1\|_\infty \leq M\) such that
\[(5.21) \quad y_n - KL(n, \psi_1) y_n = f + KG(0).\]
Since \(KG(0) \in L^\infty(\Omega)\), the right-hand side of (5.21) belongs to \(L^p(\Omega)\). Now, by Corollary 5.1 it follows that \(y_n \in L^p(\Omega)\).

We turn to the uniqueness problem for (5.7). In a similar manner to (5.19) for any \(z_1, z_2 \in L^1(\Omega)\) we can find \(\psi_2 \in L^\infty(\Omega)\) with \(\|\psi_2\|_\infty \leq M\) such that
\[(5.22) \quad KG(P_n z_1) - KG(P_n z_2) = KL(n, \psi_2)(z_1 - z_2).\]
Using (5.22) we can apply Lemma 5.1 to obtain the estimate
\[(5.23) \quad C_1 \|z_1 - z_2\|_p \leq \|(z_1 - z_2) - (KG(P_n z_1) - KG(P_n z_2))\|_p\]
for any \(z_1, z_2 \in L^p(\Omega), 1 \leq p \leq \infty\) and \(n \geq N_0\).

Hence we get immediately the uniqueness of solution of (5.7). If \(z_n\) satisfies (5.5), then \(y_n = f + KG(z_n)\) is a solution of (5.7) and \(z_n = P_n y_n\). Therefore the uniqueness for (5.7) implies the uniqueness for (5.5).

Combining (1.1) with (5.7) we obtain
\[(5.24) \quad (z_0 - y_n) - (KG(P_n z_0) - KG(P_n y_n)) = KG(z_0) - KG(P_n z_n).\]
Estimating the left-hand side of (5.24) in a similar way to (5.23) and applying Lemma 4.2 to the right-hand side we obtain (5.18).

Since by (5.6), \(z_0 - z_n = z_0 - P_n z_0 + P_n(z_0 - y_n)\), the right inequality in (5.17) follows from Remark 5.2 and (5.18).
We now turn to the left inequality in (5.17). Combining (1.1) with (5.5) we get
\[ z_0 - P_n z_0 = z_0 - z_n - P_n (K G(z_0) - K G(z_n)). \]
Therefore applying Lemma 4.2 and Remark 5.2 we get
\[ \|z_0 - P_n z_0\|_p \leq (1 + C)\|z_0 - z_n\|_p \]
if \( z_0 \in L^p(\Omega), 1 \leq p \leq \infty \). Thus it suffices to set \( C_1 = 1/(1+C) \) to complete the proof.

6. An example. In practice, the \( X_n \) are certain standard finite element spaces ([1]) and the \( P_n \) are the \( L^2 \)-projections onto \( X_n \). The stability of \( P_n \) considered as a map in \( L^p \), \( 1 \leq p \leq \infty \), is usually shown under the requirements of quasi-uniformity of the triangulations underlying the definitions of the \( X_n \) (see [2], [3]).

Now we present an example of the subspaces \( X_n \) and the projections \( P_n \). Let \( \Omega = [0,1]^{q+1} \). Consider a sequence of quasi-uniform partitions
\[ II_n : \quad 0 = a_0 < a_1 < a_2 < \ldots < a_n = 1, \quad n \in \mathbb{N}, \]
and set \( h = \max[a_{i+1} - a_i], 0 \leq i \leq n - 1 \). Thus we get regular partitions of \( \Omega \) given by the cubes
\[ A_i = \prod_{j=1}^{q+1} [a_{i_j}, a_{i_j+1}], \quad i = (i_1, \ldots, i_{q+1}). \]

We are interested in the class of tensor product splines \( S_n = \bigotimes_{n=1}^{q+1} S(II_n) \), where \( S(II_n) = \{ s \in C[0,1] : s|_{[a_j, a_{j+1}]} \text{ is linear}, j = 0, 1, \ldots, n-1 \} \).

Now we define \( X_n = \{ w \in C(\Omega) : w|_{A_i} \text{ is a polynomial of degree } \leq q + 1 \} \). It was pointed out in [3] that the \( L^2 \)-orthogonal projections \( P_n : L^2(\Omega) \to X_n \) satisfy conditions (5.2)-(5.4).

By Lemma 5.5 in [7] we have
\[ \inf_{w \in S_n} \|f - w\|_1 \leq C(h\|f\|_1 + \omega(f, h, 1)), \quad f \in L^1(\Omega), \]
where
\[ \omega(f, h, 1) = \sup_{|\delta| < h} \int_{\Omega} |f(x + \delta) - f(x)| \, dx, \quad \delta \in \mathbb{R}^{q+1}. \]
Since \( S_n \subseteq X_n \), the same estimate is valid for \( \inf_{\xi \in X_n} \|f - \xi\|_1 \). Therefore, for the solution \( z_0 \) of (1.1) as regular as in Corollary 4.3 we get, in view of Theorem 5.1, the estimate
\[ \|z_0 - z_n\|_1 \leq \text{const} \cdot h^r. \]
References


WOJCIECH MYDLARCZYK
MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCLAW
PL. GRUNWALDZKI 2/4
50-384 WROCLAW, POLAND

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