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GALERKIN METHODS FOR MULTIDIMENSIONAL NONLINEAR VOLTERRA TYPE EQUATIONS

1. Introduction. Our aim in this paper is to construct some approximate solution of the nonlinear Volterra type equation

$$(1.1)\qquad z(t,x)-\int_0^t\int_D m(t,x,s,y)k(t-s,x-y)G(z)(s,y)\,dy\,ds=f(t,x)\,,$$

where $D\subseteq\mathbf{R}^q$ is a bounded domain with Lipschitz continuous boundary, $G(z)(s,y)=g(z(s,y),s,y)$ and $t\in[0,T]$, $0<T<\infty$. We assume that m is a regular function and k is in a suitable Nikol'skiĭ space.

After proving the unique solvability of (1.1) in the Nikol'skiĭ or L^p-spaces, $1\le p\le\infty$, we discretize our problem by the Galerkin method and we seek approximate solutions in finite-dimensional spaces.

We apply a fixed point theorem due to Leray and Schauder ([4], p. 189) to prove existence theorems for the approximate problem. The estimates of approximations are given in the L^p-norm, $1\le p\le\infty$.

The results presented in this paper are a generalization of those obtained in our previous paper [8], where similar questions for a one-dimensional equation were considered.

2. Basic notations and assumptions. For $\mathcal{U}\subseteq\mathbf{R}^N$ and $\delta\in\mathbf{R}^N$ we set

$$\widetilde{\mathcal{U}}=\{z\in\mathbf{R}^N: z=x-y \text{ for some } x,y\in\mathcal{U}\}\,,\quad \mathcal{U}_\delta=\{z\in\mathcal{U}: z+\delta\in\mathcal{U}\}\,.$$

The euclidean norm of a vector δ is denoted by $|\delta|$. We put

$$\Delta_\delta f(u)=f(u+\delta)-f(u)\,.$$

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Let q be a fixed integer. We set

$$\Omega = [0,T] \times D, \quad \Omega(a) = [0,a] \times D, \quad \Omega(a,b) = [a,b] \times D \quad \text{for } a, b \in \mathbf{R}.$$

Throughout this paper C with or without a subscript always denotes a constant. We permit it to change its value from paragraph to paragraph.

Given an operator $T : L^p(\mathcal{U}) \to L^p(\mathcal{U})$ we let $\|T\|_{p,p}$ denote its operator norm.

3. Some auxiliary definitions and theorems. For the convenience of the reader we give in this section some definitions and theorems used in the sequel.

By $C^\beta(\mathcal{U})$, $0 < \beta < 1$, we denote the Hölder space of functions defined on $\mathcal{U}$.

The *Nikol'skiĭ space* $N_p^\alpha(\mathcal{U})$, $0 < \alpha < 1$, $1 \le p \le \infty$, is the set of all functions $\varphi \in L^p(\mathcal{U})$ satisfying the condition

$$|\varphi|_{p,\alpha} = \sup_{\delta \neq 0} |\delta|^{-\alpha} \|\Delta_\delta \varphi\|_{p,\mathcal{U}_\delta} < \infty .$$

It is known [9] that $N_p^\alpha(\mathcal{U})$ equipped with the norm

$$\|\varphi\|_{p,\alpha} = \|\varphi\|_p + |\varphi|_{p,\alpha}$$

is a Banach space.

LEMMA 3.1. *Let $k \in N_1^\alpha([0,1] \times \widetilde{D})$, $0 < \alpha < 1$, and put $k(u) = 0$ for $u \in [-1,0) \times \widetilde{D}$. Then $k \in N_1^\alpha(\widetilde{\Omega})$.*

Proof. For details, see [5], [6].

In the study of the integral operators appearing in the equation (1.1) we need the following lemma.

LEMMA 3.2. *Let nonnegative functions $z, f \in L^1(\Omega(d))$ and $k \in L^1([0,d] \times \widetilde{D})$ for some $0 < d < \infty$ satisfy the inequality*

$$z(u) \le \int_{\Omega(t)} k(u-v) z(v)\, dv + f(u) \tag{3.1}$$

for $u = (t,x)$, $u \in \Omega(d)$. Then $\|z\|_1 \le C\|f\|_1$, where C depends on k only.

Proof. Define

$$Z(T) = \int_{\Omega(T)} z(v)\, dv, \quad I(T) = \int_A k(v)\, dv, \quad \text{where } A = [0,T] \times \widetilde{D},$$

$$F(T) = \int_{\Omega(T)} f(v)\, dv.$$

Setting $u = (t, x)$ and $v = (s, y)$ and changing the order of integration we get

$$\int\limits_{\Omega(T)} \Big(\int\limits_{\Omega(t)} k(u-v)z(v)\,dv\Big)du$$
$$= \int\limits_0^T \int\limits_D z(s,y)\Big(\int\limits_s^T \Big(\int\limits_D k(t-s,x-y)\,dx\Big)dt\Big)dy\,ds$$
$$\le \int\limits_0^T \int\limits_D z(s,y)I(T-s)\,dy\,ds\,. \tag{3.2}$$

Let us integrate both sides of the inequality (3.1) on $\Omega(T)$. Since Z and I are nondecreasing, in view of (3.2), we get

$$Z(T) \le I(T)Z(T_1) + I(T-T_1)Z(T) + F(T)\,,$$

for every $0 \le T_1 \le T \le d$. Therefore selecting $\eta > 0$ so that $I(t) \le 1/2$ for $0 \le t \le \eta$ and applying once more the monotonicity argument we can write

$$Z(T) \le 2(I(d)Z(T_1) + F(d))$$

for every $0 \le T_1 \le T \le d$ with $T - T_1 \le \eta$.

Hence beginning with $T_1 = 0$ and $T = \eta$ we get the recurrent estimates of $Z(\eta)$, $Z(2\eta)$ and so on. After a finite number of steps we obtain the required estimate of $Z(d) = \|z\|_1$.

Let $[X]$ denote the Banach space of bounded linear operators $L : X \to X$, where X is a Banach space. Then a set $\mathcal{K} \subseteq [X]$ is *collectively compact* provided that the set $\mathcal{K}\mathcal{B} = \{Lz : L \in \mathcal{K},\ \|z\|_X \le 1\}$ is relatively compact.

The essential results concerning the integral operators which are considered in this paper follow from the properties of functions belonging to suitable Nikol'skiĭ spaces.

Let $\mathcal{U} \subseteq \mathbf{R}^N$ be an arbitrary bounded domain with Lipschitz continuous boundary. The proof of the following lemma can be found in [9].

LEMMA 3.3. *Let $0 < \alpha < 1$. Then the imbedding $N_1^\alpha(\mathcal{U}) \subseteq L^1(\mathcal{U})$ is compact.*

In the next sections we use the following fixed point theorem due to Leray and Schauder ([4], p. 189).

THEOREM 3.1. *Let X be a Banach space and let $T : X \to X$ be a continuous and compact operator. Assume that for any fixed point z_λ of the operator λT, $\lambda \in [0,1]$, we have an a priori estimate $\|z_\lambda\|_X \le C$, where C is independent of λ. Then there exists $z \in X$ such that $z = T(z)$.*

4. The exact problem. We suppose from now on that $g : \mathbf{R} \times \Omega \to \mathbf{R}$ is a function which satisfies the following conditions:

$$|g(t_1, u) - g(t_2, u)| \le M|t_1 - t_2|, \quad t_1, t_2 \in \mathbf{R}, \tag{4.1}$$

$$|g(0, u)| \le M \tag{4.2}$$

for almost every $u \in \Omega$.

We assume that m and k appearing in (1.1) satisfy

$$k \in N_1^\alpha([0,1] \times \widetilde{D}), \tag{4.3}$$

$$m \in C(\overline{\Omega} \times \overline{\Omega}), \qquad \|m\|_\infty \le M, \tag{4.4}$$

$$|m(u_1, v) - m(u_2, v)| \le M|u_1 - u_2|^\alpha \quad \text{for } u_1, u_2 \in \Omega. \tag{4.5}$$

In (4.1)–(4.5), M is a constant and $\alpha \in (0,1)$.

By Lemma 3.1 the function k extended by 0 to the whole set $\widetilde{\Omega}$ belongs to $N_1^\alpha(\widetilde{\Omega})$. Therefore it will be convenient to replace (4.3) by the condition

$$k \in N_1^\alpha(\widetilde{\Omega}) \quad \text{and} \quad k(u) = 0 \quad \text{for } u \in [-1, 0) \times \widetilde{D}. \tag{4.6}$$

We denote by $\|k\|_1$ and $\|k\|_{1,\alpha}$ the suitable norms of the kernel k over the domain Ω.

In view of (4.6), the Volterra type operator

$$Kz(u) = \int_{\Omega(t)} m(u, v) k(u - v) z(v)\, dv,$$

where $u = (t, x)$, $t \in [0, 1]$ and $x \in D$, may be written in the form of the Fredholm operator

$$Kz(u) = \int_{\Omega} m(u - v) k(u - v) z(v)\, dv \quad \text{for } u \in \Omega.$$

Define the operator

$$G(z)(u) = g(z(u), u) \quad \text{for } z \in L^1(\Omega), \ u \in \Omega.$$

Thus

$$KG(z)(u) = \int_{\Omega} m(u, v) k(u - v) G(z)(v)\, dv \quad \text{for } u \in \Omega.$$

For an arbitrary $\psi \in L^\infty(\Omega)$, $\|\psi\|_\infty \le M$, we define the operator

$$L(\psi)z = \psi z \quad \text{for } z \in L^1(\Omega).$$

The family of all these operators will be denoted by $\mathcal{L}_M$.

Let $z', z'' \in L^1(\Omega)$. Taking

$$\psi(u) = \begin{cases} \dfrac{g(z'(u), u) - g(z''(u), u)}{z'(u) - z''(u)} & \text{if } z'(u) \ne z''(u), \\ 0 & \text{if } z'(u) = z''(u), \end{cases}$$

we note that $\psi \in L^\infty(\Omega)$, $\|\psi\|_\infty \le M$ and

$$G(z') - G(z'') = L(\psi)(z' - z''). \tag{4.7}$$

LEMMA 4.1. *The operators*

$$\text{(a) } K : L^1(\Omega) \to N_1^\alpha(\Omega), \qquad \text{(b) } K : L^\infty(\Omega) \to C^\alpha(\overline{\Omega})$$

are bounded.

Proof. For simplicity we prove part (a) only. Case (b) can be treated in a similar way. For $z \in L^1(\Omega)$ we have

$$\Delta_\delta K z(u) = I_1 + I_2,$$

where

$$I_1 = \int_\Omega m(u+\delta, v)[k(u+\delta-v) - k(u-v)]z(v)\,dv, \tag{4.8}$$

$$I_2 = \int_\Omega [m(u+\delta, v) - m(u,v)]k(u-v)z(v)\,dv \tag{4.9}$$

for $u \in \Omega_\delta$, $\delta \in \mathbb{R}^{q+1}$.

Changing the order of integration we obtain

$$\begin{aligned} \int_{\Omega_\delta} |I_1|\,du &\le \int_\Omega |z(v)| \Big(\int_\Omega |k(u+\delta-v) - k(u-v)|\,du \Big) dv \\ &\le M|k|_{1,\alpha}\|z\|_{1,\Omega}|\delta|^\alpha. \end{aligned} \tag{4.10}$$

Applying (4.5) to $m(u+\delta,v) - m(u,v)$ and changing the order of integration we get

$$\int_{\Omega_\delta} |I_2|\,du \le M\|k\|_1\|z\|_{1,\Omega}|\delta|^\alpha. \tag{4.11}$$

Thus

$$|Kz|_{1,\alpha} \le M\|k\|_{1,\alpha}\|z\|_{1,\Omega}. \tag{4.12}$$

By the inequality

$$\int_\Omega |Kz(u)|\,du \le \int_\Omega |z(v)| \Big(\int_\Omega |m(u,v)k(u-v)|\,du \Big) dv$$

it follows that

$$\|Kz\|_{1,\Omega} \le M\|k\|_1\|z\|_{1,\Omega} \le M\|k\|_{1,\alpha}\|z\|_{1,\Omega}. \tag{4.13}$$

Now the inequalities (4.12) and (4.13) give

$$\|Kz\|_{1,\alpha} \le 2M\|k\|_{1,\alpha}\|z\|_{1,\Omega}, \tag{4.14}$$

which completes the proof.

As a consequence of Lemma 4.1 we get

COROLLARY 4.1. *The operators* $K : L^1(\Omega) \to L^1(\Omega)$ *and* $K : L^\infty(\Omega) \to L^\infty(\Omega)$ *are compact.*

Using the Riesz–Thorin interpolation theorem we obtain

COROLLARY 4.2. *We have* $K : L^p(\Omega) \to L^p(\Omega)$, $1 \le p \le \infty$, *with* $\|K\|_{p,p} \le M\|k\|_1$.

LEMMA 4.2. *Let* $0 < d \le 1$. *Then*

(a) *for any* $z_1, z_2 \in L^p(\Omega(d))$, $1 \le p \le \infty$,

$$\|KG(z_1) - KG(z_2)\|_{p,\Omega(d)} \le M^2 \int_A |k(v)|\,dv\, \|z_1 - z_2\|_{p,\Omega(d)},$$

where $A = [0,d] \times \widetilde{D}$,

(b) $KG(z) \in N_1^\alpha(\Omega(d))$ *for any* $z \in L^1(\Omega(d))$,

(c) $KG(z) \in C^\alpha(\Omega(d))$ *for any* $z \in L^\infty(\Omega(d))$.

Proof. First we note that for any $z \in L^p(\Omega(d))$, in view of (4.1), we have

$$|G(z)(u)| \le M|z(u)| + |G(0)(u)| \tag{4.15}$$

for almost every $u \in \Omega(d)$. Therefore by (4.2) and Corollary 4.2 the operator KG is well defined on $L^p(\Omega(d))$, $1 \le p \le \infty$.

(a) By (4.7) we get

$$\|KG(z_1) - KG(z_2)\|_p \le \|K\|_{p,p}\|L(\psi)\|_{p,p}\|z_1 - z_2\|_p$$

for $z_1, z_2 \in L^p(\Omega(d))$. It is clear that $\|L(\psi)\|_{p,p} \le M$. Therefore our assertion follows by Corollary 4.2.

(b) and (c). In view of (4.15) our assertions follow by Lemma 4.1.

LEMMA 4.3. *The operators* $KG : L^\infty(\Omega) \to L^\infty(\Omega)$ *and* $KG : L^1(\Omega) \to L^1(\Omega)$ *are continuous and compact.*

Proof. By (4.7) and (4.15) it follows that G considered both on $L^1(\Omega)$ and on $L^\infty(\Omega)$ is continuous and bounded. Therefore our assertion follows from Corollary 4.1.

We are now ready to consider the equation (1.1):

THEOREM 4.1. *Let* $f \in L^p(\Omega)$, $1 \le p \le \infty$. *Then the equation* (1.1) *has a unique solution* $z_0 \in L^p(\Omega)$. *If* $f \in C(\Omega)$, *then* $z_0 \in C(\Omega)$.

Proof. Choose a sufficiently small $d > 0$ so that $2M^2 \int_A |k(v)|\,dv < 1$, where $A = [0,d] \times \widetilde{D}$. By Lemma 4.2(a) the operator KG considered both on $L^1(\Omega(d))$ and on $L^\infty(\Omega(d))$ is a contraction. Hence there exists a unique solution z_0 of (1.1) defined on $\Omega(d)$, if f belongs to $L^1(\Omega)$ or to $L^\infty(\Omega)$.

Consider the following complete metric spaces:

$X_1 = \{z \in L^1(\Omega(2d)) : z = z_0 \text{ on } \Omega(d)\}$ in the first case,
$X_2 = \{z \in L^\infty(\Omega(2d)) : z = z_0 \text{ on } \Omega(d)\}$ in the second case.

For any z_1 and z_2 either both in X_1 or both in X_2 we have

$$KG(z_1)(u) - KG(z_2)(u) = 0 \quad \text{for } u \in \Omega(d)\,, \tag{4.16}$$

$$|KG(z_1)(u) - KG(z_2)(u)| \le M^2 \int_{\Omega(d,2d)} |k(u-v)|\,|z_1(v) - z_2(v)|\,dv \quad \text{for } u \in \Omega(d,2d)\,. \tag{4.17}$$

Combining (4.16), (4.17) with (4.6) we get

$$\|KG(z_1) - KG(z_2)\|_p \le M^2 \int_{\Omega(d)} |k(v)|\,dv \|z_1 - z_2\|_p\,,$$

where $p = 1$ and ∞ in the first and second case, respectively. All the norms are taken over $\Omega(2d)$.

Therefore KG is a contraction on X_1 and on X_2. As a consequence there exists a unique extension of z_0 to a solution of (1.1) on $\Omega(2d)$. Repeating this procedure we are led to a unique solution of our equation defined on the whole Ω, when f belongs to $L^1(\Omega)$ or to $L^\infty(\Omega)$.

We now turn to the case $1 < p < \infty$. Let $f \in L^p(\Omega) \subseteq L^1(\Omega)$. Then (1.1) has a unique solution $z_0 \in L^1(\Omega)$. According to (4.7) we have

$$KG(z_0) - KG(0) = L(\psi_0)z_0$$

for some $\psi_0 \in L^\infty(\Omega)$, $\|\psi_0\|_\infty \le M$. Therefore taking $f_0 = f + KG(0)$ we can write

$$z_0 - KL(\psi_0)z_0 = f_0\,.$$

According to Lemma 4.2(c), $KG(0) \in C^\alpha(\Omega)$. Hence $f_0 \in L^p(\Omega)$.

Now we consider the operator $I - KL(\psi_0)$. Define

$$g(t,u) = t\psi_0(u) \quad \text{for } t \in \mathbf{R}\,,\ u \in \Omega\,.$$

Since g satisfies (4.1) and (4.2) the homogeneous equation $z - KL(\psi_0)z = 0$ has no nontrivial solution. Now Lemma 4.3 allows us to use the Fredholm alternative to prove the existence of the bounded linear operators $(I - KL_0)^{-1} : L^\infty(\Omega) \to L^\infty(\Omega)$ and $(I - KL_0)^{-1} : L^1(\Omega) \to L^1(\Omega)$, where $L_0 = L(\psi_0)$. By the Riesz–Thorin interpolation theorem $(I - KL_0)^{-1}$ is bounded on $L^p(\Omega)$ for all $1 \le p \le \infty$. Since $z_0 = (I - KL_0)^{-1} f_0$, we get $z_0 \in L^p(\Omega)$.

It remains to consider the case $f \in C(\Omega)$. Since then $z_0 \in L^\infty(\Omega)$, by Lemma 4.2(c) it follows that $KG(z_0) \in C^\alpha(\overline{\Omega})$. Therefore by (1.1) we get $z_0 \in C(\overline{\Omega})$.

Corollary 4.3. *Let $f \in N_1^\beta(\Omega)$, $0 < \beta < 1$. Then the solution z_0 of (1.1) belongs to $N_1^\gamma(\Omega)$, where $\gamma = \min(\alpha, \beta)$. If $f \in C^\beta(\overline{\Omega})$, then $z_0 \in C^\gamma(\overline{\Omega})$.*

Proof. Since $z_0 = KG(z_0) + f$ our assertion follows from Theorem 4.1 and the obvious imbeddings

$$C^\alpha(\overline{\Omega}) \cup C^\beta(\overline{\Omega}) \subseteq C^\gamma(\overline{\Omega}) \subseteq C(\overline{\Omega}), \quad N_1^\alpha(\Omega) \cup N_1^\beta(\Omega) \subseteq N_1^\gamma(\Omega) \subseteq L^1(\Omega).$$

5. The approximate problem. In this section we construct a certain approximate solution of (1.1) in finite-dimensional function spaces.

Consider a sequence $\{X_n\}$, $n \in \mathbb{N}$ of finite-dimensional subspaces of $L^\infty(\Omega)$ and a sequence $\{P_n\}$, $n \in \mathbb{N}$, of projections $P_n : L^1(\Omega) \to X_n$ which satisfy the following conditions:

$$P_n \circ P_n = P_n, \quad n \in \mathbb{N}, \tag{5.1}$$

$$\|P_n\|_{1,1} \le C \quad \text{for every } n \in \mathbb{N}, \tag{5.2}$$

$$\|P_n\|_{\infty,\infty} \le C \quad \text{for every } n \in \mathbb{N}, \tag{5.3}$$

$$\|\varphi - P_n\varphi\|_\infty \to 0 \quad \text{as } n \to \infty \text{ for every } \varphi \in C(\overline{\Omega}), \tag{5.4}$$

where the constant C in (5.2) and (5.3) is independent of n.

Since $C(\overline{\Omega})$ is a dense subspace of $L^1(\Omega)$, the Banach–Steinhaus theorem yields

Remark 5.1. $\|\varphi - P_n\varphi\|_1 \to 0$ as $n \to \infty$ for every $\varphi \in L^1(\Omega)$.

By the Riesz–Thorin interpolation theorem we immediately obtain

Remark 5.2. $P_n : L^p(\Omega) \to L^p(\Omega)$ with $\|P_n\|_{p,p} \le C$ for $1 \le p \le \infty$, where C is the same constant as in (5.2) and (5.3).

Our approximate problem is formulated as follows: find $z_n \in X_n$ which satisfies

$$z_n - P_n KG(z_n) = P_n f. \tag{5.5}$$

For the study of (5.5) it will be convenient to introduce auxiliary functions y_n defined by

$$y_n = f + KG(z_n).$$

It is easy to verify that y_n satisfies the equations

$$z_n = P_n y_n, \tag{5.6}$$

$$y_n - KG(P_n y_n) = f. \tag{5.7}$$

It should also be noted that if y_n is a solution of (5.7) and $z_n = P_n y_n$, then z_n satisfies (5.5).

Before proving the existence and uniqueness theorem for (5.5) and (5.7) we prove a lemma essential for our further considerations.

For any $n \in \mathbb{N}$ and for any function $\varphi \in L^\infty(\Omega)$ with $\|\varphi\|_\infty \le M$ we denote by $L(n,\varphi)$ the linear operator defined on $L^1(\Omega)$ by the formula

$$L(n,\varphi)z = \varphi P_n z \quad \text{for } z \in L^1(\Omega). \tag{5.8}$$

The family of all these operators will be denoted by $\mathcal{L}_M(\mathbb{N})$.

LEMMA 5.1. *There exist positive constants C_1, C_2 and N_0 such that*

$$C_1\|z\|_p \le \|(I - KL(n,\psi))z\|_p \le C_2\|z\|_p \tag{5.9}$$

for any $z \in L^p(\Omega)$, $1 \le p \le \infty$, and any $L(n,\psi) \in \mathcal{L}_M(\mathbb{N})$ with $n \ge N_0$.

Proof. The right inequality in (5.9) follows immediately from Corollary 4.2 and the estimate (5.2).

Since

$$P_n z = KL(n,\psi)z + (P_n z - KL(n,\psi)z) \tag{5.10}$$

we get

$$|P_n z(u)| \le M \int_\Omega |k(u-v)|\,|P_n z(v)|\,dv + |P_n z(u) - KL(n,\psi)z(u)|$$

for $u \in \Omega$. Now applying Lemma 3.2 we conclude that

$$\|P_n z\|_1 \le C\|P_n z - KL(n,\psi)z\|_1, \tag{5.11}$$

where the constant C depends on k only.

First we consider the case $p = 1$ or ∞. Of course, it suffices to study the case $\|z\|_p = 1$. To get a contradiction assume that there exists a sequence $z_m \in L^1(\Omega)$ with $\|z_m\|_p = 1$ and a sequence of operators $L_m = L(n_m,\psi_m)$ with $n_m \to \infty$ as $m \to \infty$ such that

$$\|(I - KL_m)z_m\|_p \to 0 \quad \text{as } m \to \infty. \tag{5.12}$$

Since, by Lemma 4.1, K is compact, and by (5.2), the L_m are uniformly bounded, the set $\{KL_m z_m : m \in \mathbb{N}\}$ is relatively compact in $L^1(\Omega)$. Without loss of generality we can assume that $KL_m z_m \to z$ in $L^1(\Omega)$ as $m \to \infty$ for some z. It then follows by (5.12) that

$$z_m \to z \quad \text{as } m \to \infty, \qquad \|z\|_p = 1. \tag{5.13}$$

Since KL_m, $m \in \mathbb{N}$, are uniformly bounded we get

$$\|KL_m(z_m - z)\|_p \to 0 \quad \text{as } m \to \infty. \tag{5.14}$$

Combining (5.12), (5.13) and (5.14) we obtain

$$\|z - KL_m z\|_p \to 0 \quad \text{as } m \to \infty. \tag{5.15}$$

Noting that $P_{n_m} z - KL(n_m,\psi_m)z = (z - KL(n_m,\psi_m)z) - (z - P_{n_m} z)$ we get, in view of (5.11),

$$(1/C)\|P_{n_m} z\|_1 \le \|z - KL(n_m,\psi_m)z\|_1 + \|z - P_{n_m}\|_1.$$

By (5.15) and Remark 5.1 the right-hand side of this inequality tends to 0 as $m \to \infty$, while the left-hand side tends to $(1/C)\|z\|_1 = 1/C$. The derived contradiction completes the proof in the case $p = 1$ or ∞.

From the obtained estimates it follows immediately that the homogeneous equation

$$z - KL(n,\psi)z = 0 \tag{5.16}$$

has no nontrivial solutions in $L^1(\Omega)$, if $n > N_0$.

In view of Lemma 4.1 and the conditions (5.2) and (5.3) the operators $KL(n,\psi)$ considered both on $L^1(\Omega)$ and on $L^\infty(\Omega)$ are compact. Therefore we can apply the Fredholm alternative to prove the existence of the inverse operators $(I - KL(n,\psi))^{-1}$ for $n > N_0$ both on $L^1(\Omega)$ and on $L^\infty(\Omega)$. Taking C_1 so that

$$\|(I - KL(n,\psi))^{-1}\|_{p,p} \le 1/C_1 \quad \text{for } p = 1 \text{ and } \infty$$

from the Riesz–Thorin interpolation theorem we get

$$\|(I - KL(n,\psi))^{-1}\|_{p,p} \le 1/C_1 \quad \text{for } 1 < p < \infty\,.$$

This completes the proof.

The results obtained at the end of the proof of Lemma 5.1 are collected in

COROLLARY 5.1. *For $n > N_0$ and $\psi \in L^\infty(\Omega)$ with $\|\psi\|_\infty \le M$ there exists the inverse operator $(I - KL(n,\psi))^{-1} : L^p(\Omega) \to L^p(\Omega)$, $1 \le p \le \infty$, with $\|(I - KL(n,\psi))^{-1}\|_{p,p} \le 1/C_1$, where C_1 is the same constant as in Lemma* 5.1.

Now we estimate the errors of approximations to the solution z_0 of (1.1) by solutions of (5.5) and (5.7). The results are collected in

THEOREM 5.1. *Let $f \in L^p(\Omega)$, $1 \le p \le \infty$. Then for sufficiently large n*

(a) *the problem* (5.5) *has a unique solution $z_n \in X_n$ for which*

$$C_1\|z_0 - P_n z_0\|_p \le \|z_0 - z_n\|_p \le C_2\|z_0 - P_n z_0\|_p\,, \tag{5.17}$$

(b) *the problem* (5.7) *has a unique solution y_n for which*

$$\|z_0 - y_n\|_p \le C_3\|z_0 - P_n z_0\|_p\,, \tag{5.18}$$

where the constants C_1, C_2 and C_3 are independent of n.

Proof. Similarly to (4.7) for any z we can find $\psi \in L^\infty(\Omega)$ with $\|\psi\|_\infty \le M$ such that

$$g(P_n z(u), u) - g(0,u) = \psi(u) P_n z(u)\,.$$

Therefore for $z \in L^1(\Omega)$ we get

$$KG(P_n z) - KG(0) = KL(n,\psi)z\,. \tag{5.19}$$

Now applying Lemma 5.1 we obtain the estimate

$$(5.20)\quad C_1\|z\|_1 - \|KG(0)\|_1 \le \|(I-\lambda KGP_n)z\|_1 \le C_2\|z\|_1 + \|KG(0)\|_1$$

for $\lambda \in [0,1]$, $z \in L^1(\Omega)$ and $n > N_0$.

Note that by (5.20) any solution z_λ of the equation

$$z_\lambda - \lambda KG(P_n z_\lambda) = f\,,$$

where $\lambda \in [0,1]$ and $n > N_0$, can be estimated as follows:

$$\|z_\lambda\|_1 \le (1/C_1)(\|f\|_1 + \|KG(0)\|_1)\,.$$

By Lemma 4.2(a) and (5.2) it follows that λKGP_n, $\lambda \in [0,1]$, are continuous and compact as the operators on $L^1(\Omega)$. Therefore using Theorem 3.1 we conclude that (5.7) has a solution y_n. Then $P_n y_n$ satisfies (5.5).

We are going to demonstrate the required regularity of the solutions of (5.7) in the case of $L^p(\Omega) \subseteq L^1(\Omega)$, $1 < p < \infty$. Adding $KG(0)$ to both sides of (5.7) and proceeding similarly to (5.19) we find $\psi_1 \in L^\infty(\Omega)$ with $\|\psi_1\|_\infty \le M$ such that

$$(5.21)\qquad y_n - KL(n,\psi_1)y_n = f + KG(0)\,.$$

Since $KG(0) \in L^\infty(\Omega)$, the right-hand side of (5.21) belongs to $L^p(\Omega)$. Now, by Corollary 5.1 it follows that $y_n \in L^p(\Omega)$.

We turn to the uniqueness problem for (5.7). In a similar manner to (5.19) for any $z_1, z_2 \in L^1(\Omega)$ we can find $\psi_2 \in L^\infty(\Omega)$ with $\|\psi_2\|_\infty \le M$ such that

$$(5.22)\qquad KG(P_n z_1) - KG(P_n z_2) = KL(n,\psi_2)(z_1 - z_2)\,.$$

Using (5.22) we can apply Lemma 5.1 to obtain the estimate

$$(5.23)\qquad C_1\|z_1 - z_2\|_p \le \|(z_1 - z_2) - (KG(P_n z_1) - KG(P_n z_2))\|_p$$

for any z_1 and $z_2 \in L^p(\Omega)$, $1 \le p \le \infty$ and $n \ge N_0$.

Hence we get immediately the uniqueness of solution of (5.7). If z_n satisfies (5.5), then $y_n = f + KG(z_n)$ is a solution of (5.7) and $z_n = P_n y_n$. Therefore the uniqueness for (5.7) implies the uniqueness for (5.5).

Combining (1.1) with (5.7) we obtain

$$(5.24)\quad (z_0 - y_n) - (KG(P_n z_0) - KG(P_n y_n)) = KG(z_0) - KG(P_n z_n)\,.$$

Estimating the left-hand side of (5.24) in a similar way to (5.23) and applying Lemma 4.2 to the right-hand side we obtain (5.18).

Since by (5.6), $z_0 - z_n = z_0 - P_n z_0 + P_n(z_0 - y_n)$, the right inequality in (5.17) follows from Remark 5.2 and (5.18).

We now turn to the left inequality in (5.17). Combining (1.1) with (5.5) we get

$$z_0 - P_n z_0 = z_0 - z_n - P_n(KG(z_0) - KG(z_n)).$$

Therefore applying Lemma 4.2 and Remark 5.2 we get

$$\|z_0 - P_n z_0\|_p \le (1 + C)\|z_0 - z_n\|_p$$

if $z_0 \in L^p(\Omega)$, $1 \le p \le \infty$. Thus it suffices to set $C_1 = 1/(1+C)$ to complete the proof.

6. An example. In practice, the X_n are certain standard finite element spaces ([1]) and the P_n are the L^2-projections onto X_n. The stability of P_n considered as a map in L^p, $1 \le p \le \infty$, is usually shown under the requirements of quasi-uniformity of the triangulations underlying the definitions of the X_n (see [2], [3]).

Now we present an example of the subspaces X_n and the projections P_n. Let $\Omega = [0,1]^{q+1}$. Consider a sequence of quasi-uniform partitions

$$\Pi_n : \quad 0 = a_0 < a_1 < a_2 < \ldots < a_n = 1, \quad n \in \mathbf{N},$$

and set $h = \max[a_{i+1} - a_i]$, $0 \le i \le n-1$. Thus we get regular partitions of Ω given by the cubes

$$A_i = \prod_{j=1}^{q+1} [a_{i_j}, a_{i_{j+1}}], \quad i = (i_1, \ldots, i_{q+1}).$$

We are interested in the class of tensor product splines $S_n = \bigotimes_{n=1}^{q+1} S(\Pi_n)$, where $S(\Pi_n) = \{s \in C[0,1] : s\,|\,[a_j, a_{j+1}] \text{ is linear}, j = 0, 1, \ldots, n-1\}$.

Now we define $X_n = \{w \in C(\Omega) : w\,|\,A_i$ is a polynomial of degree $\le q+1\}$. It was pointed out in [3] that the L^2-orthogonal projections $P_n : L^2(\Omega) \to X_n$ satisfy conditions (5.2)–(5.4).

By Lemma 5.5 in [7] we have

$$\inf_{w \in S_n} \|f - w\|_1 \le C(h\|f\|_1 + \omega(f, h, 1)), \quad f \in L^1(\Omega),$$

where

$$\omega(f, h, 1) = \sup_{|\delta| < h} \int_{\Omega_\delta} |f(x + \delta) - f(x)|\,dx, \quad \delta \in \mathbf{R}^{q+1}.$$

Since $S_n \subseteq X_n$, the same estimate is valid for $\inf_{\xi \in X_n} \|f - \xi\|_1$. Therefore, for the solution z_0 of (1.1) as regular as in Corollary 4.3 we get, in view of Theorem 5.1, the estimate

$$\|z_0 - z_n\|_1 \le \text{const} \cdot h^\gamma.$$

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