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APPLICATIONES MATHEMATICAE
21,1 (1991), pp. 1–14

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LIMIT THEOREMS FOR NON-HOMOGENEOUS SEMI-MARKOV PROCESSES

Abstract. Non-homogeneous renewal processes and non-homogeneous semi-Markov processes are considered. In particular, Smith's Theorem is extended to the case of non-homogeneous renewal processes and the Central Limit Theorem for non-homogeneous semi-Markov processes is obtained.

1. Introduction. Homogeneous semi-Markov processes are not satisfactory models for many problems in reliability theory. Hence it is necessary to consider a wider class of processes, i.e. non-homogeneous semi-Markov processes.

In [10] a non-homogeneous renewal process $N(t)$ is generated by a sequence $(T_n)_{n\in\mathbf{N}}$ of independent and non-negative random variables with distributions (F^n) and expectations (m_n). If for every $n \in \mathbf{N}$ we set $S_0 = 0$, $S_n = \sum_{i=1}^n T_i$ then the *renewal function* $H^n(t - S_{n-1}, t)$ is defined as the conditional mean of the number of renewal moments over the random interval $(S_{n-1}, t]$, i.e.

$$H^n(t-S_{n-1},t) = \begin{cases} E(N(S_{n-1},t)|S_{n-1}) & \text{for } t \geq 0,\ \omega\in\{\omega : S_{n-1}(\omega) \leq t\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $N(x, y) = N(y) - N(x)$. In [10] the equality

$$H^n(x,t) = \sum_{k=n}^{\infty} F^{n,k}(x)$$

1985 *Mathematics Subject Classification*: 60K15.

Key words and phrases: Markov process, semi-Markov process, renewal equation, Smith's theorem, Blackwell theorem.

The main result of this paper was presented during the First World Congress of the Bernoulli Society in Tashkent.

is shown, where

$$F^{n,k}(x) = \begin{cases} 1 & \text{for } k < n, \\ \int_0^x F^k(du) F^{n,k-1}(x-u) & \text{for } k \geq n, \end{cases}$$

so $H^n(x,t) \equiv H^n(x)$. Moreover, some conditions for H^n to be finite are given.

In 1956 T. Kawata proved the following theorem.

THEOREM 1.1 [4]. *Suppose $T_1, T_2, \dots$ are independent random variables with distributions $F^1, F^2, \dots$ and expectations $m_1, m_2, \dots$ such that:*

(i) $\int_{-\infty}^0 e^{-sx} F^n(dx) < \infty$, *where $0 < s \leq s_0$ for some s_0, uniformly in n;*

(ii) $\lim_{A\to\infty} \int_A^\infty x F^n(dx) = 0$ *uniformly in n;*

(iii) $\lim_{A\to\infty} \int_{-\infty}^{-A} e^{-sx} F^n(dx) = 0$, *where $0 < s \leq s_0$ for some s_0, uniformly in n;*

(iv) $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n m_i = m$, *where $0 < m < \infty$.*

Then

$$\lim_{t\to\infty} \frac{1}{t} \int_{-\infty}^{t} (H^1(x+h) - H^1(x))\, dx = \frac{h}{m}.$$

H. Morimura [6] showed the equality

$$\lim_{t\to\infty} \frac{1}{t} \int_{-\infty}^{t} dx \sum_{n=1}^{\infty} \left(n - \frac{x}{M_n}\right) P\{x < S_n \leq x+h\} = \frac{h}{m^2}\left(\frac{v'}{m} - \frac{h}{2}\right)$$

where

$$M_n = \frac{1}{n}\sum_{i=1}^n m_i \to m \quad \text{as } n \to \infty,$$

$$V_n' = \frac{1}{n}\sum_{i=1}^n v_i' \to v' \quad \text{as } n \to \infty, \quad \text{where } v_i' = E(T_i^2).$$

Similar problems were considered by H.Hatori, who presented the following result in [3]. Let φ denote any Baire function integrable over $(0,\infty)$ and let $m_n \geq C$, $D(T_n) \leq K$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} n^{-1}\sum_{i=1}^n m_i = m > 0$. Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt \Big(\int_0^t \varphi(t-u)\, dN(u) \Big) = \frac{1}{m} \int_0^\infty \varphi(u)\, du \qquad \text{a.s.}$$

In this paper a similar theorem will be presented with $H^1(u)$ replacing $N(u)$ and without the assumptions $m_n \geq C$, $D(T_n) \leq K$.

On the other hand, in [9] a non-homogeneous semi-Markov process is defined as follows. Let $(T_n)_{n\in\mathbb{N}}$, $(X_n)_{n\in\mathbb{N}\cup\{0\}}$ be sequences of random variables such that for $n \in \mathbf{N}$, $T_n : \Omega \to [0,\infty)$, $X_n : \Omega \to B \subset \mathbf{N}$. Let $p = \{p_i;\ i \in B\}$ denote a distribution on B and for $n \in \mathbf{N}$, let $Q^n(t) = \{Q^n_{ij}(t);\ i,j \in B\}$ denote a semi-Markov matrix such that

$$P\{X_n = j,\ T_n \le t | X_0, X_1, \ldots, X_{n-1}, T_1, T_2, \ldots, T_{n-1}\} \\ = P\{X_n = j,\ T_n \le t | X_{n-1}\} \equiv Q^n_{X_{n-1}j}(t).$$

The sequence (T_n) generates a non-homogeneous renewal process $N(t)$. Hence the random process $X(t) = X_{N(t)}$ is called a *non-homogeneous semi-Markov process.*

This process is considered over some interval $(t, t+h]$ and therefore the matrices of transition probabilities after moment t are defined.

For $t, h \ge 0$, $0 \le y \le h$, let

$$K^0_{ij}(t,h,t+h) = \delta_{ij} \quad \text{(Kronecker's delta)},$$

$$K^1_{ij}(t,h,t+h) = P\{X_{N(t)+1} = j,\ S_{N(t)+1} \le t+h | X_{N(t)} = i\},$$

$$K^n_{ij}(t,h-y,t+h) = P\{X_{N(t)+n} = j,\ S_{N(t)+n} \le t+h | X_{N(t)+n-1} = i,\ S_{N(t)+n-1} = t+y\}.$$

By [9] the matrices $K^n(\cdot,\cdot,\cdot)$ can be obtained from the matrices Q^n. Moreover, let

$$K^{m,n}_{ij}(t,h_m,t+h) = \begin{cases} K^0_{ij}(t,h,t+h) & \text{for } n < m, \\ K^m_{ij}(t,h_m,t+h) & \text{for } n = m, \\ \sum_{k\in B} \int_0^{h_m} K^n_{ij}(t,dx,t+h-h_m+x) & \\ \qquad \times\ K^{m,n-1}_{kj}(t,h_m-x,t+h) & \\ = \sum_{k\in B} K^n_{ik} * K^{m,n-1}_{kj}(t,h_m,t+h) & \text{for } n > m, \end{cases}$$

where

$$h_n = \begin{cases} h & \text{for } n = 0,1, \\ h-y & \text{for } n > 1. \end{cases}$$

The functions $K^{m,n}_{ij}(t,h_m,t+h)$ are the transition probabilities after $n-m+1$ steps, i.e.

$$P\{X_{N(t)+n} = j,\ S_{N(t)+n} \le t+h | X(t) = i\} = K^{1,n}_{ij}(t,h,t+h), \qquad n = 1,2,\ldots,$$

$$P\{X_{N(t)+n} = j,\ S_{N(t)+n} \le t+h | X_{N(t)+m-1} = i,\ S_{N(t)+m-1} = t+y\} \\ = K^{m,n}_{ij}(t,h-y,t+h), \quad m = 2,3,\ldots,\ n = m, m+1, \ldots$$

DEFINITION 1.1. For every $n \in \mathbf{N}$, $t,h \ge 0$, let $K^n(t,\cdot,t+h) = \{K^n_{ij}(t,\cdot,t+h);\ i,j \in B\}$ be a given semi-Markov matrix; $L^n(t,\cdot,t+h) = \{L^n_{ij}(t,\cdot,t+h);\ i,j \in B\}$ be a given matrix of measurable and bounded functions on the interval $[0,h]$; and $U^n(t,\cdot,t+h) = \{U^n_{ij}(t,\cdot,t+h); i,j \in B\}$

be an unknown matrix of measurable and bounded functions on $[0,h]$. The system of linear integral equations of the form

$$(1.1)\quad U^n(t,h_n,t+h) = L^n(t,h_n,t+h) + \int_0^{h_n} K^n(t,dx,t+h-h_n+x)U^{n+1}(t,h_n-x,t+h)$$

is called the *renewal equation* for a non-homogeneous semi-Markov process.

In [9] it is proved that a unique solution of this system exists under some conditions and can be presented in the form

$$\begin{aligned} U^n(t,h_n,t+h) &= L^n(t,h_n,t+h) \\ &\quad + \sum_{k=n}^{\infty} \int_0^{h_n} K^{n,k}(t,dx,t+h-h_n+x)L^{k+1}(t,h_n-x,t+h) \\ &= L^n(t,h_n,t+h) + \sum_{k=n}^{\infty} K^{n,k} * L^{k+1}(t,h_n,t+h). \end{aligned}$$

2. Limit theorems for non-homogeneous renewal processes

THEOREM 2.1. *Let a sequence $(T_n)_{n\in\mathbf{N}}$ of non-negative, independent random variables with distributions $(F^n)_{n\in\mathbf{N}}$ and expectations $(m_n)_{n\in\mathbf{N}}$ satisfy*

(i) $\lim_{A\to\infty}\int_A^\infty xF^n(dx) = 0$ *uniformly in* n;

(ii) $m_n > 0$ *for* $n \in \mathbf{N}$ *and there exists* $0 < \mu < \infty$ *for which* $\lim_{k\to\infty}(m_1+\ldots+m_k)/k = \mu$.

If $L : \mathbf{R}_+ \to \mathbf{R}$ *is a measurable and integrable function of finite variation on* $[0,\infty)$, *then*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (H*L)(x)\,dx = \frac{1}{\mu}\int_0^\infty L(x)\,dx\,,$$

where H denotes the renewal function H^1.

Proof. Without loss of generality assume L is decreasing and non-negative. For some $x_0 > 0$ introduce the following notations:

$$\begin{aligned} I(t) &= \frac{1}{t}\int_0^t\int_0^x H(du)L(x-u)\,dx \\ &= \frac{1}{t}\Big(\int_0^{x_0}\int_0^x + \int_{x_0}^t\int_0^{x_0} + \int_{x_0}^t\int_{x_0}^x\Big)H(du)L(x-u)\,dx \\ &\equiv I_1(t,x_0) + I_2(t,x_0) + I_3(t,x_0), \end{aligned}$$

$$c(t) = \int_0^t L(x)\,dx\,.$$

Let $\varepsilon > 0$, $h > 0$ satisfy $h \le \varepsilon/L(0)$ and let $x_0 > 0$ be such that for all $x \ge x_0$

$$\left|\frac{1}{x}\int_0^x \frac{H(y+h)-H(y)}{h}\,dy - \frac{1}{\mu}\right| < \varepsilon \quad \text{and } H(x) > 0\,.$$

(Theorem 1.1 implies the existence of such an x_0.) Finally, define $v = [(t-x_0)/h]$.

By Theorem 1.1, $H(x) < \infty$ for $x \ge 0$. From the integrability of L we can choose t to satisfy

1) $\frac{1}{t}\int_0^t L(x)\,dx \le \frac{\varepsilon}{H(x_0+h)}$,
2) $\frac{1}{v}\sum_{i=1}^{v} hiL(hi) \le \varepsilon$,
3) $\frac{1}{v}\sum_{i=1}^{v} L(hi) \le \frac{\varepsilon}{x_0}$,
4) $t \ge \frac{1}{\mu L(0)}$,
5) $t \ge 2x_0 + h$.

Now bound the integrals $I_k(t,x_0)$, $k = 1,2,3$, as follows:

$$0 \le I_1(t,x_0) = \frac{1}{t}\int_0^{x_0} H(du)\int_u^{x_0} L(x-u)\,dx$$

$$= \frac{1}{t}\int_0^{x_0} H(du)\int_0^{x_0-u} L(y)\,dy \le \frac{H(x_0)}{t}\int_0^t L(y)\,dy \le \varepsilon\,,$$

$$0 \le I_2(t,x_0) \le \frac{1}{t}\int_{x_0}^t L(x-x_0)H(x_0)\,dx \le H(x_0)\frac{1}{t}\int_0^t L(y)\,dy \le \varepsilon\,.$$

For $I_3(t,x_0)$ we have

$$(2.1) \qquad I_3(t,x_0)$$
$$= \frac{1}{t}\int_{x_0}^t \Big(\sum_{i=0}^{v-1}\int_{x-h(i+1)}^{x-hi} H(du)L(x-u) + \int_{x_0}^{x-hv} H(du)L(x-u)\Big)\,dx\,,$$

so that

$$I_3(t,x_0) \le \frac{1}{t} \int\limits_{x_0}^{t} \sum_{i=0}^{v-1} hL(hi) \frac{H(x-hi)-H(x-h(i+1))}{h}\,dx$$

$$+ \frac{1}{t} \int\limits_{x_0}^{t} H(du) \int\limits_{x_0}^{x_0+h} L(x-u)\,dx \equiv A_3(t,x_0) + B_3(t,x_0)\,.$$

But

$$B_3(t,x_0) \le \frac{1}{t} \int\limits_{x_0+h}^{t} L(x-x_0-h)(H(x_0+h)-H(x_0))\,dx$$

$$\le H(x_0+h)\frac{1}{t}\int\limits_{0}^{t} L(y)\,dy \le \varepsilon$$

and if we change the order of the sum and the integral and substitute $y_i = x - h(i+1)$ for every i in $A_3(t,x_0)$ then we get

$$A_3(t,x_0) \le \frac{1}{t}\sum_{i=0}^{v} hL(hi) \int\limits_{x_0+h(i+1)}^{t} \frac{H(x-hi)-H(x-h(i+1))}{h}\,dx$$

$$= \frac{1}{t}\sum_{i=0}^{v-1} hL(hi) \int\limits_{x_0}^{t-h(i+1)} \frac{H(y_i+h)-H(y_i)}{h}\,dy_i$$

$$\le \frac{1}{t}\sum_{i=0}^{v-1} hL(hi) \int\limits_{0}^{t} \frac{H(y+h)-H(y)}{h}\,dy$$

$$\le \left(\frac{1}{\mu}+\varepsilon\right)\left(\sum_{i=1}^{v} hL(hi) + hL(0)\right) \le \left(\frac{1}{\mu}+\varepsilon\right)(c(t-x_0)+\varepsilon)\,.$$

On the other hand, using the rectangle method for bounding $c(t)$ we obtain

$$I_3(t,x_0) \ge \frac{1}{t}\sum_{i=0}^{v-1} hL(h(i+1)) \int\limits_{x_0}^{t} \frac{H(x-hi)-H(x-h(i+1))}{h}\,dx$$

$$\ge \left(\frac{1}{\mu}-\varepsilon\right)\sum_{i=1}^{v} hL(hi) - \left(\frac{1}{\mu}+\varepsilon\right)\frac{x_0}{t}\sum_{i=1}^{v} hL(hi)$$

$$- \left(\frac{1}{\mu}-\varepsilon\right)\frac{h}{t}\sum_{i=1}^{v} hiL(hi) - 2\varepsilon$$

$$\ge \left(\frac{1}{\mu}-\varepsilon\right)(c(t-x_0)-\varepsilon) - \left(\frac{1}{\mu}+\varepsilon\right)\varepsilon - \left(\frac{1}{\mu}-\varepsilon\right)\varepsilon - 2\varepsilon\,.$$

Finally, we get

$$\left(\frac{1}{\mu}-\varepsilon\right)(c(t-x_0)-\varepsilon)-\frac{2\varepsilon}{\mu}-2\varepsilon \le I_3(t,x_0)$$
$$\le \left(\frac{1}{\mu}+\varepsilon\right)(c(t-x_0)+\varepsilon)+\varepsilon\,,$$

so that

$$\lim_{t\to\infty} I(t)=\frac{1}{\mu}\int_0^\infty L(x)\,dx$$

and the proof is complete.

THEOREM 2.2. *Let the assumptions* (i)–(ii) *of Theorem* 2.1 *hold. Moreover, let* $(L^n)_{n\in\mathbb{N}}$ *be a sequence of measurable and integrable functions of finite variation on* $[0,\infty)$ *and suppose there exists an integrable and measurable function* L *of finite variation on* $[0,\infty)$ *such that the series*

$$\sum_{n=1}^\infty (L-L^n)(x)$$

is uniformly convergent. Then for every $n\in\mathbb{N}$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{k=1}^\infty (F^{1,k}*L^{k+1})(x)\,dx=\frac{1}{\mu}\int_0^\infty L(x)\,dx\,.$$

Proof. Using Theorem 2.1 it is enough to show that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{k=1}^\infty (F^{1,k}*(L-L^{k+1}))(x)\,dx=0$$

or

$$\lim_{t\to\infty}\sum_{k=1}^\infty (F^{1,k}*(L-L^{k+1}))(t)$$
$$=\lim_{t\to\infty}\sum_{k=1}^\infty\int_0^t F^{1,k}(dx)(L-L^{k+1})(t-x)=0\,.$$

Since the series

$$\sum_{n=1}^\infty (L(x)-L^n(x))$$

is uniformly convergent, and for $k\in\mathbb{N}$ the function $L(x)-L^k(x)$ is integrable and of finite variation on $[0,\infty)$, there is $C>0$ such that $|L(x)-L^k(x)|<C$ for all $k\in\mathbb{N}$, $x>0$. For any $0<\varepsilon<1$, $k\in\mathbb{N}$ take t_k such that for all $t>t_k$

1) $\sup_{x\in[t/2,t]} |L(x) - L^{k+1}(x)| < \varepsilon/2^{k+1}$,

2) $F^{1,k}(t) - F^{1,k}(t/2) < \varepsilon/(C2^{k+1})$.

For any $k \in \mathbf{N}$, $t > t_k$ we can bound

$$\Big| \int_0^t F^{1,k}(dx)(L(t-x) - L^{k+1}(t-x))\Big| \le \Big| \int_0^{t/2} \dots \Big| + \Big| \int_{t/2}^{t} \dots \Big|$$
$$\le \sup_{x\in[t/2,t]} |L(x) - L^{k+1}(x)| F^{1,k}(t/2) + C(F^{1,k}(t) - F^{1,k}(t/2))$$
$$< \frac{\varepsilon}{2^{k+1}} + C\frac{\varepsilon}{C2^{k+1}} = \frac{\varepsilon}{2^k}.$$

Hence we get the inequality

$$\lim_{t\to\infty} \sum_{k=1}^{\infty} |F^{1,k} * (L - L^{k+1})(t)| = \sum_{k=1}^{\infty} \lim_{t\to\infty} |F^{1,k} * (L - L^{k+1})(t)|$$
$$\le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

leading to the asymptotical equality

$$\sum_{k=1}^{\infty} (F^{1,k} * L^{k+1})(t) \sim \Big(\Big(\sum_{k=1}^{\infty} F^{1,k}\Big) * L\Big)(t),$$

so the theorem is proved.

3. Limit theorems for non-homogeneous semi-Markov processes. First for any $n \in \mathbf{N}$, $i, j \in B$ define the "counting process" M_{ij}^n connected with a non-homogeneous semi-Markov process $X(t)$ by

$$M_{ij}^n(t - S_{n-1}, t) = \sum_{k=1}^{N(t)-n+1} \delta_{X_{n-1+k},j}.$$

The process M_{ij}^n "counts" the number of hits of the jth state by the process $X(t)$ over the random interval $(S_{n-1}, t]$ under the condition $X_{N(t)+n-1} = i$.

Let us define the random variable

$$\tau_{ij}^{n,m}(t) = \inf\{y > S_{N(t)+n-1} : M_{ij}^{N(t)+n}(y - S_{N(t)+n-1}, y) \ge m\},$$
$$n, m \in \mathbf{N},$$

as the moment of the mth visit in the jth state for the process $X(t)$ after the moment $S_{N(t)+n-1}$ under the condition $X_{N(t)+n-1} = i$. The random variables

$$q_{ij}^{n,1}(t) = \tau_{ij}^{n,1}(t) - S_{N(t)+n-1}, \quad n \in \mathbf{N},$$
$$q_{ij}^{n,m}(t) = \tau_{ij}^{n,m}(t) - \tau_{ij}^{n,m-1}(t), \quad n \in \mathbf{N},\ m \in \mathbf{N} \setminus \{1\},$$

represent time distances between the $(m-1)$st hit and the mth hit in the jth state. Denote the expectations of these random variables by $\mu_{ij}^{n,m}(t)$.

Moreover, for $t, h \geq 0$, $y \in [0, h]$, set

$$V_{ij}^{1,k}(t,h,t+h) = \begin{cases} P\{\tau_{ij}^{1}(t) = S_{N(t)+k}, \tau_{ij}^{1}(t) \leq t+h\} & \text{for } k \in \mathbf{N}, \\ 0 & \text{for } k = 0; \end{cases}$$

$$V_{ij}^{n,k}(t,h-y,t+h) = \begin{cases} P\{\tau_{ij}^{n}(t) = S_{N(t)+k}, \\ \qquad \tau_{ij}^{n}(t) \leq t+h | S_{N(t)+n-1} = t+y\} \\ \qquad \text{for } n = 2,3,\ldots,\ k = n, n+1, \ldots, \\ 0 \qquad \text{otherwise;} \end{cases}$$

$$V_{ij}^{n,k,m}(t,h_n,t+h) = \begin{cases} V_{ij}^{n,k}(t,h_n,t+h) \\ \qquad \text{for } m = 1,\ n = 1,2,\ldots, \\ \qquad k = n, n+1, \ldots, \\ \sum_{r=n}^{k-m+1} \int_0^{h_n} V_{ij}^{n,r,1}(t,dx,t+h-h_n+x) \\ \qquad \times V_{jj}^{r+1,k,m-1}(t,h_n-x,t+h) \\ = \sum_{r=n+m-2}^{k-1} \int_0^{h_n} V_{ij}^{n,r,m-1}(t,dx,t+h-h_n+x) \\ \qquad \times V_{jj}^{r+1,k,1}(t,h_n-x,t+h) \\ \qquad \text{for } n = 1,2,\ldots,\ m = 2,3,\ldots, \\ \qquad k = n+m-1, n+m, \ldots, \\ 0 \qquad \text{otherwise.} \end{cases}$$

It is easy to show that

$$P\{\tau_{ij}^{1,m}(t) = S_{N(t)+k}, \tau_{ij}^{1,m}(t) \leq t+h\} = V_{ij}^{1,k,m}(t,h,t+h),$$
$$m \in \mathbf{N},\ k = m, m+1, \ldots,$$
$$P\{\tau_{ij}^{n,m}(t) = S_{N(t)+k}, \tau_{ij}^{n,m}(t) \leq t+h | S_{N(t)+n-1} = t+y\}$$
$$= V_{ij}^{n,k,m}(t,h-y,t+h),$$
$$n \in \mathbf{N} \setminus \{1\},\ m \in \mathbf{N},\ k = n+m-1, n+m, \ldots$$

If we denote the distributions of the random variables $\tau_{ij}^{n,m}(\cdot)$ by $W_{ij}^{n,m}(\cdot,\cdot,\cdot)$, i.e.

$$W_{ij}^{1,m}(t,h,t+h) = P\{\tau_{ij}^{1,m}(t) \leq t+h\},$$
$$W_{ij}^{n,m}(t,h-y,t+h) = P\{\tau_{ij}^{n,m}(t) \leq t+h | S_{N(t)+n-1} = t+y\},$$

then

$$W_{ij}^{n,m}(t,h_n,t+h) = \sum_{k=n+m-1}^{N(t,t+h)} V_{ij}^{n,k,m}(t,h_n,t+h), \qquad n,m \in \mathbf{N}.$$

Using the properties of the integral mean it is easy to prove the following

LEMMA 3.1. *If $f,g : \mathbf{R} \to \mathbf{R}$ are monotone functions such that $f(t) = g(t) = 0$ for $t < 0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (f*g)(x)\,dx = \lim_{t\to\infty} f(t) \lim_{t\to\infty} \frac{1}{t} \int_0^t g(x)\,dx\,.$$

THEOREM 3.1. *Let the following assumptions hold:*

(i) *the non-homogeneous semi-Markov process $X(t)$ is irreducible and regular;*

(ii) $\lim_{A\to\infty} \int_A^\infty xW_{jj}^{n,1}(t,dx,t+h-h_n+x) = 0$ *uniformly in n;*

(iii) *for any $t \geq 0$ $n \in \mathbf{N}$, $j \in B$ the finite limit*

$$\lim_{k\to\infty} (\mu_{jj}^{n,1}(t) + \ldots + \mu_{jj}^{n,k}(t))/k = v_j^n(t)$$

exists;

(iv) *for $i,j \in B$, $n \in \mathbf{N}$, $L_{ij}^n(t,h_n,t+h) = \delta_{ij}L_j^n(t,h_n,t+h)$ is a measurable and integrable function of finite variation for $h \in [0,\infty)$;*

(v) *the measurable functions $l_j^n(t,h_n,t+h) = \sup_{k\geq n} L_j^k(t,h_n,t+h)$ are integrable and have finite variation for $h \in [0,\infty)$.*

Then for every $n \in \mathbf{N}$, $i,j \in B$

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x U_{ij}^n(t,h_n,t+h)\,dh$$
$$\leq \int_0^\infty l_j^n(t,h_n,t+h)\,dh \sum_{s=n}^\infty \frac{1}{v_j^{s+1}(t)} \lim_{h\to\infty} V_{ij}^{n,s}(t,h_n,t+h)$$

where the sequence $(U^n(t,h_n,t+h))$ is the solution of the renewal equation (1.1).

Proof. Recall that

$$U_{ij}^n(t,h_n,t+h)$$
$$= \sum_{k=n-1}^\infty \sum_{m\in B} \int_0^{h_n} K_{im}^{n,k}(t,dx,t+h-h_n+x)L_{mj}^{k+1}(t,h_n-x,t+h)\,.$$

Let $n = 1$, $k \geq 1$ and $i, m \in B$. We can write

$$\begin{aligned} K_{im}^{1,k}(t,h,t+h) &= P\{X_{N(t)+k} = m,\ S_{N(t)+k} \leq t+h | X_{N(t)} = i\} \\ &= \sum_{r=1}^{k} P\{\tau_{im}^{1,r}(t) = S_{N(t)+k}, S_{N(t)+k} \leq t+h | X_{N(t)} = i\} \\ &= \sum_{r=1}^{k} V_{im}^{1,k,r}(t,h,t+h) \\ &= V_{im}^{1,k,1}(t,h,t+h) + \sum_{r=2}^{k} \sum_{s=1}^{k-r+1} (V_{im}^{1,s,1} * V_{mm}^{s+1,k,r-1}(t,h,t+h)), \end{aligned}$$

which leads to

$$\begin{aligned} U_{ij}^{1}(t,h,t+h) &= L_{ij}^{1}(t,h,t+h) + \sum_{k=1}^{\infty} (V_{ij}^{1,k} * L_{j}^{k+1})(t,h,t+h) \\ &\quad + \sum_{k=2}^{\infty} \sum_{r=2}^{k} \sum_{s=1}^{k-r+1} (V_{ij}^{1,s,1} * V_{jj}^{s+1,k,r-1} * L_{j}^{k+1})(t,h,t+h) \\ &= L_{ij}^{1}(t,h,t+h) + \sum_{k=1}^{\infty} (V_{ij}^{1,k} * L_{j}^{k+1})(t,h,t+h) \\ &\quad + \sum_{s=1}^{\infty} \sum_{r=2}^{\infty} \sum_{k=s+r-1}^{\infty} (V_{ij}^{1,s,1} * V_{jj}^{s+1,k,r-1} * L_{j}^{k+1})(t,h,t+h). \end{aligned} \tag{3.1}$$

The second term on the right-hand side can be bounded as follows:

$$\begin{aligned} \sum_{k=1}^{\infty} (V_{ij}^{1,k} * L_{j}^{k+1})(t,h,t+h) &\leq \left(\left(\sum_{k=1}^{\infty} V_{ij}^{1,k} \right) * l_{j}^{2} \right)(t,h,t+h) \\ &= (W_{ij}^{1,1} * l_{j}^{2})(t,h,t+h). \end{aligned}$$

Now let $h \to \infty$; then from the integrability and Lemma 3.1

$$\lim_{x\to\infty} \frac{1}{x} \int_{0}^{x} L_{ij}^{1}(t,h,t+h)\, dh = 0,$$

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{x} \int_{0}^{x} (W_{ij}^{1,1} * l_{j}^{2})(t,h,t+h)\, dh \\ = \lim_{h\to\infty} W_{ij}^{1,1}(t,h,t+h) \lim_{x\to\infty} \frac{1}{x} \int_{0}^{x} l_{j}^{2}(t,h,t+h)\, dh = 0. \end{aligned}$$

Using the above equalities we get

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x U_{ij}^1(t,h,t+h)\,dh$$

$$\leq \lim_{x\to\infty} \frac{1}{x} \int_0^x \sum_{s=1}^{\infty}\sum_{r=2}^{\infty}\Big(V_{ij}^{1,s,1} * \Big(\sum_{k=s+r-1}^{\infty} V_{jj}^{s+1,k,r-1}\Big) * l_j^2\Big)(t,h,t+h)\,dh$$

$$= \lim_{x\to\infty} \frac{1}{x} \int_0^x \sum_{s=1}^{\infty}\Big(V_{ij}^{1,s} * \Big(\sum_{r=2}^{\infty} W_{jj}^{s+1,r-1}\Big) * l_j^2\Big)(t,h,t+h)\,dh.$$

By the uniform convergence in s and Lemma 3.1 we have

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x U_{ij}^1(t,h,t+h)\,dh$$

$$\leq \sum_{s=1}^{\infty} \lim_{h\to\infty} V_{ij}^{1,s}(t,h,t+h) \lim_{x\to\infty} \frac{1}{x} \int_0^x \Big(\Big(\sum_{r=2}^{\infty} W_{jj}^{s+1,r-1}\Big) * l_j^2\Big)(t,h,t+h)\,dh.$$

The sum

$$\sum_{r=2}^{\infty} W_{jj}^{s+1,r-1}(t,h-y,t+h), \quad s \in \mathbf{N}, \tag{3.2}$$

is the expectation of the number of visits in the jth state over $(t+y,t+h]$ under the conditions $X_{N(t)+s} = j$, $S_{N(t)+s} = t+y$. If we consider the non-homogeneous renewal process with the renewal moments as the moments of the visits in the jth state then (3.2) is a sequence of renewal functions and hence Theorem 2.1 can be used. So

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x U_{ij}^1(t,h,t+h)\,dh$$

$$\leq \sum_{s=1}^{\infty} \lim_{h\to\infty} V_{ij}^{1,s}(t,h,t+h) \frac{1}{v_j^{s+1}(t)} \int_0^{\infty} l_j^2(t,h,t+h)\,dh.$$

For $n > 1$ the proof is similar and we omit it.

THEOREM 3.2. *Let the assumptions* (i)–(iv) *of Theorem* 3.1 *hold. Moreover, suppose for* $j \in B$ *there exists a measurable and integrable function* $l_j(t,h-y,t+h)$*of finite variation for* $h \in [0,\infty)$ *and such that the series*

$$\sum_{k=1}^{\infty}(l_j - L_j^k)(t,h-y,t+h)$$

is uniformly convergent in h. Then for $n \in \mathbb{N}$, $i, j \in B$,

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x U_{ij}^n(t, h_n, t+h)\, dh$$

$$= \int_0^\infty l_j(t, h_n, t+h)\, dh \sum_{s=n}^\infty \frac{1}{v_j^{s+1}(t)} \lim_{h\to\infty} V_{ij}^{n,s}(t, h_n, t+h).$$

Proof. Let $n = 1$. Consider the second and third terms on the right-hand side of the equality (3.1). By Theorem 2.2 we obtain

$$\lim_{h\to\infty} \sum_{k=1}^\infty (V_{ij}^{1,k} * L_j^{k+1})(t, h, t+h) = \lim_{h\to\infty} \Big(\Big(\sum_{k=1}^\infty V_{ij}^{1,k}\Big) * l_j\Big)(t, h, t+h)$$
$$= \lim_{h\to\infty} (W_{ij}^{1,1} * l_j)(t, h, t+h)$$

and

$$\lim_{h\to\infty} \sum_{k=s+r-1}^\infty (V_{jj}^{s+1,k,r-1} * L_j^k)(t, h, t+h)$$
$$= \lim_{h\to\infty} \Big(\Big(\sum_{k=s+r-1}^\infty V_{jj}^{s+1,k,r-1}\Big) * l_j\Big)(t, h, t+h)$$
$$= \lim_{h\to\infty} (W_{jj}^{s+1,r-1} * l_j)(t, h, t+h).$$

The rest of the proof is like in Theorem 3.1 but instead $\leq$ we write $=$.

COROLLARY 3.1. *Let the assumptions of Theorem 3.2 hold. If the sequence $(u_{jj}^{n,1}(t))_{n\in\mathbb{N}}$ has a finite limit $v_j(t)$ then for every $n \in \mathbb{N}$*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x U_{ij}^n(t, h_n, t+h)\, dh$$

$$= \lim_{h\to\infty} W_{ij}^{n,1}(t, h_n, t+h) \frac{1}{v_j(t)} \int_0^\infty l_j(t, h_n, t+h)\, dh.$$

Remark 3.1. Theorem 3.2 corresponds to a known theorem for homogeneous semi-Markov processes (see for instance [5]).

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Received on 5.5.1988;
revised version on 19.6.1989