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The Mellin analytic functionals and the Laplace–Beltrami operator on the Minkowski half-plane

by

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Abstract. In this paper the theory of Fourier analytic functionals is developed. These functionals are generalizations of some Fourier hyperfunctions. Then the Mellin analytic functionals theory is developed and Paley–Wiener type theorems for Fourier and Mellin analytic functionals are proved. The Mellin transform for Mellin analytic functionals is defined. These notions are applied to solve the Laplace–Beltrami equation and to study its solution in the space of Mellin analytic functionals.

0. Introduction. In this paper we introduce the notion of a Mellin analytic functional and we develop a theory of such functionals with a view to applications in the analysis of singular differential operators such as, for instance, the Laplace–Beltrami operator on a hyperbolic space.

In Section 2 we define the space of Fourier analytic functionals which are related with some equivalence classes of holomorphic functions of exponential type. These functionals are generalizations of Fourier hyperfunctions whose defining functions are of infraexponential type (Kaneko [1], Kawai [2], Zharinov [6]). More general analytic functionals with noncompact carrier were considered in Zharinov [6], Sargos–Morimoto [5] and Park–Morimoto [3].

It is shown in Section 3 that the Fourier transformation and the inverse Fourier transformation operate on Fourier analytic functionals. We prove Paley–Wiener type theorems for the Fourier transform of Fourier analytic functionals in Section 4.

In Section 5 we introduce the spaces of Mellin analytic functionals by using the substitution $w = e^{-\zeta} = (e^{-\zeta_1}, \ldots, e^{-\zeta_n})$ in some Fourier analytic functionals. Mellin analytic functionals corresponding to Fourier hyperfunctions are called *Mellin hyperfunctions*.

Section 6 contains the definitions of the Mellin transform of Mellin analytic functionals by evaluating the functional on the functions $\varphi_z(w) = w^{-z-1}$. We

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prove Paley–Wiener type theorems for the Mellin transform of Mellin analytic functionals.

The next section is devoted to the Laplace–Beltrami operator $P = x_1^2(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2) + c$ and we consider the equation

$$Pu = f \tag{1}$$

with f being a Mellin hyperfunction. We solve this equation by applying the Mellin transform and a successive approximation method. The Mellin hyperfunction f is assumed to have Mellin transform with a sufficiently large domain of holomorphy, and after a limit process we obtain a function which is the Mellin transform of a Mellin analytic functional. That functional is the solution of equation (1).

Section 8 contains some characterization of the Mellin hyperfunctions whose supports reduce to $\{0\}\times[0, t]$. We prove that if f is so then the solution of (1) is a Mellin hyperfunction with support in $\{0\}\times[0, t]$.

1. Notation and preliminary definitions. Throughout the paper we use the following notation: if $w\in \mathbf{C}^n$, $a\in\mathbf{R}^n$, then $w = (w_1, \ldots, w_n)$, $a = (a_1, \ldots, a_n)$ and $w^a = w_1^{a_1}\ldots w_n^{a_n}$. If $b = (b_1, \ldots, b_n)\in\mathbf{R}^n$ then $a < b$ (resp. $a \leqslant b$) denotes that $a_j < b_j$ (resp. $a_j \leqslant b_j$) for $j = 1, \ldots, n$, $[a, b] = [a_1, b_1]\times \ldots \times [a_n, b_n]$. For $w, z\in\mathbf{C}^n$, $wz = w_1z_1 + \ldots + w_nz_n$. If $a\in\mathbf{R}^n$ and $w\in\mathbf{R}^n$ (or $\mathbf{C}^n$) we use the notation $e^{a|w|} = e^{a_1|w_1|+\ldots+a_n|w_n|}$.

For $t\in\mathbf{R}_+^n$ we write $J_t = (0, t] = (0, t_1]\times\ldots\times(0, t_n]$. If there is no danger of misunderstanding, we drop the subscript t.

D^n will stand for the compactification of $\mathbf{R}^n$, $D^n = \mathbf{R}^n\cup S_\infty^{n-1}$, in other words D^n is $\mathbf{R}^n$ with added points at infinity in all directions.

If V is an open set, $V\subset\mathbf{C}^n$, let $\mathcal{O}(V)$ denote the space of holomorphic functions on V.

We shall use the maps $\mu(\zeta) = e^{-\zeta}$ and $\mu^{-1}(w) = -\ln w$, where $e^{-\zeta} = (e^{-\zeta_1}, \ldots, e^{-\zeta_n})$, $-\ln w = (-\ln w_1, \ldots, -\ln w_n)$. Here μ is a diffeomorphism from $\{\zeta\in\mathbf{C}^n;\ |\mathrm{Im}\,\zeta_j| < \pi/2, j = 1, \ldots, n\}$ to $\{w\in\mathbf{C}^n;\ \mathrm{Re}\,w_j > 0,\ j = 1, \ldots, n\}$.

For $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_j\in\{+, -\}$, we put $\sigma x = (\sigma_1x_1, \ldots, \sigma_nx_n)$ and $\Gamma_\sigma = \{x\in\mathbf{R}^n;\ \sigma x > 0\}$. We shall frequently consider the cone Γ_+, it is Γ_σ with $\sigma_j = +$ for all $j = 1, \ldots, n$. We define $\mathrm{sgn}\,\sigma$ to be $+1$ if the number of "$-$" in σ is even and -1 otherwise.

We assume that M is the closure in D^n of a cone $\Gamma_+ + u$ for $u = -\ln t\in\mathbf{R}^n$. If $a\in\mathbf{R}^n$ and $0\leqslant a < \pi/2$ we set $M^a = M + i[-a, a]$ and $\bar{J}^a = \bar{J}_t^a = \mu(M^a)$ (the image of M^a). Here $\mu(S_\infty^{n-1}) = \{0\}$. $\mathscr{S}^a$ will stand for the family of complex sets containing $\bar{J}^a$ defined in the following way: $U\in\mathscr{S}^a$ iff there exists a bounded neighbourhood V of $\bar{J}^a$ in $\mathbf{C}^n$ and some numbers $\alpha_j, \beta_j\in(a_j, \pi/2)$ for $j = 1, \ldots, n$ such that

$$U = V\cap\{(\{-\alpha_j < \mathrm{Arg}\,w_j < \beta_j\}\cup\{0\}),\ j = 1, \ldots, n\}.$$

2. Fourier analytic functionals

DEFINITION 1. If $V\subset D^n + i\mathbf{R}^n$ is an open set, then $\varphi\in\tilde{\mathcal{O}}^a(V)$ (the space of *exponentially increasing functions*) iff $\varphi\in\mathcal{O}(V\cap\mathbf{C}^n)$ and for every $\varepsilon > 0$ and every compact set $K\subset V$ (K is compact in $D^n + i\mathbf{R}^n$)

$$\sup\{|\varphi(z)e^{-(a+\varepsilon)|z|}|;\ z\in K\cap\mathbf{C}^n\} < \infty.$$

In the case $a = 0$ we obtain the space $\tilde{\mathcal{O}}(V) = \tilde{\mathcal{O}}^0(V)$ of *infraexponential functions*, the so-called space of *slowly increasing functions*.

DEFINITION 2. $\varphi\in \underset{\sim}{\mathcal{O}}_c^{a,\delta}(V)$ iff $\varphi\in\mathcal{O}(V\cap\mathbf{C}^n)$, φ is continuous on $\bar{V}\cap\mathbf{C}^n$ and

$$\|\varphi\|_\delta = \sup\{|\varphi(z)|e^{(a+\delta)|z|};\ z\in\bar{V}\cap\mathbf{C}^n\} < \infty.$$

$\underset{\sim}{\mathcal{O}}_c^{a,\delta}(V)$ is a Banach space with norm $\|\cdot\|_\delta$.

For a compact set K in $D^n + i\mathbf{R}^n$, let $\{V_m\}_{m\in\mathbf{N}}$ be a fundamental system of neighbourhoods of K satisfying $V_m \Supset V_{m+1}$, i.e. V_{m+1} has a compact neighbourhood in V_m with respect to the topology of $D^n + i\mathbf{R}^n$.

Now we put

$$\underset{\sim}{\mathcal{O}}^a(K) = \varinjlim_{m\to\infty} \underset{\sim}{\mathcal{O}}_c^{a,1/m}(V_m)$$

and it is a DFS-space.

More general spaces than $\underset{\sim}{\mathcal{O}}^a(K)$ were defined in Zharinov [6], Sargos–Morimoto [5] and Park–Morimoto [3] as the spaces of test functions for some analytic functionals with noncompact carrier. For our purpose it is sufficient to consider this special case.

If K is a compact set in D^n, the space $\underset{\sim}{\mathcal{O}}(K)$ (for $a = 0$) is the space of rapidly decreasing functions which are test functions for Fourier hyperfunctions with support in K. Such spaces are considered in [1], [2] and [6].

Let $K = K_1\times\ldots\times K_n$ be a compact set in D^n, and $a\in\mathbf{R}_+^n$. We set $K_j^a = K_j + i[-a_j, a_j]$, and $K^a = K_1^a\times\ldots\times K_n^a = K + i[-a, a]$. Then it is obvious that $\underset{\sim}{\mathcal{O}}^a(K^a)\subset\underset{\sim}{\mathcal{O}}^a(K)\subset\underset{\sim}{\mathcal{O}}(K)$.

We shall be concerned with the space dual to $\underset{\sim}{\mathcal{O}}^a(K^a)$.

DEFINITION 3. Analytic functionals in $[\underset{\sim}{\mathcal{O}}^a(K^a)]'$ are called *Fourier analytic functionals with carrier* K^a.

The space $[\underset{\sim}{\mathcal{O}}(K)]'$ is the space of *Fourier hyperfunctions with support* K. Since $[\underset{\sim}{\mathcal{O}}(K)]'\subset[\underset{\sim}{\mathcal{O}}^a(K^a)]'$ it follows that the Fourier analytic functionals are a generalization of the Fourier hyperfunctions.

Let $V = V_1\times\ldots\times V_n$ where V_j is an open neighbourhood of K_j^a. We have a natural generalization to Fourier analytic functionals of the theorem on characterization of compactly supported hyperfunctions and of Fourier hyperfunctions:

THEOREM 1.

$$[\underset{\sim}{\mathcal{O}}^a(K^a)]' \approx \tilde{\mathcal{O}}^a(V\# K^a)/\sum_{j=1}^n \tilde{\mathcal{O}}^a(V\#_j K^a). \tag{2}$$

Here $V \# K^a = (V_1\backslash K_1^a)\times \ldots \times (V_n\backslash K_n^a)$, *and* $V \#_j K^a = (V_1\backslash K_1^a)\times \ldots \times V_j \times \ldots \times (V_n\backslash K_n^a)$.

Proof (outline). The isomorphism is given in the following way: let $F \in \tilde{\mathcal{O}}^a(V \# K^a)$ and let $[F]^a$ be its equivalence class in the quotient space on the right side of (2). We can assign to $[F]^a$ the functional $f \in [\underset{\sim}{\mathcal{O}}^a(K^a)]'$ defined by the following formula: for $\varphi \in \underset{\sim}{\mathcal{O}}^a(U)$ $(U = U_1 \times \ldots \times U_n,\ U_j \supset K_j^a)$

$$f[\varphi] = \int_\gamma F(z)\varphi(z)\,dz$$

where $\gamma = \gamma_1 \times \ldots \times \gamma_n$, γ_j is an arbitrary curve contained in $V_j \cap U_j$, encircling K_j^a once in the counterclockwise direction.

It can be proved that the quotient space (2) is independent of the choice of the neighbourhood V.

Definition 4. Let $\sigma \in \{+,-\}^n$, $F \in \tilde{\mathcal{O}}^a(V \# K^a)$ and $F \equiv 0$ on $(V \# K^a) \cap (D^n + i\Gamma_\tau)$ for all $\tau \neq \sigma$. Then the Fourier analytic functional f assigned to F by the isomorphism (2) is called the *monomial Fourier analytic functional with defining function* F and denoted by $f = j^a_{\Gamma_\sigma}(F)$.

Like general hyperfunctions and Fourier hyperfunctions, a Fourier analytic functional f has the representation

$$f = \sum_\sigma \operatorname{sgn}\sigma\, j^a_{\Gamma_\sigma}(F_\sigma)$$

where $F_\sigma \in \tilde{\mathcal{O}}^a(V \# K^a)$ and $F_\sigma \equiv 0$ outside $D^n + i\Gamma_\sigma$.

3. Fourier transformation of Fourier analytic functionals. Let K_σ be a closed subcone of $\Gamma_\sigma + u$, for some $u \in \mathbf{R}^n$. It is easy to see that $e^{i\zeta z} \in \underset{\sim}{\mathcal{O}}^a(K_\sigma^a)$ for ζ with $\operatorname{Im}\zeta \in \Gamma_\sigma$, $|\operatorname{Im}\zeta_j| > a_j$ for $j = 1, \ldots, n$. Now, we can extend the Fourier transformation from $[\underset{\sim}{\mathcal{O}}(K_\sigma)]'$ to $[\underset{\sim}{\mathcal{O}}^a(K_\sigma^a)]'$ in the following manner: if $f \in [\underset{\sim}{\mathcal{O}}^a(K_\sigma^a)]'$ then $G(\zeta) = f[e^{-i\zeta z}]$ is a holomorphic function for ζ such that $\operatorname{Im}\zeta \in -\Gamma_\sigma$, $|\operatorname{Im}\zeta_j| > a_j$ for $j = 1, \ldots, n$, and $G \in \tilde{\mathcal{O}}^a(D^n + i\{\sigma \operatorname{Im}\zeta + a < 0\})$.

Definition 5. The Fourier analytic functional $\mathcal{F}f = j^a_{-\Gamma_\sigma}(G)$ is called the *Fourier transform* of the Fourier analytic functional $f \in [\underset{\sim}{\mathcal{O}}^a(K_\sigma^a)]'$.

We shall abbreviate the term "Fourier analytic functional" to "Fourier a.f.".

For f being a monomial Fourier a.f., $f = j^a_{\Gamma_\sigma}(F)$, we define the Fourier transform by means of an "exponential partition of unity": we set

$$\chi_+^a(z) = \frac{e^{(2a+1)z}}{1+e^{(2a+1)z}}, \qquad \chi_-^a(z) = \frac{1}{1+e^{(2a+1)z}}$$

and for $\tau = (\tau_1, \ldots, \tau_n) \in \{+,-\}^n$

$$\chi_\tau^a(z) = \chi_\tau^a(z_1, \ldots, z_n) = \chi_{\tau_1}^{a_1}(z_1)\ldots\chi_{\tau_n}^{a_n}(z_n).$$

The set of functions $\{\chi_\tau^a\}_\tau$ has the following properties:

(a) $\sum_\tau \chi_\tau^a(z) \equiv 1$,

(b) $\chi_\pm^a$ is a meromorphic function with poles at $\{\pm i(2k+1)\pi/(2a_j+1)\}$ and χ_τ^a is of infraexponential type outside the poles.

(c) $|\chi_\tau^a(z)| \leqslant Ce^{-(2a+1)|\operatorname{Re} z|}$ for $|\operatorname{Im} z_j| < \pi/(2a_j+1)$ and $\operatorname{Re} z \notin \Gamma_\tau$ with some constant C.

Every collection of functions $\{\chi_\tau^a\}_\tau$ satisfying (a)–(c) is called an *exponential partition of unity.*

Definition 6. The *Fourier transform* of a Fourier a.f. $f = j^a_{\Gamma_\sigma}(F)$ for a such that $a < \pi/(2a+1)$ is the Fourier a.f. $\mathcal{F}f = \sum_\tau \operatorname{sgn}\tau\, j^a_{-\Gamma_\tau}(G_\tau)$, where the functions G_τ are defined by

$$(3)\qquad G_\tau(\zeta) = \operatorname{sgn}\tau \int_{\operatorname{Im} z = y_0} F(z)\chi_\tau^a(z)e^{-iz\zeta}\,dz \quad \text{for } \operatorname{Im}\zeta \in -\Gamma_\tau,\ |\operatorname{Im}\zeta_j| > a_j$$

with a fixed $y_0 \in \Gamma_\sigma$ and $a_j < |y_{0j}| < \pi/(2a_j+1)$.

Remark. The restriction $a < \pi/(2a+1)$ is not necessary for the definition of the Fourier transform of a Fourier a.f., for example we can use the dual of the Fourier image of the fundamental space (as in Sargos–Morimoto [5]). But for our purposes of defining the Mellin transform of a Mellin a.f. and applying it to the Laplace–Beltrami equation it is sufficient to use this method.

Now, similarly to Fourier hyperfunctions, from properties (a)–(c) it follows that the integral in (3) is well defined and $G \in \tilde{\mathcal{O}}^a(U \cap (D^n - i\Gamma_\tau))$ for U being some neighbourhood of D^n in $D^n + i\mathbf{R}^n$.

Definition 7. We define the *inverse Fourier transform* $\mathcal{F}^{-1}$ in the following way: if $f \in [\underset{\sim}{\mathcal{O}}^a(K_\sigma^a)]'$ then $\mathcal{F}^{-1}f = j^a_{\Gamma_\sigma}(G)$ with $G(\zeta) = (2\pi)^{-n} f[e^{i\zeta z}]$, and if $f = j^a_{\Gamma_\sigma}(F)$ then $\mathcal{F}^{-1}f = \sum_\tau \operatorname{sgn}\tau\, j^a_{\Gamma_\tau}(\tilde{G}_\tau)$ with $\tilde{G}_\tau(\zeta) = (2\pi)^{-n} G_\tau(-\zeta)$, where G_τ is defined by (3).

It can be proved, just as for Fourier hyperfunctions, that if f is a Fourier a.f. then $\mathcal{F}^{-1}\mathcal{F}f = f = \mathcal{F}\mathcal{F}^{-1}f$.

Now, we shall prove Paley–Wiener type theorems for Fourier a.f.

4. Characterization of Fourier analytic functionals

Theorem 2. *Assume that M is the closure in D^n of a cone $\Gamma_+ + u \subset \mathbf{R}^n$ for some $u \in \mathbf{R}^n$, and g is a Fourier a.f., $g \in [\underset{\sim}{\mathcal{O}}^a(M^a)]'$. Then $g[e^{iz\zeta}]$ is a holomorphic function on $\mathbf{R}^n + i(\Gamma_+ + a)$, and for every $M' \Subset \Gamma_+ + a$ and every $\varepsilon > 0$ there exists some constant C_ε such that*

$$(4)\qquad |g[e^{iz\zeta}]| \leqslant C_\varepsilon e^{(a+\varepsilon)|\operatorname{Re} z| + H^a_{M,\varepsilon}(\operatorname{Im} z)} \quad \text{for } z \in \mathbf{R}^n + iM'$$

where $H^a_{M,\varepsilon}(y) = \sup\{-xy + (a+\varepsilon)|x|;\ x + \varepsilon \in M\}$.

Proof. It is a slight modification of the proof of this theorem in the case $a = 0$, i.e. the case of a Fourier hyperfunction (see Kaneko [1], Th. 8.5.7, or Kawai [2], Th. 3.3.1).

THEOREM 3. *If the function F satisfies the condition* (4) *then there exists a unique Fourier analytic functional $g \in [\underset{\sim}{\mathcal{O}}^a(M^a)]'$ such that $F(z) = g[e^{iz\zeta}]$.*

Proof. In the case $a = 0$ the proof can be found in Kaneko [1], Cor. 8.5.8, or Kawai [2], Th. 3.3.2. For $0 < a_j < \pi/(2a_j+1)$ we modify the proof from [1] as follows:

Let f be the monomial Fourier a.f. assigned to F, $f = j^a_{\Gamma_+}(F)$. Its Fourier transform is $\mathscr{F}f = \sum_\sigma \operatorname{sgn}\sigma j^a_{-\Gamma_\sigma}(G_\sigma)$ with G_σ defined by (3). It suffices to show that $g = (2\pi)^n \mathscr{F}f$ belongs to $[\underset{\sim}{\mathcal{O}}^a(M^a)]'$. To this end we deform the integration path in (3) inside $\mathbf{R}^n + i(\Gamma_+ + a)$.

We set for any $\theta = (\theta_1, \ldots, \theta_n) \in [0, \pi/2)^n$

$$R^+_{\theta_j} = \{z_j \in \mathbf{R} + i(a_j, \infty);\ z_j = i\mathring{y}_j + r_j e^{i\theta_j},\ r_j \in \mathbf{R}_+\},$$
$$R^-_{\theta_j} = \{z_j \in \mathbf{R} + i(a_j, \infty);\ z_j = i\mathring{y}_j - r_j e^{-i\theta_j},\ r_j \in \mathbf{R}_+\},$$
$$R^\pm_\theta = R^\pm_{\theta_1} \times \ldots \times R^\pm_{\theta_n} \quad \text{and} \quad R_\theta = R^+_\theta \cup R^-_\theta,$$
$$U^{\theta_j}_\pm = \{\zeta_j = \xi_j + i\eta_j \in \mathbf{C};\ (a_j \pm \eta_j)\operatorname{Re} e^{i\theta_j} - (u_j - \xi_j)\operatorname{Im} e^{i\theta_j} < 0\},$$
$$U^\theta_\sigma = U^{\theta_1}_{\sigma_1} \times \ldots \times U^{\theta_n}_{\sigma_n}.$$

We show that G_σ defined by (3) can be defined as an integral over R_θ:

$$\operatorname{sgn}\sigma G_\sigma(\zeta) = \int_{R_\theta} F(z)\chi^a_\sigma(z) e^{-iz\zeta}\,dz \quad \text{for } \zeta \in (U^{\theta_1}_{\sigma_1} \cap U^0_{\sigma_1}) \times \ldots \times (U^{\theta_n}_{\sigma_n} \cap U^0_{\sigma_n}). \tag{5}$$

Indeed, note that the integral in (3) is defined for ζ in

$$U^0_\sigma = \{|\operatorname{Im}\zeta_j| > a_j\} \cap \{\operatorname{Im}\zeta \in -\Gamma_\sigma\} = \{\zeta = \xi + i\eta;\ a + \sigma\eta < 0\}.$$

We have

$$\int_0^r F(\pm x + i\mathring{y})\chi^a_\sigma(\pm x + i\mathring{y}) e^{-i(\pm x + i\mathring{y})\zeta}\,dx$$
$$- \int_0^r F(\pm x e^{\pm i\theta} + i\mathring{y})\chi^a_\sigma(\pm x e^{\pm i\theta} + i\mathring{y}) e^{-i(\pm x e^{\pm i\theta} + i\mathring{y})\zeta} e^{\pm i\theta}\,dx$$
$$= \int_0^\theta F(\pm r e^{\pm i\alpha} + i\mathring{y})\chi^a_\sigma(\pm r e^{\pm i\alpha} + i\mathring{y}) e^{-i(\pm r e^{\pm i\alpha} + i\mathring{y})\zeta} r\,d\alpha.$$

Then, for $\zeta \in U^\theta_\sigma \cap U^0_\sigma$ and for every $\varepsilon > 0$, it follows from (4) that

$$\Big|\int_0^\theta F(\pm r e^{\pm i\alpha} + i\mathring{y})\chi^a_\sigma(\pm r e^{\pm i\alpha} + i\mathring{y}) e^{-i(\pm r e^{\pm i\alpha} + i\mathring{y})\zeta} r\,d\alpha\Big|$$
$$\leqslant C_\varepsilon \int_0^\theta e^{(a+2\varepsilon)r\operatorname{Re} e^{i\alpha} + (a+\varepsilon-\mathring{y}-r\operatorname{Im} e^{i\alpha})(u-\varepsilon-\xi)} e^{r\operatorname{Re} e^{i\alpha}\sigma\eta + (\mathring{y} + r\operatorname{Im} e^{i\alpha})\xi} r\,d\alpha$$
$$= C'_\varepsilon e^{\mathring{y}\xi} \int_0^\theta e^{[(a+2\varepsilon+\sigma\eta)\operatorname{Re} e^{i\alpha} - (u-\varepsilon-\xi)\operatorname{Im} e^{i\alpha}]r} r\,d\alpha.$$

Choosing $\varepsilon > 0$ sufficiently small we see that the last integrand converges to 0 as $r \to \infty$, uniformly with respect to α, therefore the last integral also converges to zero. This proves (5). Now one can easily see that the function G_σ can be extended holomorphically to the domain

$$U_\sigma = U_{\sigma_1} \times \ldots \times U_{\sigma_n} = \bigcup_{\theta_1} U^{\theta_1}_{\sigma_1} \times \ldots \times \bigcup_{\theta_n} U^{\theta_n}_{\sigma_n},$$

and $G_\sigma \in \tilde{\mathcal{O}}^a(U_\sigma)$.

Suppose $\sigma, \tau \in \{+, -\}^n$ and $\sigma_j = \tau_j$ for $j \neq k$, $k \in \{1, \ldots, n\}$ fixed. Then for $\zeta \in U_\sigma \cap U_\tau$ and for θ sufficiently close to $\pi/2$ we have

$$\operatorname{sgn}\sigma\big(G_\sigma(\zeta) - G_\tau(\zeta)\big) = \int_{R_\theta} F(z)\chi^a_\sigma(z) e^{-iz\zeta}\,dz + \int_{R_\theta} F(z)\chi^a_\tau(z) e^{-iz\zeta}\,dz$$
$$= \int_{R_\theta} F(z)\chi^{a_1}_{\sigma_1}(z_1)\ldots\chi^{a_k}_{+}(z_k)\ldots\chi^{a_n}_{\sigma_n}(z_n) e^{-iz\zeta}\,dz$$
$$+ \int_{R_\theta} F(z)\chi^{a_1}_{\sigma_1}(z_1)\ldots\chi^{a_k}_{-}(z_k)\ldots\chi^{a_n}_{\sigma_n}(z_n) e^{-iz\zeta}\,dz$$
$$= \int_{R_\theta} F(z)\chi^a_{\hat\sigma}(\hat z) e^{-iz\zeta}\,dz$$

where $\hat\sigma = (\sigma_1, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n)$, $\hat z = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$. Hence it follows for $\zeta \in U_\sigma \cap U_\tau$ that if $\theta_k \to \pi/2$, the last integral converges to zero. Therefore $G_\sigma(\zeta) = G_\tau(\zeta)$ on $U_\sigma \cap U_\tau$. We can fix k arbitrarily, so there exists a function G such that $G = G_\sigma$ on U_σ and $G \in \tilde{\mathcal{O}}^a(\mathbf{C}^n \# M^a) = \tilde{\mathcal{O}}^a(\bigcup_\sigma U_\sigma)$. This proves that $\mathscr{F}f = \sum_\sigma \operatorname{sgn}\sigma j^a_{-\Gamma_\sigma}(G_\sigma)$ belongs to $[\underset{\sim}{\mathcal{O}}^a(M^a)]'$. The proof of Theorem 3 is thus complete.

5. Mellin analytic functionals. First of all we will define the spaces $\underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a)$ of test functions for Mellin analytic functionals. Suppose that $M = [-\ln t_1, \infty] \times \ldots \times [-\ln t_n, \infty]$, $t \in \mathbf{R}^n_+$. Then we have

DEFINITION 8. $\varphi \in \underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a)$ iff $(w^{b+1}\varphi)\circ\mu \in \underset{\sim}{\mathcal{O}}^a(M^a)$. Here $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$ and $1 = (1, \ldots, 1)$.

We can rewrite this definition in the following equivalent form:

DEFINITION 8′. $\varphi \in \underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a)$ if and only if $\varphi \in \mathcal{O}(U \# \{0\})$ for some $U \in \mathscr{S}^a$, and for every compact $K \subset U$ there exists $\delta > 0$ such that $\sup\{|w^{b+1-\delta-a}\varphi(w)|;\ w \in K \# \{0\}\} < \infty$.

The map $\varphi \to (w^{b+1}\varphi)\circ\mu$ is an isomorphism between $\underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a)$ and $\underset{\sim}{\mathcal{O}}^a(M^a)$, so $\underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a)$ equipped with the topology induced from $\underset{\sim}{\mathcal{O}}^a(M^a)$ is a complete space. Similarly to the case of the space $\mathcal{M}_{(b)}(J)$ of test functions for Mellin distributions we see that if $b \leqslant c$ then $\underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a) \subset \underset{\sim}{\mathcal{M}}^a_{(c)}(\bar J^a)$.

DEFINITION 9. *Mellin analytic functionals* (in short Mellin a.f.) are elements of the dual space $[\underset{\sim}{\mathcal{M}}^a_{(b)}(\bar J^a)]' = \underset{\sim}{\mathcal{M}}^{a\prime}_{(b)}(\bar J^a)$.

Immediately from Definitions 8 and 9 we have the following

LEMMA 1. *$f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$ if and only if $e^{b\zeta}(f\circ\mu) \in [\mathcal{O}^a(M^a)]'$. If $c \leqslant b$, then also $e^{c\zeta}(f\circ\mu) \in [\mathcal{O}^a(M^a)]'$.*

In the case $a = 0$, we use the notation $\mathcal{M}'^{0}_{(b)}(\bar{J}^0) = \mathcal{M}'_{(b)}(\bar{J})$. A functional $f \in \mathcal{M}'_{(b)}(\bar{J})$ is a hyperfunction which we will call a *Mellin hyperfunction.* Obviously, f is a Mellin hyperfunction, $f \in \mathcal{M}'_{(b)}(\bar{J})$, if and only if $e^{b\zeta}(f\circ\mu)$ is a Fourier hyperfunction, $e^{b\zeta}(f\circ\mu) \in [\mathcal{O}(M)]'$.

If g is a Fourier a.f., $g \in [\mathcal{O}^a(M^a)]'$, then $f = (e^{-b\zeta}g)\circ\mu^{-1}$ is a Mellin a.f., $f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$. Since to every Fourier a.f. we can assign a class of defining functions, we can do the same with a Mellin a.f. Let $f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$. If G is a defining function for the Fourier a.f. $g = e^{b\zeta}(f\circ\mu)$, then $G \in \tilde{\mathcal{O}}^a(U \# M^a)$ for some $U = U_1 \times \ldots \times U_n$, where U_j is a complex neighbourhood of M_j^a. Now, define a function F by

$$F(w) = (e^{-b\zeta}G)(-\ln w) = w^b G(-\ln w).$$

Then F is holomorphic in the set $V \# \bar{J}^a$, for some V, a complex neighbourhood of $\bar{J}^a$, $V \in \mathcal{S}^a$. Moreover, F satisfies the condition resulting from the exponential property of G: for every $\varepsilon > 0$ and for every compact $K \subset V \# \bar{J}^a$, there exists a constant $C_{\varepsilon,K}$ such that

$$|F(w)| < C_{\varepsilon,K}|w|^{b-a-\varepsilon}. \tag{6}$$

DEFINITION 10. Assume that F satisfies (6) on $(V \# \bar{J}^a) \cap (\mathbf{R}^n + i\Gamma_\sigma)$, and $F \equiv 0$ outside this set. The Mellin analytic functional f assigned to the function F is called a *monomial Mellin a.f.* and denoted by $f = j^{a,M}_{\Gamma_\sigma}(F)$.

6. The Mellin transform of Mellin analytic functionals. First we note that the function $\varphi_z(w) = w^{-z-1}$ belongs to $\mathcal{M}^{a}_{(b)}(\bar{J}^a)$ iff $\mathrm{Re}\, z < b-a$. Indeed, $(w^{b+1}\varphi_z)\circ\mu = e^{-(b-z)\zeta}$, and $|e^{-(b-z)\zeta}| \leqslant Ce^{-(a\mp\delta)|\mathrm{Re}\,\zeta|}$ for some $\delta > 0$ if and only if $\mathrm{Re}\, z < b-a$. Hence every Mellin a.f. can be evaluated on the test function φ_z. Now we can define the Mellin transform of a Mellin a.f.

DEFINITION 11. If f is a Mellin a.f., $f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$, the *Mellin transform* of f is the function $\mathcal{M}f$ defined by

$$\mathcal{M}f(z) = f[w^{-z-1}] \quad \text{for } \mathrm{Re}\, z < b-a,$$

and every holomorphic extension of this function. The operation $\mathcal{M}$ assigning to each $f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$ its Mellin transform $\mathcal{M}f$ is called the *Mellin transformation.*

Similarly to the Mellin transformation in the class of Mellin distributions, the transformation defined above has the following operational properties:

$$\mathcal{M}(x^\beta f)(z) = \mathcal{M}f(z-\beta) \quad \text{for } \mathrm{Re}\, z < b-a+\mathrm{Re}\,\beta,$$

$$\mathcal{M}\left(\frac{\partial}{\partial x_j}f\right)(z) = (z_j+1)\mathcal{M}f(z+(1)_j) \quad \text{for } \mathrm{Re}\, z < b-a-(1)_j,$$

$$\mathcal{M}\left(P\left(x\frac{\partial}{\partial x}\right)f\right)(z) = P(z)\mathcal{M}f(z),$$

where P is a polynomial in $z_1, \ldots, z_n$ and $P(x\partial/\partial x)$ is the operator $P(x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n)$. Here $z+(1)_j$ denotes the vector z translated by 1 along the $\mathrm{Re}\, z_j$-axis, i.e.

$$z+(1)_j = (z_1, \ldots, z_{j-1}, z_j+1, z_{j+1}, \ldots, z_n).$$

If f is a Mellin a.f., $x^\beta f$ denotes the functional

$$x^\beta f[\varphi] = f[w^\beta \varphi]$$

and $(\partial/\partial x_j)f$ acts as $(\partial/\partial x_j)f[\varphi] = f[-(\partial/\partial w_j)\varphi]$.

For the Mellin transform of a Mellin a.f. the following Paley–Wiener type theorems hold.

THEOREM 4. *Assume that $f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$. Then for every $\varepsilon > 0$ there exists a constant C_ε such that*

$$|\mathcal{M}f(z)| \leqslant C_\varepsilon e^{(a+\varepsilon)|\mathrm{Im}\, z|}(te^\varepsilon)^{-\mathrm{Re}\, z} \quad \text{for } \mathrm{Re}\, z \leqslant b-a-\varepsilon. \tag{7}$$

Proof. This follows from Theorem 2 for the Fourier a.f. in the case of $M = M_1 \times \ldots \times M_n = [-\ln t_1, \infty] \times \ldots \times [-\ln t_n, \infty]$, since we have for $z = b-\gamma+i\beta$ (with $\gamma > a$)

$$\mathcal{M}f(z) = f[w^{-z-1}] = e^{b\zeta}(f\circ\mu)[e^{i(\beta+i\gamma)\zeta}],$$

therefore

$$|\mathcal{M}f(z)| = |e^{b\zeta}(f\circ\mu)[e^{i(\beta+i\gamma)\zeta}]| \leqslant C_\varepsilon e^{(a+\varepsilon)|\beta| + H^a_{M,\varepsilon}(\gamma)}.$$

For the cone M as above, $H^a_{M,\varepsilon}(\gamma)$ has the form

$$H^a_{M,\varepsilon}(\gamma) = \sup\{(a+\varepsilon)|x| - \gamma x;\ x+\varepsilon \in M\} = (a+\varepsilon)|\ln t + \varepsilon| - \gamma(-\ln t - \varepsilon),$$

hence

$$|\mathcal{M}f(z)| \leqslant C_\varepsilon e^{(a+\varepsilon)|\beta|} e^{(a+\varepsilon)|\ln t+\varepsilon| - \gamma(-\ln t - \varepsilon)}$$
$$= C'_\varepsilon e^{(a+\varepsilon)|\beta|} e^{-\gamma(-\ln t - \varepsilon)} = C''_\varepsilon e^{(a+\varepsilon)|\beta|}(te^\varepsilon)^{-\alpha}$$

for $\alpha \leqslant b-a-\varepsilon$.

The following converse theorem corresponds to Theorem 3.

THEOREM 5. *Assume the function F in the variables $z = \alpha+i\beta$ is holomorphic on a set $\{\alpha < b-a\}$, and satisfies condition (7). Then there exists a unique Mellin a.f. $f \in \mathcal{M}'^{a}_{(b)}(\bar{J}^a)$ such that $\mathcal{M}f(z) = F(z)$ for $\alpha < b-a$. Moreover, the functional f can be represented as $f = \sum_\sigma \mathrm{sgn}\,\sigma j^{a,M}_{\Gamma_\sigma}(H_\sigma)$, with H_σ defined by the formula*

$$H_\sigma(w) = \int_{\mathbf{R}^n} F(b-\gamma+i\beta)\chi^a_\sigma(\beta+i\gamma)w^{b-\gamma+i\beta}\,d\beta \tag{8}$$

for w such that $a < \sigma \mathrm{Arg}\, w < a+\eta$ for some $\eta \in \mathbf{R}^n_+$. The integral (8) is independent of a fixed γ, $a < \gamma < \pi/(2a+1)$.

Proof. It follows from (7) that if we consider F as a function of the variables $\beta+i\gamma$ (with $z=b-\gamma+i\beta$), setting $F_b(\beta+i\gamma)=F(b-\gamma+i\beta)$ we see that F_b satisfies the assumptions of Theorem 3. Hence there exists a unique Fourier a.f. $g_b\in[\underset{\sim}{Q}^a(M^a)]'$ such that $F_b(\beta+i\gamma)=g_b[e^{i(\beta+i\gamma)\zeta}]$. Then the functional $f=(e^{-b\zeta}g_b)\circ\mu^{-1}$ is a Mellin a.f. such that $\mathcal{M}f(z)=F(z)$, $f\in\underset{\sim}{\mathcal{M}}{}^{a\prime}_{(b)}(\bar{J}^a)$.

Now, we shall prove formula (8). Denote by f_b the Fourier a.f. $j^a_{\Gamma+}(F_b)$. We see that g_b is the Fourier transform of f_b, therefore $g_b=\mathcal{F}f_b=\sum_\sigma \operatorname{sgn}\sigma j^a_{-\Gamma_\sigma}(G^b_\sigma)$, where G^b_σ is defined by (3) with F_b in place of F. The functional g_b is independent of the choice of an exponential partition of unity $\{\chi^a_\sigma\}_\sigma$. Thus if we write $\tilde{\chi}^a_\sigma(\beta+i\gamma)=\chi^a_\sigma(\beta+i(\gamma-\tau))$, then the set of functions $\{\tilde{\chi}^a_\sigma\}_\sigma$ is again an exponential partition of unity for $|\gamma_j-\tau_j|<\pi/(2a_j+1)$, and $g_b=\sum_\sigma \operatorname{sgn}\sigma j^a_{-\Gamma_\sigma}(\tilde{G}^b_\sigma)$, where

$$\tilde{G}^b_\sigma(\zeta)=\int_{\mathbf{R}^n} F_b(\beta+i\gamma)\tilde{\chi}^a_\sigma(\beta+i\gamma)e^{-i(\beta+i\gamma)\zeta}\,d\beta\qquad\text{for } \sigma\operatorname{Im}\zeta<-a$$

and for fixed γ, $|\gamma_j-\tau_j|<\pi/(2a_j+1)$.

So, we conclude that the Mellin a.f. $f=(e^{-b\zeta}g_b)\circ\mu^{-1}$ can be represented as $f=\sum_\sigma \operatorname{sgn}\sigma j^{a,M}_{\Gamma_\sigma}(H_\sigma)$, where $H_\sigma(w)=w^b G^b_\sigma(-\ln w)$, and we obtain (8).

7. The Mellin transform of a Laplace–Beltrami equation. In this section we consider the Laplace–Beltrami partial differential operator on the hyperbolic half-plane:

$$P=x_1^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+s(1-s),$$

with some complex s, and the equation

$$Pu=f. \tag{1}$$

Assume that f is a Mellin hyperfunction such that its Mellin transform $\mathcal{M}f$ is a holomorphic function on $\Omega=\{(z_1,z_2);\ \operatorname{Re}(z_1+z_2)<0,\ \operatorname{Re}z_1<0\}$. In other words, we assume condition (7) with $a=0$, for $z\in\Omega$, and $f\in\underset{\sim}{\mathcal{M}}{}'_{(b)}(\bar{J})$ for every $b\in\bar{\Omega}$.

We shall compute the Mellin transform of both sides of equation (1). The operator P can be written as $P=R+Q$ where

$$R=x_1^2\frac{\partial^2}{\partial x_1^2}+s(1-s)=\left(x_1\frac{\partial}{\partial x_1}\right)^2-x_1\frac{\partial}{\partial x_1}+s(1-s),\qquad Q=x_1^2\frac{\partial^2}{\partial x_2^2}.$$

It follows from the operational properties of the Mellin transformation that

$$\begin{aligned}\mathcal{M}(Pu)(z)&=\mathcal{M}(Ru)(z)+\mathcal{M}(Qu)(z)\\&=R(z_1)\mathcal{M}u(z)+(z_2+1)(z_2+2)\mathcal{M}u(z_1-2,\ z_2+2),\end{aligned}$$

where $R(z_1)=z_1^2-z_1+s(1-s)$. Thus we have the equality

$$R(z_1)\mathcal{M}u(z)=-(z_2+1)(z_2+2)\mathcal{M}u(z_1-2,\ z_2+2)+\mathcal{M}f(z).$$

We shall find and study a solution u of equation (1) by finding and studying its Mellin transform $\mathcal{M}u$, so we restrict our attention to the solutions of the functional equation

$$R(z_1)F(z)=-(z_2+1)(z_2+2)F(z_1-2,\ z_2+2)+\mathcal{M}f(z). \tag{9}$$

Note that the polynomial R is zero at s and $1-s$. Suppose, for simplicity, that $0<\operatorname{Re}s\leqslant\operatorname{Re}(1-s)$; then on Ω equation (9) is equivalent to

$$F(z)=-\frac{(z_2+1)(z_2+2)}{R(z_1)}F(z_1-2,\ z_2+2)+\frac{\mathcal{M}f(z)}{R(z_1)}.$$

We solve this equation by a successive approximation method, putting

$$F_0(z)=\frac{\mathcal{M}f(z)}{R(z_1)},\qquad F_j(z)=-\frac{(z_2+1)(z_2+2)}{R(z_1)}F_{j-1}(z_1-2,\ z_2+2)+F_0(z)$$

for $j=1,2,\ldots$ It follows from the definition that

$$F_j(z)=\sum_{k=1}^{j}(-1)^k\frac{(z_2+1)(z_2+2)\ldots(z_2+2k)}{R(z_1)R(z_1-2)\ldots R(z_1-2k)}\mathcal{M}f(z_1-2k,\ z_2+2k)+\frac{\mathcal{M}f(z)}{R(z_1)}.$$

Fix an $\varepsilon>0$. We easily see ([4]) that there exists some constant B_ε such that $|R(z_1)|\geqslant B_\varepsilon$, $|R(z_1-2)\ldots R(z_1-2k)|\geqslant 2^{2k}(k!)^2$, for $z=(z_1,z_2)$ in $\{\operatorname{Re}z_1\leqslant-\varepsilon\}$. From the assumption on f it follows that

$$\begin{aligned}|\mathcal{M}f(z_1-2k,\ z_2+2k)|&\leqslant C_\varepsilon e^{\varepsilon|\operatorname{Im}z|}(t_1e^\varepsilon)^{-\operatorname{Re}z_1+2k}(t_2e^\varepsilon)^{-\operatorname{Re}z_2-2k}\\&=C_\varepsilon e^{\varepsilon|\operatorname{Im}z|}(te^\varepsilon)^{-\operatorname{Re}z}(t_1/t_2)^{2k}.\end{aligned}$$

We shall use the identity

$$(z_2+1)(z_2+2)\ldots(z_2+2k)=(-z_2-1)\ldots(-z_2-2k)=\frac{\Gamma(-z_2)}{\Gamma(-z_2-2k)}$$

for $\operatorname{Re}z_2<0$, where Γ is the Euler gamma function. We shall also apply the following asymptotic property of Γ:

$$\lim_{|\beta|\to\infty}|\Gamma(\alpha+i\beta)|e^{\pi|\beta|/2}|\beta|^{1/2-\alpha}=\sqrt{2\pi}.$$

It means that if $z=\alpha+i\beta$, then

$$\lim_{|\beta_2|\to\infty}|\Gamma(-z_2)/\Gamma(-z_2-2k)||\beta_2|^{-2k}=1,$$

so for sufficiently large $|\beta_2|$ we have

$$|(z_2+1)(z_2+2)\ldots(z_2+2k)|\leqslant 2|\beta_2|^{2k}.$$

In view of all the above remarks we obtain the following estimate: for every $\varepsilon > 0$ there exists some constant A_ε such that

$$|F_j(z)| \leqslant A_\varepsilon e^{\varepsilon|\beta|}(te^\varepsilon)^{-\alpha}\left[\sum_{k=1}^{j}\frac{1}{(k!)^2}\left(\frac{t_1}{2t_2}\right)^{2k}|\beta_2|^{2k}+1\right]$$
$$< A_\varepsilon e^{\varepsilon|\beta|}(te^\varepsilon)^{-\alpha}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(|\beta_2|\frac{t_1}{2t_2}\right)^k\right)^2$$
$$= A_\varepsilon e^{\varepsilon|\beta|}(te^\varepsilon)^{-\alpha}e^{(t_1/t_2)|\beta_2|} \quad \text{for } \alpha \leqslant -\varepsilon.$$

Hence we conclude that the sequence $\{F_j\}$ is convergent locally uniformly to a function F holomorphic on $\Omega' = \{\operatorname{Re} z < 0\}$, and satisfying the condition

$$|F(z)| \leqslant A_\varepsilon e^{\varepsilon|\beta_1|}e^{(\varepsilon+t_1/t_2)|\beta_2|}(te^\varepsilon)^{-\alpha} \quad \text{for } \alpha \leqslant -\varepsilon.$$

It follows from the definition of F_j and the last estimate that F is a solution of equation (9) on the domain $\Omega'' = \{\operatorname{Re} z_1 < 0,\ \operatorname{Re} z_2 < -2\}$, and fulfills condition (7) with $a = \tilde{a} = (0, t_1/t_2)$. According to Theorem 5 there exists a Mellin a.f. $u \in \mathcal{M}'^{\tilde{a}}_{(b)}(\bar{J}^a)$ for every b, $b - \tilde{a} \in \Omega''$, and its Mellin transform $\mathcal{M}u(z)$ equals $F(z)$ on Ω''. Since $\mathcal{M}u$ is a solution of (9) it follows that u is a solution of (1).

Moreover, the functional u can be written as $u = \sum_\sigma \operatorname{sgn}\sigma j^{\tilde{a},M}_{\Gamma_\sigma}(U_\sigma)$, where

$$\operatorname{sgn}\sigma\, U_\sigma(w) = \int_{\mathbf{R}^2} F(b-\gamma+i\beta)\chi^{\tilde{a}}_\sigma(\beta+i\gamma)w^{b-\gamma+i\beta}\,d\beta$$

for fixed γ, $\gamma_1 > 0$, $\gamma_2 > t_1/t_2$.

8. Mellin hyperfunctions with support in an $(n-k)$-dimensional space. In this section we intend to establish some characterization for Mellin hyperfunctions with support in an $(n-k)$-dimensional space. Suppose that f is a nontrivial Mellin hyperfunction, $f \in \mathcal{M}'_{(b)}(\bar{J})$ and $\operatorname{supp} f \subset \{0\}^k \times [0, t'']$, $t'' = (t_{k+1}, \ldots, t_n)$. If we set $z = (z', z'')$, $z' = (z_1, \ldots, z_k)$, $z'' = (z_{k+1}, \ldots, z_n)$, we conclude that for every $t' = (t_1, \ldots, t_k)$, $t' > 0$, $\operatorname{supp} f \subset [0, t'] \times [0, t'']$. It follows from Theorem 4 that for every $\varepsilon > 0$ and $t' > 0$ there exists a constant $C_{\varepsilon,t'}$ such that

$$|\mathcal{M}f(z)| \leqslant C_{\varepsilon,t'}e^{\varepsilon|\operatorname{Im} z|}(t'e^\varepsilon)^{-\operatorname{Re} z'}(t''e^\varepsilon)^{-\operatorname{Re} z''} \quad \text{for } \operatorname{Re} z \leqslant b-\varepsilon.$$

This condition is equivalent to the following: for every $\varepsilon > 0$ and $\delta > 0$, there exists a constant $C_{\varepsilon,\delta}$ such that

$$|\mathcal{M}f(z)| \leqslant C_{\varepsilon,\delta}e^{\varepsilon|\operatorname{Im} z|}\delta^{-\operatorname{Re} z'}(t''e^\varepsilon)^{-\operatorname{Re} z''} \quad \text{for } \operatorname{Re} z \leqslant b-\varepsilon. \tag{10}$$

Therefore it follows from Theorem 5 that if a function F is holomorphic on $\{\operatorname{Re} z < b\}$ and fulfills (10), then for every t' there exists a Mellin hyperfunction $f_{t'} \in \mathcal{M}'_{(b)}(\bar{J}_{t',t''})$ with $\operatorname{supp} f_{t'} \subset [0, t'] \times [0, t'']$ such that $\mathcal{M}f_{t'}(z) = F(z)$. In view of the uniqueness of the Mellin transformation we obtain

THEOREM 6. *A Mellin hyperfunction $f \in \mathcal{M}'_{(b)}(\bar{J})$ has support contained in $\{0\}^k \times [0, t'']$ if and only if its Mellin transform fulfills* (10).

Now, we return to the Laplace–Beltrami operator and to equation (1). Suppose that f on the right-hand side of (1) is a Mellin hyperfunction with $\operatorname{supp} f \subset \{0\} \times [0, t_2]$, and $\mathcal{M}f$ is holomorphic on Ω. Repeating the argument from Section 7 we conclude that the limit function F satisfies the following condition: for every $\varepsilon > 0$, $t_1 > 0$, there exists A_{ε,t_1} such that

$$|F(z)| \leqslant A_{\varepsilon,t_1}e^{\varepsilon|\beta_1|}e^{(t_1/2t_2+\varepsilon)|\beta_2|}(te^\varepsilon)^{-\alpha} \quad \text{for } \alpha \leqslant -\varepsilon.$$

This means that for every $\varepsilon > 0$, $\delta > 0$

$$|F(z)| \leqslant B_{\varepsilon,\delta}e^{\varepsilon|\beta|}\delta^{-\alpha_1}(t_2e^\varepsilon)^{-\alpha_2} \quad \text{for } \alpha \leqslant -\varepsilon$$

and for some constant $B_{\varepsilon,\delta}$. In other words, F satisfies (10). It follows from Theorems 5 and 6 that the functional u for which $\mathcal{M}u(z) = F(z)$ is a Mellin hyperfunction with support in $\{0\} \times [0, t_2]$.

We conclude that in the case of f with support in $\{0\} \times [0, t_2]$ equation (1) has a solution which is a Mellin hyperfunction with support in $\{0\} \times [0, t_2]$.

EXAMPLES. Set $K = \{(\xi_1, \xi_2);\ 0 < \xi_2 < \xi_1\}$. Suppose that g is a Fourier hyperfunction, $g \in [\mathcal{Q}(\bar{K})]'$, where $\bar{K}$ is the closure of K in D^2. Then it follows from Kawai [2] that $g[e^{iz\zeta}]$ is a holomorphic function for $\operatorname{Im} z \in K^\circ$ (dual cone) and satisfies (4) with $a = 0$ and $M = \bar{K}$, for every cone $M' \Subset K^\circ$. Therefore, if we take $f = g \circ \mu^{-1}$, then $f \in \mathcal{M}'_{(0)}(\bar{J})$, $\operatorname{supp} f \subset L = \{(x_1, x_2);\ 0 < x_1 < x_2 < 1\}$, and $\mathcal{M}f(z) = \mathcal{M}f(-\gamma+i\beta) = g[e^{i(\beta+i\gamma)}]$, holomorphic for $\gamma_1 > 0$, $\gamma_1+\gamma_2 > 0$. Hence $\mathcal{M}f$ is holomorphic for $\operatorname{Re} z_1 < 0$, $\operatorname{Re} z_1 + \operatorname{Re} z_2 < 0$ and satisfies (7) there.

Now, let $M_t = [-\ln t_1, \infty] \times [-\ln t_2, \infty]$ and $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Set $K_t = A(M_t)$, i.e.

$$K_t = \{(\xi_1, \xi_2);\ A^{-1}(\xi_1, \xi_2) \in M_t\}$$
$$= \{(\xi_1, \xi_2);\ \xi_1 - \xi_2 > -\ln t_1,\ \xi_2 > -\ln t_2\} \subset K.$$

We can see that $\varphi \in \mathcal{Q}(M_t)$ iff $\varphi \circ A^{-1} \in \mathcal{Q}(K_t)$, therefore it is easy to see that if g is a Fourier hyperfunction with $\operatorname{supp} g \subset M_t$, then $g \circ A^{-1}$ can be defined by $g \circ A^{-1}[\psi] = g[\psi \circ A]$ (here $\det A = 1$), and $g \circ A^{-1} \in [\mathcal{Q}(K_t)]'$, i.e. $g \circ A^{-1}$ is a Fourier hyperfunction with $\operatorname{supp}(g \circ A^{-1}) \subset K_t$.

Let h_1 be a one-dimensional Fourier hyperfunction with $\operatorname{supp} h_1 = \{\infty\}$, let h_2 be a one-dimensional Fourier hyperfunction with $\operatorname{supp} h_2 \subset [-\ln t_2, \infty]$, and let $h(x_1, x_2) = h_1(x_1)h_2(x_2)$. Then $g = h \circ A^{-1}$ is a Fourier hyperfunction with support included in the cone K_t for every $t_1 < 1$. Since $K_t \subset K$, $g[e^{iz\zeta}]$ is a function holomorphic for $\operatorname{Im} z \in K^\circ$ and satisfies (4), so for $f = g \circ \mu^{-1}$, $\mathcal{M}f$ is a function holomorphic on $\{\operatorname{Re} z_1 < 0,\ \operatorname{Re}(z_1 + z_2) < 0\}$, satisfies (7) for every t_1 and $\operatorname{supp} f \subset \{0\} \times [0, t_2]$.

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