STUDIA MATHEMATICA 99 (3) (1991)

Generalized Taylor expansions of singular functions

by

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Abstract. A definition of hyperfunctions on a sum of linear subspaces of $\mathbf{C}^n$ is given. Methods for computing the Mellin transforms of singular functions and their boundary values are presented with application to asymptotic expansions of solutions of certain differential equations.

The Mellin transform of a bounded function f on $\mathbf{R}_+$ (vanishing at infinity) was defined in the classical form as the integral

$$\int_0^\infty f(x)x^{z-1}\,dx \qquad (\operatorname{Re} z > 0).$$

In his monograph [10] A. H. Zemanian extended the above definition to certain classes of distributions on $\mathbf{R}_+$. The Mellin transformation so defined does not distinguish the singularities of distributions at zero from those at infinity. To study the singularities of distributions at zero B. Ziemian defined in [11] the space $M'(J_r)$ of Mellin distributions on the cube $J_r = (0, r]^n$. By the Mellin transform of a function f bounded on J_r he understands any holomorphic extension of the function

$$\mathscr{M}f(z) = \int_{J_r} f(x)x_1^{-z_1-1}\ldots x_n^{-z_n-1}\,dx$$

defined for $\operatorname{Re} z_j < 0$, $j = 1, \ldots, n$. By using this formula he defines the Mellin transform $\mathscr{M}U$ of a distribution $U \in M'(J_r)$. The singularities of the function $\mathscr{M}U$ serve to obtain a generalized asymptotic expansion at zero of U.

The aim of this paper is to establish generalized asymptotic expansions at zero for certain classes of Mellin distributions. We rely on the Taylor–Ziemian formula (see [12], Th. 5), valid for distributions U such that $\mathscr{M}U$ is a holomorphic function on suitable wedges in $\mathbf{C}^n$. The boundary value of $\mathscr{M}U$ is the spectral hyperfunction which occurs in the Taylor–Ziemian formula.

In Section 1 we present selected results from the theory of hyperfunctions. The space of hyperfunctions can be defined on a real analytic manifold in $\mathbf{C}^n$ of dimension n ([7]). A real linear subspace $L \subset \mathbf{C}^n$ of dimension n in real position

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1980 *Mathematics Subject Classification* (1985 *Revision*): 46F10, 46F15, 46F20.

in $\mathbf{C}^n$ is a special case of such a manifold. We treat a hyperfunction g on $L = A^{-1}(\mathbf{R}^n)$ as an equivalence class $[G]_A$ of an A-meromorphic function G, where A is a nonsingular complex matrix. Afterwards we define hyperfunctions on a sum $L_1 \cup \ldots \cup L_k$ of linear subspaces of $\mathbf{C}^n$ in real position. We assume that $L_j = A_j^{-1}(\mathbf{R}^n)$, where all A_j $(j = 1, \ldots, k)$ are diagonal complex matrices with positive determinants and $L_j \cap L_l = \{0\}$ for $j, l = 1, \ldots, k$, $j \neq l$. This extension of the definition of hyperfunctions is needed to define and compute (in Section 4) the boundary value of the Mellin transform of the fundamental solution of the singular Laplace operator

$$\tilde{\Delta} = \left(x_1 \frac{\partial}{\partial x_1}\right)^2 + \left(x_2 \frac{\partial}{\partial x_2}\right)^2.$$

In Sections 2–4 we present methods for computing the Mellin transforms of distributions and their boundary values. In Section 3 we reduce the computation of certain n-dimensional Mellin transforms to one-dimensional ones (computed in Section 2). The results obtained are used to find the generalized asymptotic expansions of the fundamental solution of the Laplace operator and of solutions of certain singular differential equations of the first order.

Throughout the paper we use many definitions and facts stated in [12] and we assume that the reader is familiar with that paper.

The author wishes to express his gratitude to his teacher Prof. Z. Szmydt for her unestimable amount of help and criticism during the preparation of this paper. Furthermore, he wishes to thank Prof. B. Ziemian for suggesting the problem and many stimulating conversations.

0. Notation. Throughout the paper we use the following vector notation: if $a, b \in (\mathbf{R} \cup \{-\infty\} \cup \{+\infty\})^n$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ then $a < b$ (resp. $a \leqslant b$) denotes that $a_j < b_j$ (resp. $a_j \leqslant b_j$) for $j = 1, \ldots, n$.

We set $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n\colon 0 < x\}$, $J = (0, 1]^n$, $J_r = (0, r]^n$ for $r > 0$.

If $r > 0$ then $\mathbf{r}$ denotes the vector $\mathbf{r} = (r, \ldots, r) \in \mathbf{R}^n_+$; in particular $\mathbf{1} = (1, \ldots, 1)$.

$\mathbf{Z}$ is the set of integers, $\mathbf{N}$ the set of positive integers and $\mathbf{N}_0$ the set of nonnegative integers.

We write

$$x^z = x_1^{z_1} \ldots x_n^{z_n} \quad \text{for } x \in \mathbf{R}^n_+,\ z \in \mathbf{C}^n;$$
$$z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n} \quad \text{for } z \in (\mathbf{C} \setminus \{0\})^n,\ \alpha \in \mathbf{Z}^n;$$
$$\ln x = (\ln x_1, \ldots, \ln x_n) \quad \text{for } x \in \mathbf{R}^n_+;$$
$$e^y = (e^{y_1}, \ldots, e^{y_n}) \quad \text{for } y \in \mathbf{R}^n.$$

The vector notation is also used for differentiation. Namely, if $\alpha \in \mathbf{N}_0^n$ then

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \quad \text{where } D_j = \partial/\partial x_j,\ j = 1, \ldots, n;$$
$$\tilde{D}^\alpha = \tilde{D}_1^{\alpha_1} \ldots \tilde{D}_n^{\alpha_n}, \quad \text{where } \tilde{D}_j = x_j \partial/\partial x_j,\ j = 1, \ldots, n.$$

In particular, we write $\tilde{\Delta} = \sum_{j=1}^2 (x_j \partial/\partial x_j)^2$.

χ_Ω stands for the characteristic function of a set $\Omega \subset \mathbf{R}^n$.

We apply the notation commonly used in the theory of generalized functions and in complex analysis. $\mathcal{O}(W)$ denotes the space of holomorphic functions on an open set $W \subset \mathbf{C}^n$. The value of a generalized function U on a test function φ is denoted by $U[\varphi]$.

1. Hyperfunctions. The Mellin transforms of distributions studied in this paper are holomorphic on suitable wedges. It is natural to regard them as defining functions of hyperfunctions on certain linear submanifolds of $\mathbf{C}^n$. To this end we recall some necessary definitions.

DEFINITION 1. Let L be an $\mathbf{R}$-linear subspace of $\mathbf{C}^n$ of dimension n. We say that L is *in real position* in $\mathbf{C}^n$ if $L = Q(\mathbf{R}^n)$, where Q is a linear complex isomorphism of $\mathbf{C}^n$.

Then $L + iL = \mathbf{C}^n$ and L does not contain any complex line.

Throughout this section L stands for $L = A^{-1}(\mathbf{R}^n)$, where A is an $n \times n$ nonsingular complex matrix.

Let V_L be an open subset of L. Set

$$W_L = V_L + iL; \quad W_{A,j} = \{z \in W_L\colon \operatorname{Im}(Az)_j \neq 0\} \quad \text{for } j = 1, \ldots, n;$$
$$W_A = \bigcap_{j=1}^n W_{A,j}; \quad \hat{W}_{A,k} = \bigcap_{\substack{j=1\\ j\neq k}}^n W_{A,j} \quad \text{for } k = 1, \ldots, n.$$

A function $G \in \mathcal{O}(W_A)$ is called *A-meromorphic* on W_L.

DEFINITION 2. We define the space $B_A(V_L)$ of hyperfunctions on V_L as the quotient space

$$B_A(V_L) = \mathcal{O}(W_A) / \sum_{k=1}^n \mathcal{O}(\hat{W}_{A,k}).$$

Thus, a hyperfunction $g \in B_A(V_L)$ is regarded as a class $g = [G]_A$ of a function G, A-meromorphic on $V_L + iL$.

Let A and $\tilde{A}$ be nonsingular complex matrices, $L = A^{-1}(\mathbf{R}^n)$, $\tilde{L} = \tilde{A}^{-1}(\mathbf{R}^n)$. Let $G \in \mathcal{O}(W_A)$, $g = [G]_A \in B_A(V_L)$. Observe that $G \circ A^{-1} \circ \tilde{A} \in \mathcal{O}(W_{\tilde{A}})$. This ensures the correctness of the following definition:

$$g \circ A^{-1} \circ \tilde{A} = [G \circ A^{-1} \circ \tilde{A}]_{\tilde{A}} \in B_{\tilde{A}}(V_{\tilde{L}}), \tag{1}$$

where $V_{\tilde{L}} = \tilde{A}^{-1} \circ A(V_L)$. In particular, $g \circ A^{-1} = [G \circ A^{-1}]_{\mathrm{Id}} \in B_{\mathrm{Id}}(\mathbf{R}^n)$.

If $L = \tilde{L}$ then the map

$$B_A(L) \ni g \to g \circ A^{-1} \circ \tilde{A} \in B_{\tilde{A}}(L) \tag{2}$$

gives an isomorphism between $B_A(L)$ and $B_{\tilde{A}}(L)$.

Let us set

$$\Gamma_\varepsilon = \{y \in \mathbf{R}^n\colon\ \varepsilon_j y_j > 0 \text{ for } j = 1, \ldots, n\},$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$;

$$\Gamma_{A,\varepsilon} = A^{-1}(\Gamma_\varepsilon).$$

Observe that the set W_A splits into $\bigcup_{\varepsilon \in \{-1,1\}^n} W_{A,\varepsilon}$, where $W_{A,\varepsilon} = V_L + i\Gamma_{A,\varepsilon}$.

Definition 3. Let $\Gamma_L \subset L$ be a convex cone and suppose $H \in \mathcal{O}(V_L + i\Gamma_L)$. Choose a matrix A such that $\Gamma_{A,\varepsilon} \subset \Gamma_L$ for some $\varepsilon \in \{-1, 1\}^n$. We define $b_{\Gamma_L}(H)$, the boundary value of H as an element of $B_A(V_L)$ defined by the function

$$H_\varepsilon = \begin{cases} \operatorname{sgn}(\varepsilon)\cdot H & \text{on } W_{A,\varepsilon}, \\ 0 & \text{on } W_{A,\varepsilon'} \text{ for } \varepsilon' \neq \varepsilon. \end{cases}$$

It can be proved that this definition is independent of A and Γ_L in the sense that if $\Gamma'_L \subset \Gamma_L$ is another convex cone, $A'^{-1}(\Gamma_\varepsilon) \subset \Gamma'_L$, $H' = H|_{V_L + i\Gamma'_L}$ then $b_{\Gamma'_L}(H') = b_{\Gamma_L}(H)$ in the sense of the isomorphism (2).

With this definition of boundary value, if $g = [G]_A \in B_A(V_L)$ then

$$g = \sum_{\varepsilon \in \{-1,1\}^n} \operatorname{sgn}(\varepsilon)\cdot b_{\Gamma_{A,\varepsilon}}(G_{A,\varepsilon}), \tag{3}$$

where $G_{A,\varepsilon} = G|_{W_{A,\varepsilon}}$.

Also if Γ is a convex cone in $\mathbf{R}^n$, $H \in \mathcal{O}(V + i\Gamma)$ then

$$b_\Gamma(H) \circ A = b_{A^{-1}(\Gamma)}(H \circ A). \tag{4}$$

Let K_L be a compact subset of L. By $B_{K_L}(L)$ we denote the space of hyperfunctions on L with support contained in K_L.

Let $K \subset K_1 \times \ldots \times K_n$ be a compact subset of $\mathbf{R}^n$, where all K_j $(j = 1, \ldots, n)$ are compact in $\mathbf{R}$. Set

$$U_j = \mathbf{C}^{j-1} \times (\mathbf{C} \setminus K_j) \times \mathbf{C}^{n-j} \quad \text{for } j = 1, \ldots, n;$$

$$\hat{U}_k = \bigcap_{\substack{j=1 \\ j \neq k}}^{n} U_j \quad \text{for } k = 1, \ldots, n.$$

We define the space of hyperfunctions on K as

$$B(K) = \mathcal{O}\Big(\prod_{j=1}^{n} (\mathbf{C} \setminus K_j)\Big) \Big/ \sum_{k=1}^{n} \mathcal{O}(\hat{U}_k).$$

We have

Proposition 1. *The natural mapping $B(K) \ni f \to f \in B_K(\mathbf{R}^n)$ induced by the map*

$$\mathcal{O}\Big(\prod_{j=1}^{n} (\mathbf{C} \setminus K_j)\Big) \ni F \to F \in \mathcal{O}((\mathbf{C} \setminus \mathbf{R})^n)$$

is an isomorphism.

Let $K_{\mathbf{C}^n}$ be a compact subset of $\mathbf{C}^n$. We denote by $\mathscr{A}'(K_{\mathbf{C}^n})$ the space of continuous linear functionals on $\mathscr{A}(K_{\mathbf{C}^n}) = \operatorname{ind\,lim} \mathcal{O}(U)$, where U runs through complex neighbourhoods of $K_{\mathbf{C}^n}$.

Set $K_A = A^{-1}(K)$, where K is compact in $\mathbf{R}^n$, A is a nonsingular complex matrix.

Let $f \in \mathscr{A}'(K)$. We define a composition $f \circ A \in \mathscr{A}'(K_A)$ by

$$f \circ A[\psi] = \frac{1}{|\det A|} f[\psi \circ A^{-1}] \quad \text{for } \psi \in \mathscr{A}(K_A).$$

Theorem 1. *Let $K \subset K_1 \times \ldots \times K_n$ be compact in $\mathbf{R}^n$, where all K_j $(j = 1, \ldots, n)$ are compact in $\mathbf{R}$, $K_A = A^{-1}(K)$. Then*

$$B(K_A) \overset{I_A}{\simeq} \mathscr{A}'(K_A). \tag{5}$$

The isomorphism I_A is defined as follows: if $g_h \in B(K_A)$, $g_h = [G]_A$ with $G \in \mathcal{O}\big(A^{-1}(\prod_{j=1}^n (\mathbf{C} \setminus K_j))\big)$ then

$$I_A(g_h) = g_a \in \mathscr{A}'(K_A)$$

with g_a defined by

$$g_a[\psi] = (-1)^n \frac{\det A}{|\det A|} \int_{\mathscr{C}_A} G(\zeta)\psi(\zeta)\,d\zeta \quad \textit{for } \psi \in \mathscr{A}(K_A), \tag{6}$$

where $\mathscr{C}_A = A^{-1}(\mathscr{C})$, $\mathscr{C} = \mathscr{C}_1 \times \ldots \times \mathscr{C}_n$, $\mathscr{C}_j$ is a curve encircling K_j once in the anticlockwise direction $(j = 1, \ldots, n)$ and $\mathscr{C}_A$ is contained in the set of holomorphy of ψ.

Conversely, if $g_a \in \mathscr{A}'(K_A)$ then

$$I_A^{-1}(g_a) = g_h, \quad \textit{where} \quad g_h = [G]_A,$$

$$G(z) = \frac{|\det A|}{(2\pi i)^n} g_a\big[(A(\cdot - z))^{-1}\big]. \tag{7}$$

The function $G \in \mathcal{O}\big(A^{-1}(\prod_{j=1}^n (\mathbf{C} \setminus K_j))\big)$ defined by (7) is called the *standard defining function* of the hyperfunction g with respect to the matrix A.

Proof. By Theorems 2.14, 2.15 of [4] there exists an isomorphism I between $B(K)$ and $\mathscr{A}'(K)$ given by $I(f_h) = f_a \in \mathscr{A}'(K)$, where $f_h = [F]_{\mathrm{Id}} \in B(K)$, $F \in \mathcal{O}(\prod_{j=1}^n (\mathbf{C} \setminus K_j))$, and

$$f_a[\varphi] = (-1)^n \int_{\mathscr{C}} F(z)\varphi(z)\,dz \quad \text{for } \varphi \in \mathscr{A}(K_A),$$

$\mathscr{C} = \mathscr{C}_1 \times \ldots \times \mathscr{C}_n$ is contained in the set of holomorphy of φ. Thus I_A defined on $B(K_A)$ by $I_A(g_h) = g_a$, where $g_a = I(g_h \circ A^{-1}) \circ A$, establishes an isomorphism between $B(K_A)$ and $\mathscr{A}'(K_A)$. Let $\psi \in \mathscr{A}(K_A)$. We obtain

$$g_a[\psi] = \frac{1}{|\det A|} I(g_h \circ A^{-1})[\psi \circ A^{-1}] = \frac{(-1)^n}{|\det A|} \int_{\mathscr{C}} G \circ A^{-1}(z) \psi \circ A^{-1}(z)\, dz$$

$$\overset{z=A\zeta}{=} (-1)^n \frac{\det A}{|\det A|} \int_{A^{-1}(\mathscr{C})} G(\zeta)\psi(\zeta)\, d\zeta.$$

An easy proof of the second part of Theorem 1 is left to the reader.

The representation (3) of a hyperfunction g as a sum of boundary values of its defining function G is very convenient because under some growth conditions, the boundary values $b_{\Gamma_{A,\varepsilon}}(G_{A,\varepsilon})$ are distributions.

DEFINITION 4. Let Γ_L be an open cone in $L = A^{-1}(\mathbf{R}^n)$, $H \in \mathcal{O}(L + i\Gamma_L)$. We say that H is *of polynomial growth* (near L) if for every open set $V_L \Subset L$ and a cone $\tilde{\Gamma}_L \Subset \Gamma_L$ there exist constants C, N such that

$$|H(z)| \leqslant C|\mathrm{Im}(Az)|^{-N} \quad \text{for } z \in V_L + i\tilde{\Gamma}_L,\ |\mathrm{Im}(Az)| \text{ close to zero},$$

or equivalently

$$|H \circ A^{-1}(z)| \leqslant C|\mathrm{Im}\, z|^{-N} \quad \text{for } z \in A(V_L + i\tilde{\Gamma}_L),\ |\mathrm{Im}\, z| \text{ close to zero}.$$

By Theorems 9.3.3, 3.1.15 of [2] and (4) we get the following

THEOREM 2. *Let Γ_L be an open cone in L. If $H \in \mathcal{O}(L + i\Gamma_L)$ is of polynomial growth then $b_{\Gamma_L}(H) \in D'(L)$ and*

(8) $$b_{\Gamma_L}(H)[\psi] = \lim_{A(\tilde{\Gamma}_L) \ni y \to 0} \frac{1}{|\det A|} \int H \circ A^{-1}(x+iy) \psi \circ A^{-1}(x)\, dx$$

for $\psi \in D(L)$.

EXAMPLE 1 ([12], Ex. 3).

(9) $$[z^{-1}]_{\mathrm{Id}} \overset{I}{\simeq} (-2\pi i)^n \delta_{(0)}.$$

More generally, for a nonsingular complex matrix A we have $[(Az)^{-1}]_A \circ A^{-1} = [z^{-1}]_{\mathrm{Id}}$. Thus

(10) $$B_A(L) \ni [(Az)^{-1}]_A \overset{I_A}{\simeq} \frac{(-2\pi i)^n}{|\det A|} \delta_{(0)} \in \mathscr{A}'(\{0\}).$$

Furthermore, if H is a function holomorphic in a neighbourhood of a point $\zeta \in A^{-1}(\mathbf{R}^n)$ then

(11) $$[H(z)(A(z-\zeta))^{-1}]_A \overset{I_A}{\simeq} \frac{(-2\pi i)^n}{|\det A|} H(\zeta) \delta_{(\zeta)} \in \mathscr{A}'(\{\zeta\}).$$

EXAMPLE 2. Let $\alpha \in \mathbf{N}_0^n$. Then

(12) $$[(-1)^{|\alpha|}\alpha!\, z^{-1-\alpha}]_{\mathrm{Id}} = D^\alpha [z^{-1}]_{\mathrm{Id}} \overset{I}{\simeq} (-2\pi i)^n \delta_{(0)}^{(\alpha)}.$$

More generally,

(13) $$[(-1)^{|\alpha|}\alpha!\,(Az)^{-1-\alpha}]_A = [(-1)^{|\alpha|}\alpha!\, z^{-1-\alpha}]_{\mathrm{Id}} \circ A \overset{I_A}{\simeq} (-2\pi i)^n \delta_{(0)}^{(\alpha)} \circ A.$$

Observe that if $A = \mathrm{diag}(a_j)_{j=1,\ldots,n}$ then

(14) $$\delta_{(0)}^{(\alpha)} \circ A = \Big(\prod_{j=1}^n a_j^{-\alpha_j}\Big) \frac{1}{|\det A|} \delta_{(0)}^{(\alpha)}.$$

EXAMPLE 3. Let A be a nonsingular, diagonal complex matrix, $\alpha \in \mathbf{N}_0^n$. Then

(15) $$B_A(L) \ni [(-1)^{|\alpha|}\alpha!\, z^{-1-\alpha}]_A \overset{I_A}{\simeq} (-2\pi i)^n \frac{\det A}{|\det A|} \delta_{(0)}^{(\alpha)} \in \mathscr{A}'(\{0\}).$$

Proof. It is sufficient to note that for a matrix $A = \mathrm{diag}(a_j)_{j=1,\ldots,n}$ the function

$$z^{-1-\alpha} = \det A \Big(\prod_{j=1}^n a_j^{\alpha_j}\Big)(Az)^{-1-\alpha}$$

is A-meromorphic and then use the formulae (13), (14).

THEOREM 3. *Let A_1, A_2 be nonsingular, diagonal complex matrices. If a function G is A_1-meromorphic on $\mathbf{C}^n$ and $\mathrm{supp}[G]_{A_1} \subset \{0\}$ then for some A_1-meromorphic function ψ such that $[\psi]_{A_1} = 0$, we have*

(16) *the function $G - \psi$ is A_2-meromorphic, $\mathrm{supp}[G-\psi]_{A_2} \subset \{0\}$, and*

$$[G-\psi]_{A_2} = \frac{\det A_2 \circ A_1^{-1}}{|\det A_2 \circ A_1^{-1}|} [G]_{A_1}$$

in the sense of equality of the corresponding analytic functionals.

Proof. To begin with we assume $A_1 = \mathrm{Id}$ and write $A_2 = A$. Let F be an Id-meromorphic function with $\mathrm{supp}[F]_{\mathrm{Id}} \subset \{0\}$. Acccording to the characterization of hyperfunctions with support at zero (see [3]) there exist constants C_α ($\alpha \in \mathbf{N}_0^n$) such that $\lim_{|\alpha| \to \infty} (|C_\alpha| \alpha!)^{1/|\alpha|} = 0$ and

$$[F]_{\mathrm{Id}} \overset{I}{\simeq} \sum_{\alpha \in \mathbf{N}_0^n} C_\alpha \delta_{(0)}^{(\alpha)}.$$

So by (12) we have

$$F(z) = \sum_{\alpha \in \mathbf{N}_0^n} \frac{C_\alpha (-1)^{|\alpha|}\alpha!}{(-2\pi i)^n} z^{-1-\alpha} + \varphi(z),$$

where $[\varphi]_{\mathrm{Id}} = 0$.

Applying (15) we find that the function $F-\varphi$ is A-meromorphic, $\operatorname{supp}[F-\varphi]_A \subset \{0\}$ and

$$[F-\varphi]_A \overset{I_A}{\simeq} \frac{\det A}{|\det A|}\sum_{\alpha\in\mathbf{N}_0^n} C_\alpha \delta^{(\alpha)}_{(0)} \overset{I}{\simeq} \frac{\det A}{|\det A|}[F]_{\mathrm{Id}}.$$

Suppose now that G is A_1-meromorphic with $\operatorname{supp}[G]_{A_1} \subset \{0\}$. Then the function $F = G\circ A_1^{-1}$ is Id-meromorphic with $\operatorname{supp}[F]_{\mathrm{Id}} \subset \{0\}$. Applying the case just proved with the matrix $A = A_2\circ A_1^{-1}$ we see that for some Id-meromorphic function φ such that $[\varphi]_{\mathrm{Id}} = 0$, the function $F-\varphi$ is $A_2\circ A_1^{-1}$-meromorphic, $\operatorname{supp}[F-\varphi]_{A_2\circ A_1^{-1}} \subset \{0\}$ and

$$[F-\varphi]_{A_2\circ A_1^{-1}} = \frac{\det A_2\circ A_1^{-1}}{|\det A_2\circ A_1^{-1}|}[F]_{\mathrm{Id}}.$$

Since $[F]_{\mathrm{Id}} = [G]_{A_1}\circ A_1^{-1}$ and

$$[F-\varphi]_{A_2\circ A_1^{-1}} = [G-\varphi\circ A_1]_{A_2}\circ A_1^{-1}, \quad [\varphi\circ A_1]_{A_1} = [\varphi]_{\mathrm{Id}}\circ A_1 = 0,$$

we get the assertion with $\psi = \varphi\circ A_1$.

LEMMA 1. *Let A_l $(l = 1, \ldots, k,\ k \geqslant 2)$ be diagonal, complex matrices. Set $L_l = A_l^{-1}(\mathbf{R}^n)$ and suppose $L_1 \cap L_l = \{0\}$ for $l = 2, \ldots, k$. If G_l is an A_l-meromorphic function on $\mathbf{C}^n$ $(l = 2, \ldots, k)$ and the function $G = G_2+\ldots+G_k$ is A_1-meromorphic on $\mathbf{C}^n$ then $\operatorname{supp}[G]_{A_1} \subset \{0\}$.*

Proof. By (1) it is sufficient to prove the lemma when $A_1 = \mathrm{Id}$. Let $A_l = \operatorname{diag}(a_{l,j})_{j=1,\ldots,n}$ for $l = 2, \ldots, k$, $a_{l,j} = \alpha_{l,j}+i\beta_{l,j}$, $z_j = x_j+iy_j$ for $l = 2, \ldots, k$, $j = 1, \ldots, n$. By assumption G_l $(l = 2, \ldots, k)$ is holomorphic on the set

$$\{z\in\mathbf{C}^n\colon \operatorname{Im}(A_l z)_j \neq 0 \text{ for } j = 1, \ldots, n\} = \{z\in\mathbf{C}^n\colon \alpha_{l,j}y_j+\beta_{l,j}x_j \neq 0 \text{ for } j = 1, \ldots, n\}.$$

So

$$G\in\mathcal{O}\Big(\bigcap_{l=2}^{k}\prod_{j=1}^{n}\{z_j\in\mathbf{C}\colon \alpha_{l,j}y_j+\beta_{l,j}x_j \neq 0\}\Big).$$

At the same time, $G\in\mathcal{O}(\prod_{j=1}^n\{z_j\in\mathbf{C}\colon y_j\neq 0\})$. Because $\mathbf{R}^n \cap L_l = \{0\}$ $(l = 2, \ldots, k)$ and the matrices A_l are diagonal we have $\beta_{l,j} \neq 0$ for $l = 2, \ldots, k$, $j = 1, \ldots, n$. Using these facts and the Hartogs theorem we get $G\in\mathcal{O}((\mathbf{C}\setminus\{0\})^n)$ and this proves the lemma.

LEMMA 2. *Let A_l $(l = 1, \ldots, k,\ k \geqslant 2)$ be diagonal complex matrices such that $L_l \cap L_{l'} = \{0\}$ for $l, l' = 1, \ldots, k$, $l \neq l'$, where $L_l = A_l^{-1}(\mathbf{R}^n)$ $(l = 1, \ldots, k)$. Let ψ_l $(l = 2, \ldots, k)$ be A_l-meromorphic functions on $\mathbf{C}^n$ such that the function $\psi_1 = \psi_2+\ldots+\psi_k$ is A_1-meromorphic on $\mathbf{C}^n$. If $[\psi_l]_{A_l} = 0$ for $l = 2, \ldots, k$ then $[\psi_1]_{A_1} = 0$.*

Proof. Let $A_l = \operatorname{diag}(a_{l,j})_{j=1,\ldots,n}$ $(l = 1, \ldots, k)$. Set $L_{l,j} = a_{l,j}^{-1}\cdot\mathbf{R} \subset \mathbf{C}$ for $l = 1, \ldots, k$, $j = 1, \ldots, n$. Observe that $L_l = \prod_{j=1}^n L_{l,j}$ $(l = 1, \ldots, k)$. We have $\psi_l\in\mathcal{O}(\prod_{j=1}^n(\mathbf{C}\setminus L_{l,j}))$ $(l = 1, \ldots, k)$. Since $L_l \cap L_{l'} = \{0\}$ for $l \neq l'$ it follows by the proof of the above lemma that $\psi_l\in\mathcal{O}((\mathbf{C}\setminus\{0\})^n)$ $(l = 1, \ldots, k)$. So by assumption $[\psi_l]_{A_l} = 0$ $(l = 2, \ldots, k)$ and by a version of Proposition 1 for the matrix A_l, we conclude that $[\psi_l]_{A_l}$, treated as elements of $B(\{0\})$, are equal to zero. Thus, the same is true for $[\psi_1]_{A_1}$ and we get the assertion.

Now we are in a position to prove the main result of this section.

THEOREM 4. *Let A_l $(l = 1, \ldots, k,\ k \geqslant 2)$ be diagonal complex matrices with positive determinants. Let $L_l = A_l^{-1}(\mathbf{R}^n)$ $(l = 1, \ldots, k)$ and $L_l \cap L_{l'} = \{0\}$ for $l, l' = 1, \ldots, k$, $l \neq l'$. Let G_l $(l = 1, \ldots, k)$ be A_l-meromorphic functions on $\mathbf{C}^n$. If $G_1+\ldots+G_k = 0$ then $\operatorname{supp}[G_l]_{A_l} \subset \{0\}$ for $l = 1, \ldots, k$ and*

$$[G_1]_{A_1}+\ldots+[G_k]_{A_k} = 0$$

regarded as the sum of the corresponding analytic functionals.

Proof. Since $G_1 = -G_2-\ldots-G_k$ is an A_1-meromorphic function, by Lemma 1 we find that $\operatorname{supp}[G_1]_{A_1} \subset \{0\}$. Analogously, $\operatorname{supp}[G_l]_{A_l} \subset \{0\}$ for $l = 2, \ldots, k$. By Theorem 3 there exist A_l-meromorphic functions ψ_l such that $[\psi_l]_{A_l} = 0$, $G_l-\psi_l$ are A_1-meromorphic functions, $\operatorname{supp}[G_l-\psi_l]_{A_1} \subset \{0\}$ and

$$[G_l-\psi_l]_{A_1} = [G_l]_{A_l} \quad \text{for } l = 2, \ldots, k.$$

Thus

$$\begin{aligned}[G_1]_{A_1}+[G_2]_{A_2}+\ldots+[G_k]_{A_k} &= [G_1]_{A_1}+[G_2-\psi_2]_{A_1}+\ldots+[G_k-\psi_k]_{A_1}\\ &= -[\psi_2+\ldots+\psi_k]_{A_1} = 0,\end{aligned}$$

because the function $\psi_1 = \psi_2+\ldots+\psi_k$ is A_1-meromorphic and by Lemma 2, $[\psi_1]_{A_1} = 0$.

The above theorem ensures the correctness of the following definition.

DEFINITION 5. Under the assumptions of Theorem 4, let $\operatorname{supp}[G_l]_{A_l}$ be compact for $l = 1, \ldots, k$. Let $G = G_1+\ldots+G_k$.

We define the boundary value $b(G)$ of G as a hyperfunction on $L_1\cup\ldots\cup L_k$ by

$$b(G) = [G_1]_{A_1}+\ldots+[G_k]_{A_k} \tag{17}$$

where the right hand side is regarded as the sum of the corresponding analytic functionals.

2. Generalized Taylor expansions of functions in dimension $n = 1$. The theory of hyperfunctions simplifies essentially in dimension $n = 1$. A hyperfunction f on an interval $[a, \infty)$ $(a\in\mathbf{R})$ can be regarded as an element of the space

$$B_{[a,\infty)} = \mathcal{O}(\mathbf{C}\setminus[a, \infty))/\mathcal{O}(\mathbf{C}).$$

Thus $f\in B_{[a,\infty)}$ is represented by a function $F\in\mathcal{O}(\mathbf{C}\setminus[a,\infty))$, which is obviously Id-meromorphic on $\mathbf{C}$. One can write the Taylor–Ziemian formula ([12], Th. 5) in the following manner.

THEOREM 5. *Let $U\in M'_{(a)}((0,r])$ be a Mellin distribution such that $\mathcal{M}U\in\mathcal{O}(\mathbf{C}\setminus[a,\infty))$. Let $T=b(\mathcal{M}U)=[\mathcal{M}U]_{\mathrm{Id}}$ be the boundary value of $\mathcal{M}U$. Then for every $\varrho>0$ there exists a Mellin distribution $R\in M'_{(a+\varrho)}$ such that*

$$U=T_\varrho[x^\alpha\chi^r(x)]+R, \tag{18}$$

where χ^r is the characteristic function of the interval $(0,r]$ and T_ϱ is a hyperfunction supported by $[a,a+\varrho]$ (regarded as an analytic functional in the variable α) equal to

$$T_\varrho=\frac{1}{2\pi i}T\Big|_{\Omega_{a,\varrho}}\quad\text{with}\quad \Omega_{a,\varrho}=\{\alpha\in\mathbf{R}:\ \alpha<a+\varrho\}.$$

The decomposition (18) is unique modulo hyperfunctions with support at $a+\varrho$.

PROPOSITION 2. *Let $c\in\mathbf{R}$, $0<r<e^c$. If $\operatorname{Re}\theta<0$ then*

$$\mathcal{M}(\chi^r(x)(-\ln x+c)^\theta)(z)=\frac{1}{\Gamma(-\theta)}\int_0^\infty \varrho^{-\theta-1}\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}\,d\varrho \tag{19}$$

for $z\in\mathbf{C}\setminus[0,\infty)$.

More generally, if $\operatorname{Re}\theta<k\in\mathbf{N}_0$ then

$$\mathcal{M}(\chi^r(x)(-\ln x+c)^\theta)(z)=\frac{(-1)^k}{\Gamma(-\theta+k)}\int_0^\infty \varrho^{-\theta+k-1}\frac{d^k}{d\varrho^k}\left(\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}\right)d\varrho \tag{20}$$

for $z\in\mathbf{C}\setminus[0,\infty)$.

$\mathcal{M}(\chi^r(x)(-\ln x+c)^\theta)$ is an Id-meromorphic function with the boundary value

$$\frac{2\pi i}{\Gamma(-\theta)}e^{-c\varrho}\varrho_+^{-\theta-1}\qquad \text{for } \theta\neq 0,1,\ldots, \tag{21}$$

$$2\pi i\sum_{k=0}^{m}\binom{m}{k}c^k\delta_{(0)}^{(m-k)}\qquad\text{for } \theta=m\in\mathbf{N}_0, \tag{21'}$$

where

$$\varrho_+^{-\theta-1}=\begin{cases}\varrho^{-\theta-1} & \text{for } \varrho>0\\ 0 & \text{for } \varrho<0\end{cases}\qquad\text{for } \operatorname{Re}\theta<0,$$

$$\frac{\varrho_+^{-\theta-1}}{\Gamma(-\theta)}=\left(\frac{d}{d\varrho}\right)^k\left(\frac{(-1)^k\varrho_+^{-\theta+k-1}}{\Gamma(-\theta+k)}\right)\qquad\text{for } \operatorname{Re}\theta<k\in\mathbf{N}_0.$$

Proof. By substitution $t=-\ln(xe^{-c})\varrho$ $(x<e^c)$ in the integral

$$\Gamma(-\theta)=\int_0^\infty t^{-\theta-1}e^{-t}\,dt\qquad(\operatorname{Re}\theta<0)$$

we get

$$(-\ln x+c)^\theta=\frac{1}{\Gamma(-\theta)}\int_0^\infty \varrho^{-\theta-1}e^{-c\varrho}x^\varrho\,d\varrho\quad \text{for } 0<x<e^c,\ \operatorname{Re}\theta<0. \tag{22}$$

Hence, we have (19) for $\operatorname{Re}z<0$. Next we observe that the integral on the right hand side of (19) defines a holomorphic function on $\mathbf{C}\setminus[0,\infty)$.

To get (20) we use the formula

$$(-\ln x+c)^\theta=\frac{(-1)^k}{\Gamma(-\theta+k)}\int_0^\infty \varrho^{-\theta+k-1}\frac{d^k}{d\varrho^k}(e^{-c\varrho}x^\varrho)\,d\varrho \tag{23}$$

for $0<x<e^c$, $\operatorname{Re}\theta<k$,

which is an easy consequence of (22).

Since

$$b\left(\frac{r^{\varrho-z}}{\varrho-z}\right)=\left[\frac{r^{\varrho-z}}{\varrho-z}\right]_{\mathrm{Id}}=2\pi i\delta_{(\varrho)}\qquad\text{for } \varrho>0,$$

(21) follows from (20).

If $\theta=m$, $m\in\{0,\ldots,k-1\}$, then formula (20) takes the form

$$\begin{aligned}\mathcal{M}(\chi^r(x)(-\ln x+c)^m)(z)&=(-1)^{m+1}\int_0^\infty\frac{d^{m+1}}{d\varrho^{m+1}}\left(\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}\right)d\varrho\\&=e^{-cz}\frac{d^m}{dz^m}\left(\frac{e^{cz}r^{-z}}{-z}\right)\end{aligned} \tag{24}$$

and consequently we get (21').

PROPOSITION 3. *Let $c\in\mathbf{R}$, $0<r\leqslant e^c$. Then*

$$\mathcal{M}(\chi^r(x)\ln(-\ln x+c))(z)=\frac{\gamma}{z}r^{-z}+\int_0^\infty\ln\varrho\frac{d}{d\varrho}\left(\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}\right)d\varrho \tag{25}$$

for $z\in\mathbf{C}\setminus[0,\infty)$, where γ is the Euler constant, and

$$b(\mathcal{M}(\chi^r(x)\ln(-\ln x+c)))=-2\pi i(\gamma\delta_{(0)}-e^{-c\varrho}\varrho_+^{-1}), \tag{26}$$

where $\varrho_+^{-1}=\dfrac{d}{d\varrho}\ln\varrho_+$, $\ln\varrho_+=\begin{cases}\ln\varrho & \text{for } \varrho>0,\\ 0 & \text{for } \varrho<0.\end{cases}$

Proof. Observe that

$$\ln(-\ln x+c)=\frac{\partial}{\partial\theta}(-\ln x+c)^\theta\Big|_{\theta=0}.$$

So we use (20) with $k=1$ to get

$$\mathcal{M}(\chi^r(x)\ln(-\ln x+c))(z) = \int_0^\infty \frac{\partial}{\partial\theta}\left(\frac{-\varrho^{-\theta}}{\Gamma(-\theta+1)}\right)\Bigg|_{\theta=0}\frac{d}{d\varrho}\left(\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}\right)d\varrho$$

$$= \frac{\gamma}{z}r^{-z}+\int_0^\infty \ln\varrho\,\frac{d}{d\varrho}\left(\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}\right)d\varrho$$

$$\text{for } r<e^c,\ \operatorname{Re} z<0.$$

Next we observe that the integral on the right hand side of (25) is convergent for $r\leqslant e^c$, $z\in\mathbf{C}\setminus[0,\infty)$. The boundary value (26) is derived immediately from (25).

THEOREM 6. *Let* $c\in\mathbf{R}$, $k\in\mathbf{N}$, $\operatorname{Re}\theta<0$. *Let* $a_j\in\mathbf{C}$ $(j\in\mathbf{N}_0)$ *be such that* $\lambda=\limsup|a_j|^{1/kj}<\infty$. *Let*

$$f(x)=\sum_{j=0}^\infty a_j(-\ln x+c)^{\theta-kj} \quad \textit{for } 0<x<e^{-\lambda+c},$$

$$f^*(\varrho)=\sum_{j=0}^\infty \frac{a_j}{\Gamma(kj-\theta)}\varrho^{kj} \qquad \textit{for } \varrho\in\mathbf{C}.$$

Then

(i) f^* *is an entire function,*

(ii) *for every* $\varepsilon>0$ *there exists* C_ε *such that*

$$|f^*(\varrho)|\leqslant C_\varepsilon e^{(\lambda+\varepsilon)|\varrho|} \quad \textit{for } \varrho\in\mathbf{C},$$

(iii) $\mathcal{M}(\chi^r f)(z)=\int_0^\infty \varrho^{-\theta-1}f^*(\varrho)\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}d\varrho$ *for* $r<e^{-\lambda+c}$, $z\in\mathbf{C}\setminus[0,\infty)$,

(iv) $b(\mathcal{M}(\chi^r f))=2\pi i e^{-c\varrho}f^*(\varrho)\varrho_+^{-\theta-1}$.

Proof. (i) and (ii) follow by Theorem 5.3.1 of [1]. Let $\operatorname{Re} z<0$. Using (20) we get

$$\mathcal{M}(\chi^r f)(z)=\sum_{j=0}^\infty a_j\mathcal{M}(\chi^r(x)(-\ln x+c)^{\theta-kj})(z)$$

$$=\sum_{j=0}^\infty \frac{a_j}{\Gamma(kj-\theta)}\int_0^\infty \varrho^{-\theta+kj-1}\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}d\varrho$$

$$=\int_0^\infty \varrho^{-\theta-1}f^*(\varrho)\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}d\varrho.$$

Since f^* satisfies (ii) the last integral defines a holomorphic function on $\mathbf{C}\setminus[0,\infty)$. The boundary value of $\mathcal{M}(\chi^r f)$ is derived from (iii).

EXAMPLE 4. Let $c\in\mathbf{R}$, $\operatorname{Re}\theta<0$, $b>0$, $r<e^{c-b}$. Then

$$\mathcal{M}(\chi^r(x)((-\ln x+c)^2+b^2)^\theta)(z)$$

$$(27)\qquad =\int_0^\infty \varrho^{-2\theta-1}\left(\sum_{j=0}^\infty\binom{\theta}{j}\frac{(b\varrho)^{2j}}{\Gamma(2j-2\theta)}\right)\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}d\varrho$$

$$(27')\qquad =(2b)^{\theta+1/2}\frac{\sqrt{\pi}}{\Gamma(-\theta)}\int_0^\infty \varrho^{-\theta-1/2}J_{-\theta-1/2}(b\varrho)\frac{e^{-c\varrho}r^{\varrho-z}}{\varrho-z}d\varrho,$$

where

$$J_\nu(\zeta)=\left(\frac{\zeta}{2}\right)^\nu\sum_{j=0}^\infty\frac{(-1)^j}{j!\,\Gamma(j+\nu+1)}\left(\frac{\zeta}{2}\right)^{2j} \qquad (\zeta\in\mathbf{C})$$

is the Bessel function of order $\nu\in\mathbf{C}$.

Proof. Let $x<e^{c-b}$. Then

$$((-\ln x+c)^2+b^2)^\theta=\sum_{j=0}^\infty\binom{\theta}{j}b^{2j}(-\ln x+c)^{2\theta-2j}.$$

Hence by Theorem 6 we have (27). To get (27′) we use the definition of the Bessel function and the duplication formula for the Γ function ([9], formula 6.3.13)

$$\Gamma(2\zeta)=\frac{2^{2\zeta-1}}{\sqrt{\pi}}\Gamma(\zeta)\Gamma(\zeta+1/2) \quad \text{for } 2\zeta\neq 0,-1,\ldots$$

EXAMPLE 5. Let $c\geqslant b\geqslant 0$. Then

$$\mathcal{M}(\chi^1(x)\ln((-\ln x+c)^2+b^2))(z)$$

$$(28)\qquad =\frac{2\gamma}{z}+2\int_0^\infty\left[\ln\varrho\frac{d}{d\varrho}\left(\frac{e^{-c\varrho}}{\varrho-z}\right)+\frac{1}{\varrho}\left(\sum_{j=1}^\infty\frac{(-1)^{j+1}}{(2j)!}(b\varrho)^{2j}\right)\frac{e^{-c\varrho}}{\varrho-z}\right]d\varrho$$

$$(28')\qquad =\frac{2\gamma}{z}+2\int_0^\infty\left[\ln\varrho\frac{d}{d\varrho}\left(\frac{e^{-c\varrho}}{\varrho-z}\right)+\frac{1-\cos b\varrho}{\varrho}\cdot\frac{e^{-c\varrho}}{\varrho-z}\right]d\varrho.$$

Proof. Let $c>b\geqslant 0$, $x\leqslant 1$. Then

$$\ln((-\ln x+c)^2+b^2)=2\ln(-\ln x+c)+\sum_{j=1}^\infty\frac{(-1)^{j+1}}{j}b^{2j}(-\ln x+c)^{-2j}.$$

Hence by Theorem 6 and Proposition 3 we get (28). Letting $c\to b$ we get (28) for $c\geqslant b$.

3. Generalized Taylor expansions of functions in arbitrary dimension. To find the Mellin transform of a given distribution $U\in M'(J)$ it is necessary to compute an n-fold integral with a parameter. That integral can be computed only in special cases. In this section we find it for the distributions of the following two forms:

1. $f\left(\sum_{j=1}^{n}(a_j x_j)^{\alpha_j}\right)$,
2. $f\left(-\sum_{j=1}^{n} a_j \ln x_j\right)$,

where f is a one-dimensional Mellin distribution.

Let us observe that a fundamental solution of the iterated Laplace operator is of the first form. Furthermore, one can derive its generalized asymptotic expansion at zero. Distributions of the second form are solutions of certain singular differential equations of the first order.

3.1. *Functions of the form* $f\left(\sum_{j=1}^{n}(a_j x_j)^{\alpha_j}\right)$. To compute the Mellin transform of a function h of the form $h = f\left(\sum_{j=1}^{n}(a_j x_j)^{\alpha_j}\right)$, where $f\colon \mathbf{R}_+^1 \to \mathbf{C}$, $a_j > 0$, $\alpha_j > 0$ for $j = 1, \ldots, n$, we observe that for $t > 0$ the set

$$\Omega_t = \left\{x \in \mathbf{R}_+^n \colon \sum_{j=1}^{n} (a_j x_j)^{\alpha_j} \leqslant t\right\}$$

is a bounded neighbourhood of zero in $\mathbf{R}_+^n$. Since our aim is to find the asymptotic expansion of h at zero we compute the Mellin transform of the function $\chi_{\Omega_t} h$ and its boundary value. To this end by integration along the level sets of the function h we reduce the calculation of $\mathcal{M}(\chi_{\Omega_t} h)$ to $\mathcal{M}f$ and then find the boundary value of $\mathcal{M}(\chi_{\Omega_t} h)$.

First of all we establish the following

Proposition 4. *Let $t > 0$, $\alpha > 0$. Let $g\colon \mathbf{R}_+^n \to \mathbf{R}_+$ be a homogeneous function of degree α such that the set*

$$\Omega_t = \left\{x \in \mathbf{R}_+^n \colon g(x) \leqslant t\right\}$$

is a bounded neighbourhood of zero in $\mathbf{R}_+^n$. Let $h = f \circ g$ where $f\colon (0, t] \to \mathbf{R}$ is a function such that $f \in M'_{(\omega)}((0, t])$ for some $\omega \in \mathbf{R}$. Then

$$\begin{aligned}(29)\qquad & \mathcal{M}(\chi_{\Omega_t} h)(z) \\ & = \frac{1}{\alpha}\mathcal{M}f(\zeta_1/\alpha) \int\limits_{(0,\pi/2)^{n-1}} (\cos\varphi')^{-z'-1} (\sin\varphi')^{-(\zeta_2,\ldots,\zeta_n)-1'} \left(\tilde{g}(\varphi')\right)^{\zeta_1/\alpha} d\varphi',\end{aligned}$$

for $z \in \mathbf{C}^n$ such that $\operatorname{Re}(z_1 + \ldots + z_n) < \alpha\omega$, $\operatorname{Re} z_j < 0$ $(j = 1, \ldots, n)$, where $\tilde{g}(\varphi') = g\left(\cos\varphi_1, \sin\varphi_1 \cos\varphi_2, \ldots, (\sin\varphi')^{1'}\right)$, with $\varphi' = (\varphi_1, \ldots, \varphi_{n-1})$, $z' = (z_1, \ldots, z_{n-1})$, $1' \in \mathbf{R}_+^{n-1}$, $\zeta_k = \sum_{j=k}^{n} z_j$ for $k = 1, \ldots, n$.

Proof. Let us first assume $n = 2$. Introducing the polar coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, the variable $w = r^\alpha g(\cos\varphi, \sin\varphi)$ and setting

$r_\varphi = [t/g(\cos\varphi, \sin\varphi)]^{1/\alpha}$ we find

$$\begin{aligned}&\int\limits_0^\infty \int\limits_0^\infty \chi_{\Omega_t}(x_1, x_2) f\left(g(x_1, x_2)\right) x_1^{-z_1-1} x_2^{-z_2-1}\, dx_1\, dx_2 \\ &\quad = \int\limits_0^{\pi/2} \int\limits_0^{r_\varphi} f\left(r^\alpha g(\cos\varphi, \sin\varphi)\right)(r\cos\varphi)^{-z_1-1}(r\sin\varphi)^{-z_2-1} r\, dr\, d\varphi \\ &\quad = \int\limits_0^t \frac{1}{\alpha} f(w) w^{(-z_1-z_2)/\alpha-1}\, dw \\ &\qquad\quad \times \int\limits_0^{\pi/2} (\cos\varphi)^{-z_1-1}(\sin\varphi)^{-z_2-1}[g(\cos\varphi, \sin\varphi)]^{(z_1+z_2)/\alpha}\, d\varphi.\end{aligned}$$

Observe that the second integral is absolutely convergent for $\operatorname{Re} z_1 < 0$, $\operatorname{Re} z_2 < 0$. If $n > 2$ the proof goes along the same lines.

Corollary 1. *Let $f\colon (0, t] \to \mathbf{C}$ be a function such that $f \in M'_{(\omega)}((0, t])$ for some $\omega \in \mathbf{R}$. Then*

$$(30)\qquad \mathcal{M}\left(\chi_{B_{\sqrt{t}}^+} f(x_1^2 + \ldots + x_n^2)\right)(z) = \frac{1}{2^n}\mathcal{M}f\left(\frac{z_1 + \ldots + z_n}{2}\right) B\left(\frac{-z_1}{2}, \ldots, \frac{-z_n}{2}\right)$$

for $z \in \mathbf{C}^n$ such that $\operatorname{Re}(z_1 + \ldots + z_n) < 2\omega$, $z_j \neq 0, 2, 4, \ldots$, $j = 1, \ldots, n$, where

$$B_{\sqrt{t}}^+ = \left\{x \in \mathbf{R}_+^n \colon |x| \leqslant \sqrt{t}\right\},$$

and

$$B(\zeta_1, \ldots, \zeta_n) = \frac{\Gamma(\zeta_1)\ldots\Gamma(\zeta_n)}{\Gamma(\zeta_1 + \ldots + \zeta_n)}$$

is the Euler function in n variables.

Proof. It suffices to apply Proposition 4 to the function $g(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$ and use the formula (see [9])

$$\int\limits_0^{\pi/2} (\cos\varphi)^{-\zeta_1-1}(\sin\varphi)^{-\zeta_2-1}\, d\varphi = \tfrac{1}{2}B(-\zeta_1/2, -\zeta_2/2).$$

By the change of variables $y_j = (a_j x_j)^{\alpha_j/2}$, $dx_j = 2(a_j\alpha_j)^{-1} y_j^{2/\alpha_j - 1}\, dy_j$ $(j = 1, \ldots, n)$ we get from the above corollary

Corollary 2. *Let $a_j > 0$, $\alpha_j > 0$ for $j = 1, \ldots, n$. Let f be as in Corollary 1. Set*

$$\Omega_t = \left\{x \in \mathbf{R}_+^n \colon \sum_{j=1}^{n} (a_j x_j)^{\alpha_j} \leqslant t\right\}.$$

Then

(31) $$\mathcal{M}\Big(\chi_{\Omega_t}\cdot f\big(\sum_{j=1}^{n}(a_j x_j)^{\alpha_j}\big)\Big)(z) = \frac{a_1^{z_1}\ldots a_n^{z_n}}{\alpha_1\ldots\alpha_n}\mathcal{M}f\Big(\frac{z_1}{\alpha_1}+\ldots+\frac{z_n}{\alpha_n}\Big)B\Big(\frac{-z_1}{\alpha_1},\ldots,\frac{-z_n}{\alpha_n}\Big)$$

for $z\in\mathbf{C}^n$ *such that* $\operatorname{Re}(z_1/\alpha_1+\ldots+z_n/\alpha_n)<\omega$, $z_j/\alpha_j\neq 0,1,\ldots$ $(j=1,\ldots,n)$.

PROPOSITION 5. *Let* $\theta\in\mathbf{R}$. *Then*

(32) $$\mathcal{M}\big(\chi_{B_1^+}\cdot(x_1^2+\ldots+x_n^2)^{\theta}\big)(z)=\frac{1}{2^{n-1}}\,\frac{B(-z_1/2,\ldots,-z_n/2)}{2\theta-z_1-\ldots-z_n}$$

for $z_1+\ldots+z_n\neq 2\theta$, $z_j\neq 0,2,4,\ldots$ $(j=1,\ldots,n)$. *Denote by* H *the function on the right hand side of* (32). H *is an* $\mathcal{A}=(A_1,\ldots,A_n)$*-meromorphic function* (*for the definition of* $\mathcal{A}$*-meromorphicity see* [12], *Section* 3), *where* (¹)

$$A_j=\begin{bmatrix}1&0&\ldots&0\\0&1&\ldots&0\\ \cdot&\cdot&\cdot&\cdot\\ 1&1&\ldots&1\\ \cdot&\cdot&\cdot&\cdot\\ 0&0&\ldots&1\end{bmatrix}\ j\text{-th row}.$$

The boundary value of H *equals*

(33) $$b^{A}(H)=(2\pi i)^n(T_1,\ldots,T_n)$$

$$\text{with}\quad T_j=\sum{}'\binom{\theta}{k_1,\ldots,\hat{k}_j,\ldots,k_n}\delta_{(\varkappa_1,\ldots,\varkappa_n)},$$

where $\sum'$ *denotes the summation over all* $l=1,\ldots,n$, $l\neq j$ *and over all* $k_l\in\mathbf{N}_0$; $\varkappa_l=2k_l$ *for* $l\neq j$, $\varkappa_j=2\theta+2k_j-\sum_{l=1}^{n}2k_l$ *and*

$$\binom{\theta}{k_1,\ldots,\hat{k}_j,\ldots,k_n}=\prod_{\substack{l=1\\l\neq j}}^{n}\binom{\theta-\tilde{k}_l}{k_l},$$

$$\tilde{k}_l=\sum_{\substack{\nu=1\\ \nu\neq j}}^{l-1}k_\nu\quad\text{for } j=1,\ldots,n$$

(*the hat denotes suppression of the factor under it*).

(¹) For $n=3$ see Corollary 3.

If $\theta=m\in\mathbf{N}_0$ *then* H *is an* Id*-meromorphic function with the boundary value*

(33′) $$b^{\mathrm{Id}}(H)=(2\pi i)^n\sum{}'\binom{m}{k_1,\ldots,k_{n-1}}\delta_{(\varkappa_1,\ldots,\varkappa_n)}$$

where $\sum'$ *denotes the summation over all* $l=1,\ldots,n-1$, $k_l=0,1,\ldots,m-k_1-\ldots-k_{l-1}$; $\varkappa_l=2k_l$ *for* $l=1,\ldots,n-1$, $\varkappa_n=m-2k_1-\ldots-2k_{l-1}$.

Proof. Let $f(r)=r^\theta$ for $r\in(0,1]$. Then $f\in M'_{(\theta)}((0,1])$ and $\mathcal{M}f(z)=1/(\theta-z)$.

Applying (30) with $t=1$ we get (32). Now we study the function H. Recall that the function Γ is holomorphic outside the points $0,-1,-2,\ldots$, where it has simple poles. So it can be represented as

$$\Gamma(\zeta)=\frac{(-1)^k}{k!}\,\frac{\Phi_k(\zeta)}{\zeta+k}\qquad(k\in\mathbf{N}_0)$$

with $\Phi_k\in\mathcal{O}((\mathbf{C}\setminus\{-\mathbf{N}_0\})\cup\{-k\})$, $\Phi_k(-k)=1$ for $k\in\mathbf{N}_0$. Moreover, $\Gamma(\zeta)\neq 0$ for $\zeta\in\mathbf{C}\setminus(-\mathbf{N}_0)$. Using the definition of the Euler function in n variables we infer that the possible singularities of H may occur only if $z_1+\ldots+z_n=2\theta$ or $z_j=2k_j$ for some $k_j\in\mathbf{N}_0$, $j=1,\ldots,n$. We also see that H is $\mathcal{A}$-meromorphic. Next, we observe that the boundary value $b^{\mathcal{A}}(H)$ is zero at the following points:

(i) $z_1+\ldots+z_n=2\theta$ and $z_j\notin 2\mathbf{N}_0$ and $z_l\notin 2\mathbf{N}_0$ for some $l,j\in\{1,\ldots,n\}$, $l\neq j$;

(ii) $z_1+\ldots+z_n\neq 2\theta$ (in this case, if $\mathring{z}_j\in 2\mathbf{N}_0$ for $j=1,\ldots,n$ then H can be written in the form $H(z)=\tilde{H}(z)(z-\mathring{z})^{-1}$, where $\tilde{H}(\mathring{z})=0$. So by (11) its boundary value at $\mathring{z}$ is zero).

It remains to consider the points $z\in\mathbf{C}^n$ such that $z_1+\ldots+z_n=2\theta$, $z_j=\varkappa_j=2\theta+2k_j-\sum_{l=1}^{n}2k_l$ for some $j\in\{1,\ldots,n\}$, $z_l=2k_l$ for $l=1,\ldots,n$, $l\neq j$, where $k_l\in\mathbf{N}_0$ $(l=1,\ldots,n)$.

In a neighbourhood of such a point the function admits the following representation:

$$H(z)=\frac{1}{2^{n-1}}\prod_{\substack{l=1\\l\neq j}}^{n}\frac{(-1)^{k_l}}{k_l!}\,\frac{\Phi_{k_l}(-z_l/2)}{(-z_l/2)+k_l}\cdot\frac{\Gamma(-\varkappa_j/2)}{\Gamma(-\theta)}\cdot\frac{1}{2\theta-z_1-\ldots-z_n}$$
$$=\binom{\theta}{k_1,\ldots,\hat{k}_j,\ldots,k_n}\frac{1}{2\theta-z_1-\ldots-z_n}\cdot\prod_{\substack{l=1\\l\neq j}}^{n}\frac{\Phi_{k_l}(-z_l/2)}{-z_l+2k_l}.$$

Note that

$$b^{A_j}\Big(\frac{1}{2\theta-z_1-\ldots-z_n}\cdot\prod_{\substack{l=1\\l\neq j}}^{n}\frac{1}{-z_l+2k_l}\Big)=(2\pi i)^n\delta_{(\varkappa_1,\ldots,\varkappa_n)}$$

for $j=1,\ldots,n$ with $\varkappa_l=2k_l$ for $l\neq j$, $\varkappa_j=2\theta+2k_j-\sum_{l=1}^{n}2k_l$.

So summing over $l = 1, \ldots, n$, $l \neq j$, $k_l \in \mathbf{N}_0$ we get formula (33).

If $\theta = m \in \mathbf{N}_0$ then we use the properties of the generalized Newton symbols to get (33′).

Applying Proposition 5 with specific values of θ we obtain the Mellin transform of the fundamental solution of the iterated Laplace operator Δ_n^k $(2k < n)$ and its boundary value. In particular for $n = 3$, $\theta = -1/2$ we have

COROLLARY 3. *We have*

$$\mathcal{M}\left(\chi_{B_1^+} \cdot (x_1^2+x_2^2+x_3^2)^{-1/2}\right)(z) = \frac{1}{4}\,\frac{B(-z_1/2,\, -z_2/2,\, -z_3/2)}{-1-z_1-z_2-z_3}. \tag{34}$$

$\mathcal{M}\left(\chi_{B_1^+} \cdot (x_1^2+x_2^2+x_3^2)^{-1/2}\right)$ is an $\mathscr{A} = (A_1, A_2, A_3)$-meromorphic function, where

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

and its boundary value equals $(2\pi i)^3 (T_1, T_2, T_3)$ with

$$T_1 = \sum_{j,l=0}^{\infty} \binom{-1/2}{j,\, l} \delta_{(2j,2l,-1-2j-2l)},$$

$$T_2 = \sum_{l,k=0}^{\infty} \binom{-1/2}{l,\, k} \delta_{(2l,-1-2l-2k,2k)},$$

$$T_3 = \sum_{k,j=0}^{\infty} \binom{-1/2}{k,\, j} \delta_{(-1-2k-2j,2k,2j)}.$$

PROPOSITION 6. *Let $n = 2$. We have*

$$\mathcal{M}\left(\chi_{B_1^+} \cdot \ln(x_1^2+x_2^2)\right)(z) = \frac{-B(-z_1/2,\, -z_2/2)}{(z_1+z_2)^2} \tag{35}$$

for $z_1+z_2 \neq 0$, $z_1 \notin 2\mathbf{N}_0$, $z_2 \notin 2\mathbf{N}_0$. Moreover, $\mathcal{M}\left(\chi_{B_1^+} \cdot \ln(x_1^2+x_2^2)\right)$ is an $\mathscr{A} = (A_1, A_2)$-meromorphic function, where

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and its boundary value equals $-4\pi^2 (T_1, T_2)$ with

$$\begin{aligned} T_1 &= -2\delta_{(0,0)}^{(1,0)} + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \delta_{(-2j,2j)}, \\ T_2 &= -2\delta_{(0,0)}^{(0,1)} + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \delta_{(2j,-2j)}. \end{aligned} \tag{36}$$

Proof. Let $f(r) = \ln r$ for $r \in (0, 1]$. Then $f \in M'_{(0)}((0, 1])$ and $\mathcal{M}f(z) = -1/z^2$. So by applying (30) we get (35). Now we study the function

$$H(z_1, z_2) = \frac{-B(-z_1/2,\, -z_2/2)}{(z_1+z_2)^2} = \frac{-\Gamma(-z_1/2)\Gamma(-z_2/2)}{(z_1+z_2)^2\,\Gamma((-z_1-z_2)/2)}.$$

In the same manner as in the proof of Proposition 5 we deduce that H is $\mathscr{A}$-meromorphic with possible essential singularities only at the points $z \in \mathbf{C}^2$ such that $z_1 + z_2 = 0$ and both coordinates are even integers. Next, we observe that H admits the following representations:

$$H(z_1, z_2) = \frac{(-1)^{j+1}}{j!}\,\frac{\Gamma(-z_1/2)\Phi_j(-z_2/2)}{(z_1+z_2)(z_2-2j)\Phi_0((-z_1-z_2)/2)}$$

in a neighbourhood of the point $z_1 = -2j$, $z_2 = 2j$, $j \in \mathbf{N}$ (Φ_j are defined in the proof of Proposition 5);

$$H(z_1, z_2) = \frac{(-1)^{j+1}}{j!}\,\frac{\Phi_j(-z_1/2)\Gamma(-z_2/2)}{(z_1-2j)(z_1+z_2)\Phi_0((-z_1-z_2)/2)}$$

in a neighbourhood of the point $z_1 = 2j$, $z_2 = -2j$, $j \in \mathbf{N}$;

$$H(z_1, z_2) = \frac{2}{(z_1+z_2)z_1 z_2}\,\frac{\Phi_0(-z_1/2)\Phi_0(-z_2/2)}{\Phi_0((-z_1-z_2)/2)}$$

in a neighbourhood of the point $z_1 = z_2 = 0$. We have

$$B^{A_1}\left(\frac{1}{(z_1+z_2)(z_2-2j)}\right) = -4\pi^2\,\delta_{(-2j,2j)} \quad \text{for } j \in \mathbf{N};$$

$$B^{A_2}\left(\frac{1}{(z_1-2j)(z_1+z_2)}\right) = -4\pi^2\,\delta_{(2j,-2j)} \quad \text{for } j \in \mathbf{N};$$

$$b^{\mathscr{A}}\left(\frac{1}{(z_1+z_2)z_1 z_2}\right) = \left(b^{A_1}\left(\frac{1}{(z_1+z_2)^2 z_2}\right),\, b^{A_2}\left(\frac{1}{z_1(z_1+z_2)^2}\right)\right) = 4\pi^2\left(\delta_{(0,0)}^{(1,0)},\, \delta_{(0,0)}^{(0,1)}\right).$$

Since $\Phi_j(-j) = 1$ and $\Gamma(j) = (j-1)!$ for $j \in \mathbf{N}$, we find that the boundary value of H is given by (36).

By Theorem 5 of [12] and the above proposition we have the following generalized asymptotic expansion at zero of the function $\ln(x_1^2+x_2^2)$:

COROLLARY 4. *For every $k\in\mathbf{N}$ there exists a Mellin distribution $R\in M'_{(-2k)}$, $(\mathscr{A}; 0, 2k)$-flat at zero, where $\mathscr{A}=(A_1, A_2)$, $A_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $A_2=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, such that*

$$\ln(x_1^2+x_2^2)=\begin{cases}2\ln x_1+\sum\limits_{j=1}^{k-1}\dfrac{(-1)^{j+1}}{j}\left(\dfrac{x_2}{x_1}\right)^{2j}+R(x) & \text{for } x\in B_1^+,\ x_2\leqslant x_1;\\ 2\ln x_2+\sum\limits_{j=1}^{k-1}\dfrac{(-1)^{j+1}}{j}\left(\dfrac{x_1}{x_2}\right)^{2j}+R(x) & \text{for } x\in B_1^+,\ x_1\leqslant x_2.\end{cases}$$

Now we present without proof another interesting example.

EXAMPLE 6. Let $n=2$, $-1<\theta<0$. Then

$$\mathscr{M}\big(\chi_{B_1^+}\cdot(-\ln(x_1^2+x_2^2))^\theta\big)(z)=\frac{1}{2\Gamma(-\theta)}\int_0^\infty \varrho^{-\theta-1}\frac{B(-z_1/2,\,-z_2/2)}{2\varrho-z_1-z_2}\,d\varrho. \tag{37}$$

Moreover, $\mathscr{M}\big(\chi_{B_1^+}\cdot(-\ln(x_1^2+x_2^2))^\theta\big)$ is an $\mathscr{A}=(A_1, A_2)$-meromorphic function where $A_1=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $A_2=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ with the boundary value $(-4\pi^2/\Gamma(-\theta))\times(T_1, T_2)$, where

$$T_1=\sum_{j=0}^\infty\binom{\theta}{j}\int_0^\infty\varrho^{-\theta-1}\delta_{(-2j+\varrho,2j)}\,d\varrho,\qquad T_2=\sum_{j=0}^\infty\binom{\theta}{j}\int_0^\infty\varrho^{-\theta-1}\delta_{(2j,-2j+\varrho)}\,d\varrho.$$

3.2. *Functions of the form $f(-\sum_{j=1}^n a_j\ln x_j)$.* The method given in Section 3.1 cannot be applied to compute the Mellin transforms of functions g of the form $f(-\sum_{j=1}^n a_j\ln x_j)$, with $f\colon \mathbf{R}_+^1\to\mathbf{C}$, $a_j>0$ for $j=1,\ldots,n$ since the set $\Omega_t=\{x\in\mathbf{R}_+^n:\ -\sum_{j=1}^n a_j\ln x_j\leqslant t\}$ is not a bounded neighbourhood of zero in $\mathbf{R}_+^n$. In this case we can integrate over $B_t^+=B(0,t)\cap\mathbf{R}_+^n$ passing to polar coordinates (as was done in dimension $n=2$ in Example 12 of [12]). As it leads to laborious calculations we present only the final result for the function $f(r)=r^\theta$, $\operatorname{Re}\theta<0$:

$$\mathscr{M}\big(\chi_{B_1^+}\cdot(-\sum_{j=1}^n a_j\ln x_j)^\theta\big)(z)=\frac{1}{2^n\Gamma(-\theta)}\int_0^\infty\varrho^{-\theta-1}\frac{B\big((a_1\varrho-z_1)/2,\ldots,(a_n\varrho-z_n)/2\big)}{(a_1+\ldots+a_n)\varrho-z_1-\ldots-z_n}\,d\varrho.$$

One may compute the boundary value of this function by using the properties of the function B.

Here, we give another method by choosing the cut-off function $\chi^r=\chi_{J_r}$ instead of $\chi_{B_r^+}$. We divide the cube J_r into $n!$ disjoint (modulo sets of measure zero) subsets $Z_{\sigma,r}$, where $\sigma\in\pi(n)$ ($\pi(n)$ denotes the set of permutations).

PROPOSITION 7. *Let $A_j=(0, 1)$ with $0\in\mathbf{R}^{j-1}$, $1\in\mathbf{R}^{n-j+1}$ $(j=1,\ldots,n)$. Let $\sigma\in\pi(n)$ and*

$$A_\sigma=\begin{bmatrix}A_{\sigma(1)}\\ \vdots\\ A_{\sigma(n)}\end{bmatrix}=(a_{jl})_{j,l=1,\ldots,n}.$$

Set

$$Z_\sigma=\{x\in\mathbf{R}_+^n:\ x_j=\prod_{l=1}^n t_l^{a_{lj}},\ 0<t_l\leqslant 1,\ l=1,\ldots,n,\ j=1,\ldots,n\},$$

$$Z_{\sigma,r}=Z_\sigma\cap J_r\quad \text{for } r\leqslant 1.$$

If $\operatorname{Re}\theta<0$ then

$$\mathscr{M}\big(\chi_{Z_{\sigma,r}}\cdot(-\sum_{j=1}^n\ln x_j)^\theta\big)(z)=\frac{1}{\Gamma(-\theta)}\int_0^\infty\varrho^{-\theta-1}(-A_\sigma(z-\varrho))^{-1}r^{n\varrho-z_1-\ldots-z_n}\,d\varrho \tag{38}$$

for $z\in\mathbf{C}^n$ such that $(A_\sigma z)_j\notin[0,\infty)$, $j=1,\ldots,n$.

Moreover, $\mathscr{M}\big(\chi_{Z_{\sigma,r}}\cdot(-\sum_{j=1}^n\ln x_j)^\theta\big)$ is an A_σ-meromorphic function with the boundary value

$$\frac{(2\pi i)^n}{\Gamma(-\theta)}\int_0^\infty\varrho^{-\theta-1}\delta_{(\varrho,\ldots,\varrho)}\,d\varrho. \tag{39}$$

Proof. Let $\sigma=\mathrm{id}$. We change the variables $x_l=\prod_{j=1}^l y_j$ $(l=1,\ldots,n)$, $dx=(\prod_{j=1}^{n-1}y_j^{n-j})\,dy$ and apply (19) with $c=-n^{-1}\sum_{j=2}^n\ln y_j^{n-j+1}$. Set $y'=(y_2,\ldots,y_n)$, $z'=(z_2,\ldots,z_n)$. We obtain

$$\begin{aligned}\mathscr{M}\big(\chi_{Z_{\mathrm{id},r}}\cdot(-\sum_{j=1}^n\ln x_j)^\theta\big)(z)&=n^\theta\int_{(0,1]^{n-1}}\int_0^r\Big(-\ln y_1-n^{-1}\sum_{j=2}^n\ln y_j^{n-j+1}\Big)^\theta(y)^{-(A_{\mathrm{id}}z)-1}\,dy\\&=\frac{n^\theta}{\Gamma(-\theta)}\int_{(0,1]^{n-1}}\int_0^\infty\varrho^{-\theta-1}(y')^{-(A_{\mathrm{id}}(z-(\varrho/n)))'-1'}\frac{r^{\varrho-z_1-\ldots-z_n}}{\varrho-z_1-\ldots-z_n}\,d\varrho\,dy'.\end{aligned}$$

Now we change the order of integration to get the right hand side of (38) with $\sigma=\mathrm{id}$.

For an arbitrary permutation $\sigma\in\pi(n)$ the proof of (38) goes along the same lines. To compute the boundary value of the function given by (38) we observe that $\det A_\sigma=1$. So by (11) we have

$$b^{A_\sigma}\big((-A_\sigma(z-\varrho))^{-1}r^{n\varrho-z_1-\ldots-z_n}\big)=(2\pi i)^n\delta_{(\varrho,\ldots,\varrho)}\quad\text{for } \varrho\geqslant 0.$$

Thus, the boundary value of $\mathscr{M}\big(\chi_{Z_{\sigma,r}}\cdot(-\sum_{j=1}^n\ln x_j)^\theta\big)$ is given by (39).

Summing both sides of (38) over all permutations $\sigma \in \pi(n)$ we get

COROLLARY 5. *Let* $\operatorname{Re}\theta < 0$, $r < 1$. *Then*

$$(40)\qquad \mathcal{M}\Big(\chi^r\cdot\Big(-\sum_{j=1}^{n}\ln x_j\Big)^{\theta}\Big)(z) = \frac{1}{\Gamma(-\theta)}\int_0^{\infty} \varrho^{-\theta-1}\frac{r^{n\varrho-z_1-\ldots-z_n}}{(\varrho-z_1)\ldots(\varrho-z_n)}\,d\varrho$$

for $z\in\mathbf{C}^n$ *such that* $z_j\notin[0,\infty)$, $j=1,\ldots,n$. *Moreover*, $\mathcal{M}\big(\chi^r\cdot(-\sum_{j=1}^{n}\ln x_j)^{\theta}\big)$ *is an* Id-*meromorphic function with the boundary value given by* (39).

Now, assume that $\operatorname{Re}\theta < k$, $k\in\mathbf{N}$. Applying (20) instead of (19) in the proof of Proposition 7 and summing the result thus obtained over all permutations $\sigma\in\pi(n)$ we get

PROPOSITION 8. *Let* $\operatorname{Re}\theta < k$, $k\in\mathbf{N}_0$, $r<1$. *Then*

$$(41)\qquad \mathcal{M}\Big(\chi^r\cdot\Big(-\sum_{j=1}^{n}\ln x_j\Big)^{\theta}\Big)(z) = \frac{(-1)^k}{\Gamma(-\theta+k)}\int_0^{\infty}\varrho^{-\theta+k-1}\frac{d^k}{d\varrho^k}\Big(\frac{r^{n\varrho-z_1-\ldots-z_n}}{(\varrho-z_1)\ldots(\varrho-z_n)}\Big)d\varrho$$

for $z_j\notin[0,\infty)$, $j=1,\ldots,n$. *The boundary value of this function is equal to*

$$(42)\qquad \frac{(2\pi i)^n}{\Gamma(-\theta+k)}\int_0^{\infty}\varrho^{-\theta+k-1}\sum_{\substack{\alpha\in\mathbf{N}_0^n\\ |\alpha|=k}}\binom{k}{\alpha}\delta^{(\alpha)}_{(\varrho,\ldots,\varrho)}\,d\varrho$$

where $\displaystyle\binom{k}{\alpha} = \frac{k!}{\alpha_1!\ldots\alpha_n!}$.

By the change of variables $x_j^{a_j} = y_j$, where $a_j>0$ for $j=1,\ldots,n$, we get from (40)

COROLLARY 6. *Let* $\operatorname{Re}\theta<0$, $r<1$, $a_j>0$ *for* $j=1,\ldots,n$. *Then*

$$(43)\qquad \mathcal{M}\Big(\chi^r\cdot\Big(-\sum_{j=1}^{n}a_j\ln x_j\Big)^{\theta}\Big)(z) = \frac{1}{\Gamma(-\theta)}\int_0^{\infty}\varrho^{-\theta-1}\frac{r^{(a_1+\ldots+a_n)\varrho-z_1-\ldots-z_n}}{(a_1\varrho-z_1)\ldots(a_n\varrho-z_n)}\,d\varrho$$

for $z_j\notin[0,\infty)$, $j=1,\ldots,n$. *Moreover*, $\mathcal{M}\big(\chi^r\cdot(-\sum_{j=1}^{n}a_j\ln x_j)^{\theta}\big)$ *is an* Id-*meromorphic function with the boundary value*

$$(44)\qquad \frac{(2\pi i)^n}{\Gamma(-\theta)}\int_0^{\infty}\varrho^{-\theta-1}\delta_{(a_1\varrho,\ldots,a_n\varrho)}\,d\varrho.$$

PROPOSITION 9. *Let* $r\leqslant 1$. *Then*

$$(45)\qquad \mathcal{M}\Big(\chi^r\cdot\ln\Big(-\sum_{j=1}^{n}\ln x_j\Big)\Big)(z) = \frac{-\gamma r^{-z_1-\ldots-z_n}}{(-z_1)\ldots(-z_n)} + \int_0^{\infty}\ln\varrho\,\frac{d}{d\varrho}\Big(\frac{r^{n\varrho-z_1-\ldots-z_n}}{(\varrho-z_1)\ldots(\varrho-z_n)}\Big)d\varrho$$

for $z_j\notin[0,\infty)$, $j=1,\ldots,n$. *The boundary value of this function equals*

$$(46)\qquad (2\pi i)^n\Big(-\gamma\delta_{(0)} - \int_0^{\infty}\ln\varrho\sum_{\substack{\alpha\in\mathbf{N}_0^n\\ |\alpha|=1}}\delta^{(\alpha)}_{(\varrho,\ldots,\varrho)}\,d\varrho\Big).$$

Proof. We have

$$\ln\Big(-\sum_{j=1}^{n}\ln x_j\Big) = \frac{\partial}{\partial\theta}\Big(\Big(-\sum_{j=1}^{n}\ln x_j\Big)^{\theta}\Big)\Big|_{\theta=0}.$$

Let $r<1$. Using (41) with $k=1$ we derive

$$\mathcal{M}\Big(\chi^r\cdot\ln\Big(-\sum_{j=1}^{n}\ln x_j\Big)\Big)(z) = \int_0^{\infty}\frac{\partial}{\partial\theta}\Big(\frac{-\varrho^{-\theta}}{\Gamma(-\theta+1)}\Big)\Big|_{\theta=0}\frac{d}{d\varrho}\Big(\frac{r^{n\varrho-z_1-\ldots-z_n}}{(\varrho-z_1)\ldots(\varrho-z_n)}\Big)d\varrho = \int_0^{\infty}(\gamma+\ln\varrho)\frac{d}{d\varrho}\Big(\frac{r^{n\varrho-z_1-\ldots-z_n}}{(\varrho-z_1)\ldots(\varrho-z_n)}\Big)d\varrho.$$

Hence we get (45) for $r<1$. Next we observe that the integral in (45) is convergent also for $r=1$. From (45) we immediately deduce the boundary value (46).

Now we can write the generalized asymptotic expansion at zero of the function $\ln(-\sum_{j=1}^{n}\ln x_j)$.

COROLLARY 7. *For every* $p>0$ *there exists a Mellin distribution* $R\in M'_{(-\mathbf{p})}$, (Id; 0, $\mathbf{p}$)-*flat at zero such that*

$$(47)\qquad \ln\Big(-\sum_{j=1}^{n}\ln x_j\Big) = -\gamma + \Big(\sum_{j=1}^{n}\ln x_j\Big)\int_0^{p}\ln\varrho\, x_1^{\varrho}\ldots x_n^{\varrho}\,d\varrho + R(x)$$

for $x\in(0,1)^n$.

Remark 1. Functions of the form $f(-\sum_{j=1}^{n}a_j\ln x_j)$, where f is a differentiable function, are solutions of singular differential equations of the type $\sum_{j=1}^{n}b_j(x_j\partial/\partial x_j)U=0$. For instance if $n=2$ then

$$\Big(x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\Big)\big((-\ln x_1-\ln x_2)^{\theta}\big) = 0$$

in a neighbourhood of zero in $\mathbf{R}^2_+$.

4. The Mellin transform the fundamental solution of the two-dimensional singular Laplace operator and its boundary value. It is well known that the distribution $E(y_1,y_2) = (4\pi)^{-1}\ln(y_1^2+y_2^2)$ is a fundamental solution of the two-dimensional Laplace operator. Changing the variables $y_1=-\ln x_1$, $y_2=-\ln x_2$ we see that the distribution

$$\frac{1}{4\pi}F(x_1,x_2) = \frac{1}{4\pi}\ln(\ln^2x_1+\ln^2x_2)\in D'(\mathbf{R}^2_+)$$

is a fundamental solution of the singular Laplace operator, i.e.

$$\tilde{\Delta}\left(\frac{1}{4\pi}F\right) = \left(\left(x_1\frac{\partial}{\partial x_1}\right)^2 + \left(x_2\frac{\partial}{\partial x_2}\right)^2\right)\left(\frac{1}{4\pi}F\right) = \delta_{(1,1)}.$$

One of the goals of this section is to study the regularity at zero of solutions of the equation

$$(48)\qquad \tilde{\Delta}U = f, \quad \text{where } f \in M'(J_r).$$

To this end we localize F to a neighbourhood of zero in $\mathbf{R}^2_+$. First we recall some definitions ([5], [6], [8]).

For $\omega \in (\mathbf{R} \cup \{-\infty\})^n$ set

$$M_{[\omega]}(J_r) = \bigcap_{a>\omega} M_a(J_r),$$

$$\mathring{M}_{[\omega]}(J_r) = \left\{\varphi \in M_{[\omega]}(J_r)\colon \left.\frac{\partial^p \varphi(x)}{\partial x_j^p}\right|_{x_j=r_j} = 0 \text{ for } j = 1, \ldots, n,\ p \in \mathbf{N}_0\right\}.$$

By a *Mellin multiplier* we understand a function φ smooth on J_r such that multiplication by φ is a linear continuous operation from $M_{(\omega)}$ to $M_{(\omega)}$ for every $\omega \in \mathbf{R}^n$.

By duality multiplication by φ is continuous from $M'_{(\omega)}$ to $M'_{(\omega)}$. The space of Mellin multipliers coincides with the space $M_{[-1]}$ ([8], Th. 4).

THEOREM and DEFINITION ([5]). *Let $\omega_1, \omega_2 \in (\mathbf{R} \cup \{+\infty\})^n$, $\omega = \min(\omega_1, \omega_2)$, $r, t > 0$, $U \in M'_{(\omega_1)}(J_r)$, $V \in M'_{(\omega_2)}(J_t)$. Write $xy = (x_1y_1, \ldots, x_ny_n)$ for $x, y \in \mathbf{R}^n$. Let $W[\varphi] = U_x[V_y[\varphi(xy)]]$ for $\varphi \in M_{(\omega)}(J_{rt})$. Then W is a well defined distribution, $W \in M'_{(\omega)}(J_{rt})$. We call W the Mellin convolution of U and V and denote it by $U *_m V$.*

We also have

$$\mathscr{M}(U *_m V)(z) = \mathscr{M}U(z)\cdot\mathscr{M}V(z) \quad \text{for } \operatorname{Re} z < \omega.$$

THEOREM 7. *Let $\omega_1 \in (\mathbf{R} \cup \{+\infty\})^2$, $\omega = \min(\omega_1, 0)$, $t > 0$, $f \in M'_{(\omega_1)}(J_t)$. Let $\varphi(x_1, x_2) = \varphi_1(x_1)\varphi_1(x_2)$ with $\varphi_1 \in C^\infty(\mathbf{R}_+)$, $\varphi_1(x) = 1$ for $x \in (0, 1]$, $\varphi_1(x) = 0$ for $x \geqslant r$, $r > 1$. Then*

$$(49)\qquad \tilde{\Delta}\left(\frac{1}{4\pi}\varphi F *_m f\right) = f + g,$$

where $g \in \mathring{M}_{[-\omega-1]}(J_{rt})$.

It is natural to call the operator $(4\pi)^{-1}\varphi F *_m \cdot$ a parametrix of the singular Laplace operator.

Proof. Observe that $F \in C^\infty(\mathbf{R}^2_+ \setminus \{1, 1\})$, $\chi_{J_r}F \in M'_{(0)}(J_r)$. Since φ is a Mellin multiplier we have $\varphi F \in M'_{(0)}(J_r)$. By the linearity of $\tilde{\Delta}$ we get

$$(50)\qquad \tilde{\Delta}\left(\frac{1}{4\pi}\varphi F\right) = \delta_{(1,1)} + R \quad \text{with} \quad R = \tilde{\Delta}((\varphi - 1)F).$$

Observe that $R \in \mathring{M}_{[-1]}(J_r)$. Let $g = R *_m f$. By Corollary 3 of [5] and (50) we have

$$\tilde{\Delta}\left(\frac{1}{4\pi}\varphi F *_m f\right) = \tilde{\Delta}\left(\frac{1}{4\pi}\varphi F\right) *_m f = f + g.$$

By Corollary 4 of [5] it follows that $g \in \mathring{M}_{[-\omega-1]}(J_r)$.

Remark 2. Since the operator $\tilde{\Delta}$ is hypoelliptic on $\mathbf{R}^2_+$ there exists a function $U_1 \in C^\infty(\mathbf{R}^2_+)$ such that $\tilde{\Delta}U_1 = g$ (see [2]). Thus, the distribution $U = (4\pi)^{-1}\varphi F *_m f - U_1$ is a solution of the equation (48).

PROPOSITION 10. *$\mathscr{M}(\chi_{J_1}F)$ is the function defined by the formula*

$$(51)\qquad \mathscr{M}(\chi_{J_1}F)(z_1, z_2) = \frac{-2\gamma - \ln 2}{z_1 z_2} + \int_0^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho-i\varrho)(z_2-\varrho+i\varrho)} + \frac{1}{(z_1-\varrho+i\varrho)(z_2-\varrho-i\varrho)}\right)d\varrho$$

and holomorphic on the set

$$\mathbf{C}^2 \setminus \bigcup_{\varrho=0}^{\infty} (\{z_1 = \varrho + i\varrho \text{ or } z_2 = \varrho - i\varrho\} \cup \{z_1 = \varrho - i\varrho \text{ or } z_2 = \varrho + i\varrho\}).$$

Proof. Denote by $\chi_{0,1}$ (resp. $\chi_{1,\infty}$) the characteristic function of the set $\{0 < x_1 \leqslant 1, 0 < x_2 \leqslant x_1\}$ (resp. $\{0 < x_2 \leqslant 1, 0 < x_1 \leqslant x_2\}$). Changing the variables $x_1 = x$, $x_2 = xy$, $dx_1dx_2 = x\,dx\,dy$ and applying (28) with $c = b = -\frac{1}{2}\ln y$ we obtain for $\operatorname{Re} z_1 < 0$, $\operatorname{Re} z_2 < 0$

$$\begin{aligned}\mathscr{M}(\chi_{0,1}F)(z_1, z_2) &= \ln 2\int_0^1\int_0^1 x^{-z_1-z_2-1}y^{-z_2-1}\,dx\,dy \\ &\quad + \int_0^1\int_0^1 \ln((-\ln x - \tfrac{1}{2}\ln y)^2 + \tfrac{1}{4}\ln^2 y)x^{-z_1-z_2-1}\,dx\,y^{-z_2-1}\,dy \\ &= \frac{\ln 2 - 2\gamma}{(z_1+z_2)z_2} + 2\int_0^1\int_0^\infty \left\{\ln\alpha \frac{d}{d\alpha}\left(\frac{y^{\alpha/2}}{\alpha - z_1 - z_2}\right)\right. \\ &\quad \left. + \frac{1}{\alpha}\left[\sum_{j=1}^\infty \frac{(-1)^{j+1}}{(2j)!}\left(\frac{\alpha}{2}\ln y\right)^{2j}\right]\frac{y^{\alpha/2}}{\alpha - z_1 - z_2}\right\}d\alpha\, y^{-z_2-1}\,dy.\end{aligned}$$

Now we change the order of integration, put $\alpha/2 = \varrho$ and use the formulae

$$\int_0^1 (\ln y)^{2j} y^{\varrho - z_2 - 1}\,dy = (2j)!\left(\frac{-1}{z_2 - \varrho}\right)^{2j+1}$$

(see (24)) and

$$\sum_{j=1}^\infty (-1)^{j+1}\left(\frac{\varrho}{z_2-\varrho}\right)^{2j} = \frac{\varrho^2}{(z_2-\varrho-i\varrho)(z_2-\varrho+i\varrho)}$$

to get

$$\mathscr{M}(\chi_{0,1}F)(z_1, z_2) = \frac{-\ln 2 - 2\gamma}{(z_1+z_2)z_2} + 2\int_0^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(2\varrho - z_1 - z_2)(\varrho - z_2)}\right) d\varrho$$

$$+2\int_0^\infty \frac{\varrho}{(z_2-\varrho-i\varrho)(z_2-\varrho+i\varrho)(z_2-\varrho)(z_1+z_2-2\varrho)} d\varrho.$$

The absolute convergence of the integrals on the right hand side legitimates our calculations.

Analogously, we have

$$\mathscr{M}(\chi_{1,\infty}F)(z_1, z_2) = \frac{-\ln 2 - 2\gamma}{(z_1+z_2)z_1} + 2\int_0^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(2\varrho - z_1 - z_2)(\varrho - z_1)}\right) d\varrho$$

$$+2\int_0^\infty \frac{\varrho}{(z_1-\varrho-i\varrho)(z_1-\varrho+i\varrho)(z_1-\varrho)(z_1+z_2-2\varrho)} d\varrho.$$

Define

$$H(z_1, z_2, \varrho) = \frac{\varrho^2}{z_1+z_2-2\varrho} \sum_{k=1}^{2} \frac{1}{(z_k-\varrho-i\varrho)(z_k-\varrho+i\varrho)(z_k-\varrho)}.$$

Adding $\mathscr{M}(\chi_{0,1}F)$ to $\mathscr{M}(\chi_{1,\infty}F)$ and integrating by parts we find

$$\mathscr{M}(\chi_{J_1}F)(z_1, z_2) = \frac{-\ln 2 - 2\gamma}{z_1 z_2} + 2\int_0^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho)(z_2-\varrho)}\right) d\varrho$$

$$+2\int_0^\infty \frac{1}{\varrho} H(z_1, z_2, \varrho) d\varrho$$

$$= \frac{-\ln 2 - 2\gamma}{z_1 z_2} + 2\int_0^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho)(z_2-\varrho)}\right) d\varrho$$

$$-2\int_0^\infty \ln\varrho \frac{d}{d\varrho}(H(z_1, z_2, \varrho)) d\varrho$$

which is equal to the right hand side of (51), because for $k = 1, 2$

$$\frac{1}{(z_k-\varrho)(z_k-\varrho-i\varrho)(z_k-\varrho+i\varrho)} = \frac{1}{2\varrho^2}\left(\frac{2}{z_k-\varrho} - \frac{1}{z_k-\varrho-i\varrho} - \frac{1}{z_k-\varrho+i\varrho}\right).$$

Let

$$A_1 = \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}.$$

Define for $t > 0$

$$H_{1,t}(z_1, z_2) = \frac{-\ln 2 - 2\gamma}{z_1 z_2} + \int_0^t \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho-i\varrho)(z_2-\varrho+i\varrho)}\right) d\varrho,$$

$$H_{2,t}(z_1, z_2) = \int_0^t \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho+i\varrho)(z_2-\varrho-i\varrho)}\right) d\varrho.$$

By Example 3 we see that $H_{1,t}$ is A_1-meromorphic on $\mathbf{C}^2$ and

$$[H_{1,t}]_{A_1} = 4\pi^2(\ln 2 + 2\gamma)\delta_{(0,0)}$$

$$+4\pi^2 \int_0^t \ln\varrho\left((1+i)\delta^{(1,0)}_{(\varrho+i\varrho,\varrho-i\varrho)} + (1-i)\delta^{(0,1)}_{(\varrho+i\varrho,\varrho-i\varrho)}\right) d\varrho.$$

Analogously, $H_{2,t}$ is A_2-meromorphic and

$$[H_{2,t}]_{A_2} = 4\pi^2 \int_0^t \ln\varrho\left((1-i)\delta^{(1,0)}_{(\varrho-i\varrho,\varrho+i\varrho)} + (1+i)\delta^{(0,1)}_{(\varrho-i\varrho,\varrho+i\varrho)}\right) d\varrho.$$

Thus, we have

THEOREM 8. *For every $t > 0$ the boundary value (in the sense of Definition 5) of the function*

$$\mathscr{M}(\chi_{J_1}F) - \int_t^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho-i\varrho)(z_2-\varrho+i\varrho)}\right) d\varrho$$

$$-\int_t^\infty \ln\varrho \frac{d}{d\varrho}\left(\frac{1}{(z_1-\varrho+i\varrho)(z_2-\varrho-i\varrho)}\right) d\varrho$$

is equal to

$$(52)\qquad 4\pi^2(\ln 2 + 2\gamma)\delta_{(0,0)} + 4\pi^2 \int_0^t \ln\varrho\left((1+i)\delta^{(1,0)}_{(\varrho+i\varrho,\varrho-i\varrho)} + (1-i)\delta^{(0,1)}_{(\varrho+i\varrho,\varrho-i\varrho)}\right) d\varrho$$

$$+4\pi^2 \int_0^t \ln\varrho\left((1-i)\delta^{(1,0)}_{(\varrho-i\varrho,\varrho+i\varrho)} + (1-i)\delta^{(1,0)}_{(\varrho-i\varrho,\varrho+i\varrho)}\right) d\varrho.$$

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Received May 22, 1990
Revised version September 28, 1990 (2693)

STUDIA MATHEMATICA 99 (3) (1991)

The Mellin analytic functionals and the Laplace–Beltrami operator on the Minkowski half-plane

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Abstract. In this paper the theory of Fourier analytic functionals is developed. These functionals are generalizations of some Fourier hyperfunctions. Then the Mellin analytic functionals theory is developed and Paley–Wiener type theorems for Fourier and Mellin analytic functionals are proved. The Mellin transform for Mellin analytic functionals is defined. These notions are applied to solve the Laplace–Beltrami equation and to study its solution in the space of Mellin analytic functionals.

0. Introduction. In this paper we introduce the notion of a Mellin analytic functional and we develop a theory of such functionals with a view to applications in the analysis of singular differential operators such as, for instance, the Laplace–Beltrami operator on a hyperbolic space.

In Section 2 we define the space of Fourier analytic functionals which are related with some equivalence classes of holomorphic functions of exponential type. These functionals are generalizations of Fourier hyperfunctions whose defining functions are of infraexponential type (Kaneko [1], Kawai [2], Zharinov [6]). More general analytic functionals with noncompact carrier were considered in Zharinov [6], Sargos–Morimoto [5] and Park–Morimoto [3].

It is shown in Section 3 that the Fourier transformation and the inverse Fourier transformation operate on Fourier analytic functionals. We prove Paley–Wiener type theorems for the Fourier transform of Fourier analytic functionals in Section 4.

In Section 5 we introduce the spaces of Mellin analytic functionals by using the substitution $w = e^{-\zeta} = (e^{-\zeta_1}, \ldots, e^{-\zeta_n})$ in some Fourier analytic functionals. Mellin analytic functionals corresponding to Fourier hyperfunctions are called *Mellin hyperfunctions*.

Section 6 contains the definitions of the Mellin transform of Mellin analytic functionals by evaluating the functional on the functions $\varphi_z(w) = w^{-z-1}$. We

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 46F15.