[9] N. A. Friedman and M. Katz, *Additive functionals on L^p spaces*, Canad. J. Math. 18 (1966), 1264–1271.
[10] Yu. I. Gribanov, *On measurability of kernels of integral operators*, Izv. Vyssh. Uchebn. Zaved. Mat. 7 (1972), 31–34 (in Russian).
[11] P. R. Halmos, *Measure Theory*, Springer, New York 1974.
[12] M. A. Krasnosel'skiĭ, P. P. Zabreĭko, E. I. Pustyl'nik and P. E. Sobolevskiĭ, *Integral Operators in Spaces of Summable Functions*, Noordhoff, Leiden 1976 [First edition Moscow 1966].
[13] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland, Amsterdam 1971.
[14] J. M. Mazón and S. Segura de León, *Order bounded orthogonally additive operators*, Rev. Roumaine Math. Pures Appl. 35 (4) (1990), 329–353.
[15] V. J. Mizel, *Characterization of non-linear transformations possessing kernels*, Canad. J. Math. 22 (1970), 449–471.
[16] V. J. Mizel and K. Sundaresan, *Representation of additive and biadditive functionals*, Arch. Rational Mech. Anal. 30 (1968), 102–126.
[17] J. von Neumann, *Charakterisierung des Spektrums eines Integraloperators*, Actualités Sci. Indust. 229, Hermann, Paris 1935.
[18] B. de Pagter, *A note on integral operators*, Acta Sci. Math. (Szeged) 50 (1986), 225–230.
[19] A. V. Ponosov, *On the Nemytskiĭ conjecture*, Dokl. Akad. Nauk SSSR 289 (6) (1986), 1308–1311 (in Russian) [= Soviet Math. Dokl. 34 (1) (1987), 231–233].
[20] W. Schachermayer, *Integral operators on L^p spaces I*, Indiana Univ. Math. J. 30 (1981), 123–140.
[21] H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin 1974.
[22] A. R. Schep, *Kernel operators*, Indag. Math. 41 (1979), 39–53.
[23] S. Segura de León, *Representación integral de operadores no lineales: Operadores de Uryson*, Thesis, University of València, 1988.
[24] K. Sundaresan, *Additive functionals on Orlicz spaces*, Studia Math. 32 (1968), 270–276.
[25] I. Vrkoč, *The representation of Carathéodory operators*, Czechoslovak Math. J. 19 (1969), 99–109.
[26] B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, Wolters-Noordhoff, Groningen 1967 [First edition Moscow 1961].
[27] W. A. Woyczyński, *Additive functionals on Orlicz spaces*, Colloq. Math. 19 (1968), 319–326.
[28] A. C. Zaanen, *Integration*, North-Holland, Amsterdam 1967.
[29] —, *Riesz Spaces II*, North-Holland, Amsterdam 1983.

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
UNIVERSITAT DE VALÈNCIA
Dr. Moliner 50, 46100 Burjassot, València, Spain

Received June 12, 1989 (2573)

STUDIA MATHEMATICA 99 (3) (1991)

Inequalities relative to two-parameter Vilenkin–Fourier coefficients

by

FERENC WEISZ (Budapest)

Abstract. The inequality

$$(*)\qquad \Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(nm)^{p-2}|\hat f(n,m)|^p\Big)^{1/p}\leqslant C_p\|f\|_{H_p^-}\qquad (0<p\leqslant 2)$$

and its dual inequality are proved for two-parameter Vilenkin–Fourier coefficients and for two-parameter martingale Hardy spaces H_p^- defined by means of the L^p-norm of the conditional quadratic variation. The inequality $(*)$ is extended to bounded Vilenkin systems and monotone coefficients for all p. The converse of the last inequality is also true for all p. From this it follows easily that under the same conditions the two-parameter Vilenkin–Fourier series of an arbitrary L^p function $(p>1)$ converges a.e. to that function.

1. Introduction. Up to now inequality $(*)$ has been known for one-parameter systems only. The proof for $p=1$ is due to Hardy, and, for the trigonometric system, it can be found e.g. in Coifman–Weiss [9]. For the Walsh system it was proved first by Ladhawala [13] and for another proof see the book [22] written by Schipp, Wade, Simon and Pál. For Vilenkin systems it was proved by Fridli and Simon [11] but for another Hardy space. The inequality for $1<p\leqslant 2$ can be found in Edwards's book [10].

First we establish the results of two-parameter martingale theory that will be used later. Our proof of $(*)$ for $0<p\leqslant 1$ is based on the atomic description of H_p^- (see [27]) and for $1<p\leqslant 2$ it can be obtained by interpolation (see [24]).

In the next section a direct proof of the dual inequality to $(*)$ is given. The analogue to this inequality for the BMO space and for the one-parameter Walsh system can be found in [13] and in [22].

Next $(*)$ will be extended to bounded Vilenkin systems and monotone coefficients for all $p>2$ (for the exact conditions see (10) and (11)). This proof is based on the proof for one-parameter systems given by Móricz in [16]. Under the above-mentioned conditions the converse of the last inequality is also true similarly to [16]; moreover, it is proved that the supremum of the absolute

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 60G42; Secondary 42C10.

values of the rectangle partial sums of a Vilenkin series is in L^p if and only if the left side of (*) is finite ($0 < p < \infty$). From this it follows easily that under the above conditions the two-parameter Vilenkin–Fourier series of an arbitrary H_1^- or L^p function ($p > 1$) converges a.e. to that function.

2. Vilenkin orthonormal systems. In our paper $\Omega = [0, 1) \times [0, 1)$, $\mathcal{A}$ is the σ-algebra of Borel subsets of Ω and P is Lebesgue measure. Let $(p_n, n \in \mathbf{N})$ and $(q_n, n \in \mathbf{N})$ be two sequences of natural numbers whose terms are at least 2. Introduce the notations $P_0 = Q_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k, \quad Q_{n+1} := \prod_{k=0}^{n} q_k \quad (n \in \mathbf{N}).$$

Every $x \in [0, 1)$ can be uniquely written in the following way:

$$x = \sum_{k=0}^{\infty} x_k/P_{k+1}, \quad 0 \leqslant x_k < p_k, \quad x_k \in \mathbf{N}.$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp\frac{2\pi i x_n}{p_n}, \quad r'_n(y) := \exp\frac{2\pi i y_n}{q_n}$$

are called *generalized Rademacher functions.*

Let $\mathcal{A}_n$ and $\mathcal{A}'_m$ be the σ-algebras generated by $\{r_0, \ldots, r_{n-1}\}$ and $\{r'_0, \ldots, r'_{m-1}\}$, respectively, and let $\mathcal{F}_{n,m}$ be the σ-algebra generated by $\mathcal{A}_n \times \mathcal{A}'_m$, i.e. $\mathcal{F}_{n,m} = \sigma(\mathcal{A}_n \times \mathcal{A}'_m)$, $\mathcal{F}_{n,\infty} := \sigma(\bigcup_{k=0}^{\infty} \mathcal{F}_{n,k})$ and $\mathcal{F}_{\infty,m} := \sigma(\bigcup_{k=0}^{\infty} \mathcal{F}_{k,m})$. It is easy to see that

$$(1) \quad \mathcal{F}_{n,m} := \sigma\{[kP_n^{-1}, (k+1)P_n^{-1}) \times [lQ_m^{-1}, (l+1)Q_m^{-1}): \; 0 \leqslant k < P_n, \; 0 \leqslant l < Q_m\}.$$

The Kronecker product system of one-parameter Vilenkin systems (see [23]) is called a *two-parameter Vilenkin system* $(w_{n,m};\ n, m \in \mathbf{N})$, i.e.

$$w_{n,m}(x, y) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \prod_{l=0}^{\infty} r'_l(y)^{m_l}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $m = \sum_{l=0}^{\infty} m_l Q_l$, $0 \leqslant n_k < p_k$, $0 \leqslant m_l < q_l$ and $n_k, m_l \in \mathbf{N}$.

Denote by $E_{n,m}$ the conditional expectation operator relative to $\mathcal{F}_{n,m}$ ($n, m \in \mathbf{N} \cup \{\infty\}$). Instead of (complex) $L^p(\Omega, \mathcal{A}, P)$ we use the shorter notation L^p and finally, for $0 < p \leqslant \infty$ let

$$L_0^p := \{f \in L^p\colon\ E_{0,n} f = E_{n,0} f = 0, \ n \in \mathbf{N}\}.$$

For $f \in L_1$ we shall denote by

$$\hat{f}(n, m) := E(f\bar{w}_{n,m}) \quad (n, m \in \mathbf{N})$$

the (n, m)th Vilenkin–Fourier coefficient of f. Similarly to the one-parameter case (see e.g. [10]) a partial generalization of the Parseval formula and the Riesz–Fischer theorem, the so-called Hausdorff–Young theorem can be proved for two parameters as well.

THEOREM 1 (Hausdorff–Young). *Suppose that* $1 \leqslant p \leqslant 2$ *and* $1/p + 1/p' = 1$.

(i) *If* $f \in L_p$ *then*

$$\|\hat{f}\|_{l_{p'}} \leqslant \|f\|_p := (E|f|^p)^{1/p}.$$

(ii) *If* $a = (a_{n,m};\ n, m \in \mathbf{N}) \in l_p$ *then the sequence*

$$s_{n,m} = \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} a_{k,l} w_{k,l}$$

converges in L^p*-norm as* $\min(n, m) \to \infty$ *to a function* f*, for which*

$$\|f\|_{p'} \leqslant \|a\|_{l_p}$$

where $\hat{f} = (\hat{f}(n, m);\ n, m \in \mathbf{N})$ *and* l_p *denotes the space of those sequences of numbers* $a = (a_{n,m};\ n, m \in \mathbf{N})$ *for which* $\|a\|_{l_p} := (\sum_{n,m\in\mathbf{N}} |a_{n,m}|^p)^{1/p} < \infty$.

3. Martingales. It is easy to see that the sequence of σ-algebras $(\mathcal{F}_{n,m})$ above satisfies the requirement that is usual in martingale theory. Namely, $(\mathcal{F}_{n,m})$ is clearly nondecreasing, i.e. if $(k, l) \leqslant (n, m)$ then $\mathcal{F}_{k,l} \subset \mathcal{F}_{n,m}$ (where $(k, l) \leqslant (n, m)$ means that $k \leqslant n$ and $l \leqslant m$). Moreover, $\mathcal{A} = \sigma\{(\mathcal{F}_{n,m};\ n, m \in \mathbf{N})\}$ and the condition F_4 introduced by Cairoli and Walsh [7] is also satisfied: for an arbitrary pair $(n, m) \in \mathbf{N}^2 := \mathbf{N} \times \mathbf{N}$ the σ-algebras $\mathcal{F}_{n,\infty}$ and $\mathcal{F}_{\infty,m}$ are *conditionally independent* relative to $\mathcal{F}_{n,m}$. An integrable sequence $f = (f_{n,m};\ n, m \in \mathbf{N})$ is said to be a *martingale* if

(i) it is adapted (i.e. $f_{n,m}$ is $\mathcal{F}_{n,m}$ measurable for all $n, m \in \mathbf{N}$),

(ii) $E_{k,l} f_{n,m} = f_{k,l}$ for all $(k, l) \leqslant (n, m)$.

For simplicity we always suppose that for a martingale f we have $f_{n,0} = f_{0,n} = 0$ ($n \in \mathbf{N}$). Of course, the theorems that are to be proved later are true without this condition.

The following notations will be used for a martingale $f = (f_{n,m};\ n, m \in \mathbf{N})$:

$$d_{n,m} f := f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}, \qquad d_{n,0} f := d_{0,n} f := 0,$$

$$f^* := \sup_{n,m} |f_{n,m}|, \qquad S(f) := \Big(\sum_{n,m} |d_{n,m} f|^2\Big)^{1/2},$$

$$s(f) := \Big(\sum_{n,m} E_{n-1,m-1} |d_{n,m} f|^2\Big)^{1/2}.$$

We now introduce Hardy spaces for $0 < p \leqslant \infty$: denote by H_p, H_p^- and $\mathbf{H}_p$ the spaces of martingales for which

$$\|f\|_{H_p} := \|S(f)\|_p < \infty, \quad \|f\|_{H_p^-} := \|s(f)\|_p < \infty \quad \text{and} \quad \|f\|_{\mathbf{H}_p} := \|f^*\|_p < \infty,$$

respectively. In martingale theory it is well known that if $f \in \mathbf{H}_p$ or $f \in H_p$ then $f_{n,m}$ converges a.e. and in L^p-norm as $\min(n, m) \to \infty$ $(p \geq 1$, see [18]). Therefore, two of the Hardy spaces above can be identified with certain subspaces of L_0^p $(p \geq 1)$. Moreover, a sharper assertion can be shown:

THEOREM 2. (i) *For* $p > 1$ *one has* $\mathbf{H}_p \sim H_p \sim L_0^p$ *where* $\sim$ *denotes the equivalence of spaces and norms* (see [6], [14], [18]).

(ii) *If* (p_n) *and* (q_n) *are bounded* (*i.e.* $p_n = O(1)$ *and* $q_n = O(1)$) *then* $\mathbf{H}_p \sim H_p \sim H_p^-$ *for every* $0 < p < \infty$ (see [3], [4], [24], [27]).

If either (p_n) or (q_n) is unbounded then the H_p^- space is different from all the other spaces introduced above $(p \neq 2)$ though the following inequalities are true:

$$\|\cdot\|_{H_p} \leq C_p \|\cdot\|_{H_p^-} \qquad (0 < p \leq 2),$$

$$\|\cdot\|_{H_p^-} \leq C_p \|\cdot\|_{H_p} \qquad (2 \leq p < \infty).$$

These inequalities also hold for $\mathbf{H}_p$ instead of H_p (see [4], [24], [27]).

Let us extend the definition of Vilenkin–Fourier coefficients from L^1 functions to H_p^- martingales $(0 < p < \infty)$ with the help of the previous two inequalities:

$$\hat{f}(n, m) := \lim_{\min(k,l) \to \infty} E(f_{k,l} \bar{w}_{n,m}) \qquad (n, m \in \mathbf{N})$$

if $f = (f_{k,l};\, k, l \in \mathbf{N}) \in H_p^-$. It is easy to see that this limit exists for $0 < p < 1$ as well.

Let us introduce the concept of a stopping time analogously to [21]. A mapping ν which maps Ω into the set of subspaces of $\mathbf{N}^2 \cup \{\infty\}$ is said to be a *stopping time* relative to $(\mathcal{F}_{n,m})$ if the elements of $\nu(\omega)$ are incomparable (i.e. for distinct $(k, l), (n, m) \in \nu(\omega)$, neither $(k, l) \leq (n, m)$ nor $(n, m) \leq (k, l)$; of course, $(k, l) < \infty$ for all $k, l \in \mathbf{N}$) and if for every $(n, m) \in \mathbf{N}^2$

$$\{\omega \in \Omega\colon (n, m) \in \nu(\omega)\} =: \{(n, m) \in \nu\} \in \mathcal{F}_{n,m}.$$

We use the notation $(k, l) \ll (n, m)$ if $k < n$ and $l < m$. For $H \subset \mathbf{N}^2$ we write $H \ll (n, m)$ if there exists $(k, l) \in H$ such that $(k, l) \ll (n, m)$. So we immediately see that if ν is a stopping time then

$$\{\nu \not\ll (n, m)\} \in \mathcal{F}_{n-1,m-1} \qquad (n, m \in \mathbf{N}).$$

On the other hand, the converse of the previous assertion comes from the equality

$$\{(n, m) \in \nu\} = \{\nu \ll (n+1, m+1)\} \cap \{\nu \not\ll (n+1, m)\} \cap \{\nu \not\ll (n, m+1)\}.$$

As in the one-parameter case, we can define a *stopped martingale* $(f_{n,m}^{\nu})$ for a martingale f and a stopping time ν:

$$f_{n,m}^{\nu} := \sum_{i \leq n} \sum_{j \leq m} \chi(\{\nu \not\ll (i, j)\}) d_{i,j} f$$

where $\chi(A)$ denotes the characteristic function of a set A. Since $\{\nu \not\ll (i, j)\} \in \mathcal{F}_{i-1,j-1}$, $(f_{n,m}^{\nu};\, n, m \in \mathbf{N})$ is a martingale (see [27]).

The base of the following section will be the atomic description of H_p^- spaces. For this we first define an atom. A function $a \in L_0^q$ is said to be a *(p, q)-atom* if there exists a stopping time ν such that

(i) $a_{n,m} := E_{n,m} a = 0$ if $\nu \not\ll (n, m)$,

(ii) $\|a^*\|_q \leq P(\nu \neq \infty)^{1/q - 1/p}$.

Now an *atomic decomposition* of H_p^- martingales can be formulated:

THEOREM 3 [27]. *A martingale* $f = (f_{n,m};\, n, m \in \mathbf{N})$ *is an element of* H_p^- $(0 < p \leq 1)$ *iff there exist a sequence of* $(p, 2)$*-atoms* $(a_k, k \in \mathbf{N})$ *and a sequence of real numbers* $\mu = (\mu_k, k \in \mathbf{N})$ *such that for all* $n, m \in \mathbf{N}$

$$\sum_{k=0}^{\infty} \mu_k E_{n,m} a_k = f_{n,m} \quad \text{a.e. and} \quad \|\mu\|_{l_p} < \infty. \tag{2}$$

Moreover, $\|f\|_{H_p^-} \sim \inf \|\mu\|_{l_p}$ *where the infimum is taken over all decompositions* (2) *of* f.

4. Hardy type inequalities. The following theorem, which is the main result of this paper, can be found in [9] and in [10] for the Fourier coefficients of the one-parameter trigonometric system for $1 \leq p \leq 2$, furthermore, in [8], [11], [13] and in [22] for bounded one-parameter Vilenkin systems (i.e. $p_n = O(1)$) for $p = 1$. In [11] a similar inequality is proved for $p = 1$ for one-parameter unbounded Vilenkin systems; the Hardy space used there is different from the ones above. Moreover, it is proved there that for an unbounded Vilenkin system there exists a function $f \in \mathbf{H}_1$ such that

$$\sum_{k=1}^{\infty} |\hat{f}(k)|/k = \infty.$$

THEOREM 4. *For an arbitrary martingale* $f \in H_p^-$

$$\Big(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\hat{f}(n,m)|^p/(nm)^{2-p}\Big)^{1/p} \leq C_p \|f\|_{H_p^-} \qquad (0 \leq p \leq 2). \tag{$*$}$$

Proof. (i) First let $0 < p \leq 1$. From the proof of Theorem 3 in [27] it follows that there exists a decomposition (2) of $f \in H_p^-$ such that

$$\|\mu\|_{l_p} \leq C_p \|f\|_{H_p^-} \quad \text{and} \quad |\hat{f}(n, m)| \leq \sum_{k=0}^{\infty} |\mu_k| |\hat{a}_k(n, m)|.$$

Having this the only thing we have to prove is that for an arbitrary $(p, 2)$-atom a

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\hat{a}(n, m)|^p/(nm)^{2-p} \leq C_p. \tag{3}$$

If ν is the stopping time belonging to a fixed atom a then the support of a^* is obviously $F := \{\nu \neq \infty\}$. The rectangles in (1) are called the *atoms* of the σ-algebra $\mathscr{F}_{n,m}$. For the time being let m be fixed. To this m choose n such that there exists an atom $A \in \mathscr{F}_{n,m}$ for which $A \subset F$ but $B \cap F^c \neq \emptyset$ for every atom $B \in \mathscr{F}_{n-1,m}$ (F^c denotes the complement of F); denote this number by $N(m)$. If there is no such n then let $N(m) = \infty$. The sequence $(N(m))$ is obviously nonincreasing. Moreover, let

$$m_1 := \min\{m\colon N(m) < \infty\}, \qquad n_1 := N(m_1).$$

We define a sequence (n_k, m_k) recursively (if it does exist):

$$m_k := \min\{m\colon N(m) < n_{k-1}\}, \qquad n_k := N(m_k).$$

Since (n_k) is decreasing and (m_k) is increasing, we have only finitely many pairs (n_k, m_k); denote the number of these pairs by K. Let

$$G := \{(n_k, m_k)\colon 1 \leqslant k \leqslant K\}, \qquad H := \{(P_{n_k}, Q_{m_k})\colon 1 \leqslant k \leqslant K\}.$$

If $G \not\leqslant (n, m)$ then it follows from the construction that there is no atom $A \in \mathscr{F}_{n,m}$ such that $A \subset F$, consequently, for all $\omega \in \Omega$ we have $(n, m) \notin \nu(\omega)$. Thus for all ω

$$G \leqslant \nu(\omega).$$

If $G \not\leqslant (n, m)$ then for all ω one has $\nu(\omega) \not\leqslant (n, m)$. Consequently, using the definition of the atom we find that $a_{n,m}(\omega) = 0$ $(\omega \in \Omega)$ if $G \not\leqslant (n, m)$. Next, it is easy to show that $\hat{a}(n, m) = 0$ if $H \not\leqslant (n, m)$. So by Hölder's inequality

$$\text{(4)}\qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\hat{a}(n, m)|^p/(nm)^{2-p} = \sum_{H \leqslant (n,m)} |\hat{a}(n, m)|^p/(nm)^{2-p}$$
$$\leqslant \Big(\sum_{H \leqslant (n,m)} |\hat{a}(n, m)|^2\Big)^{p/2} \Big(\sum_{H \leqslant (n,m)} 1/(nm)^2\Big)^{(2-p)/2}.$$

It is obvious by the definition of the atom and from the Parseval inequality that

$$\text{(5)}\qquad \Big(\sum_{H \leqslant (n,m)} |\hat{a}(n, m)|^2\Big)^{p/2} \leqslant \|a\|_2^p \leqslant P(F)^{p/2-1}.$$

We shall show that

$$\text{(6)}\qquad \sum_{H \leqslant (n,m)}{}' 1/(nm)^2 \leqslant 2P(F).$$

Combining (4)–(6) yields (3), and the theorem follows.

Using the inequality

$$\sum_{k \geqslant n} 1/k^2 \leqslant \int_{n-1}^{\infty} (1/x^2)\,dx \leqslant 2/n \qquad (n \geqslant 2)$$

we get immediately

$$\text{(7)}\qquad \tfrac{1}{2} \sum_{H \leqslant (n,m)} 1/(nm)^2 \leqslant \sum_{k=1}^{K} Q_{m_k}^{-1} (P_{n_k}^{-1} - P_{n_{k-1}}^{-1}) =: |H|$$

where $P_{n_0}^{-1} := 0$. By the construction of the set H for $1 \leqslant k \leqslant K$ there exists an atom $A_k \in \mathscr{F}_{n_k,m_k}$ such that $A_k \subset F$. Let $A := \bigcup_{k=1}^{K} A_k$; then $A \subset F$. We shall show that

$$\text{(8)}\qquad |H| \leqslant P(A).$$

For $1 \leqslant k \leqslant K$ choose an atom $B_k \in \mathscr{F}_{n_k,m_k}$ such that the intersection of any B_{k_1} and B_{k_2} is nonempty. Then it can easily be shown that the Lebesgue measure of $B := \bigcup_{k=1}^{K} B_k$ is equal to $|H|$. Let $C := \bigcup_{k=1}^{K} C_k$ where $C_k \in \mathscr{F}_{n_k,m_k}$ is an atom. By induction on K we show that C has minimal area if and only if the intersection of any C_{k_1} and C_{k_2} is nonempty. For $k = 1$ or $k = 2$ this is trivial. Let $1 \leqslant l \leqslant K$ be the minimal index for which $C_{l+1} \cap C_l = \emptyset$. (If there is no such index then the intersection of any two sets C_k is nonempty.) It can be seen that

$$P\Big(C_l - \bigcup_{\substack{k=1\\k \neq l}}^{K} C_k\Big) > P\Big(B_l - \bigcup_{\substack{k=1\\k \neq l}}^{K} B_k\Big).$$

Nevertheless, by the induction hypothesis we have

$$P\Big(\bigcup_{\substack{k=1\\k \neq l}}^{K} C_k\Big) \geqslant P\Big(\bigcup_{\substack{k=1\\k \neq l}}^{K} B_k\Big),$$

consequently, $P(C) > P(B)$. Thus we have proved (8), and hence (6), so the proof of the theorem is complete for $0 < p \leqslant 1$.

(ii) Secondly, let $1 < p \leqslant 2$. Denote by $\mathbf{P}$ the set of positive natural numbers and introduce on $\mathbf{P}^2$ the measure $\mu(n, m) = 1/(n^2 m^2)$. If

$$Tf(n, m) = nm\hat{f}(n, m)$$

then it follows by the Parseval formula and by the previous theorem for $p = 1$ that both

$$T\colon L_0^2 \to L^2(\mathbf{P}^2, \mu) \quad \text{and} \quad T\colon H_1^- \to L^1(\mathbf{P}^2, \mu)$$

are bounded. (In contrast to the one-parameter case it is not true that T is of weak type (1, 1).) By a well known interpolation theorem (see [1]) the operator

$$T\colon (H_1^-, L_0^2)_{\theta,p} \to (L^1(\mathbf{P}^2, \mu), L^2(\mathbf{P}^2, \mu))_{\theta,p} \qquad (0 < \theta < 1)$$

is bounded as well. However, on the one hand,

$$(L^1(\mathbf{P}^2, \mu), L^2(\mathbf{P}^2, \mu))_{\theta,p} = L^p(\mathbf{P}^2, \mu)$$

(see [1]), and, on the other hand, we have proved in [24] that

$$(H_1^-, L_0^2)_{\theta,p} = H_p^-$$

where in both cases $0 < \theta < 1$ and $1/p = (1-\theta)+\theta/2$. This completes the proof of Theorem 4. ∎

Of course, it can similarly be proved that the theorem is true even if we do not suppose for a martingale $f \in H_p^-$ that $f_{n,0} = f_{0,n} = 0$.

In [27] we have introduced an atomic Hardy space $\mathscr{H}^{1,q}$ generated by some special atoms $(1 < q \leqslant \infty)$. Applying Theorem 1(i) we can show a theorem similar to Theorem 4 for $\mathscr{H}^{1,q}$ $(1 < q \leqslant \infty,\ p = 1)$:

THEOREM 5. *For an arbitrary martingale* $f \in \mathscr{H}^{1,q}$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\hat{f}(n, m)|/(nm) \leqslant \|f\|_{\mathscr{H}^{1,q}} \quad (1 < q \leqslant \infty).$$

5. Dual theorems. In [27] we have shown that the dual of H_p^- is $\Lambda_2(\alpha)$ $(0 < p \leqslant 1,\ \alpha = 1/p - 1)$ where $\Lambda_2(\alpha)$ denotes the space of functions $\varphi \in L_0^2$ for which

$$\|\varphi\|_{\Lambda_2(\alpha)} := \sup_{\nu} \{P(\nu \neq \infty)^{-1/2-\alpha} \|\varphi - \varphi^{\nu}\|_2\} < \infty \qquad (\alpha \geqslant 0)$$

(the supremum is taken over all stopping times). As in the one-parameter case, $\Lambda_2(0)$ is also denoted by BMO_2. Denote the set of sequences $\{(P_{n_k}, Q_{m_k}):\ 1 \leqslant k \leqslant K\}$ by $\mathscr{H}$ where $K \in \mathbf{N}$, (n_k) is decreasing and (m_k) is increasing. Now we give a direct proof for the dual inequality to Theorem 4 for $0 < p \leqslant 1$.

THEOREM 6. *If* $(b_{n,m};\ n, m \in \mathbf{P})$ *is a sequence of complex numbers such that*

$$M := \sup_{H \in \mathscr{H}} |H|^{-1/2-\alpha} \Big(\sum_{H \leqslant (n,m)} |b_{n,m}|^2\Big)^{1/2} < \infty$$

then there exists $\varphi \in \Lambda_2(\alpha)$ *for which* $\hat{\varphi}(n, m) = b_{n,m}$ $(n, m \in \mathbf{P})$ *and* $\|\varphi\|_{\Lambda_2(\alpha)} \leqslant M$. (The definition of $|H|$ is given in (7).)

Proof. It follows from the Riesz–Fischer theorem that there exists a function $\varphi \in L_0^2$ such that $\hat{\varphi}(n, m) = b_{n,m}$ $(n, m \in \mathbf{P})$. It is easy to show that for every stopping time ν

$$\|\varphi - \varphi^{\nu}\|_2^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E[\chi(\nu \ll (i, j)) |d_{i,j}\varphi|^2].$$

Let us use again the sets G and H constructed for the stopping time ν in the proof of Theorem 4. It is clear that

$$\|\varphi - \varphi^{\nu}\|_2^2 = \sum_{G \ll (i,j)} E[\chi(\nu \ll (i, j)) |d_{i,j}\varphi|^2] \leqslant \sum_{G \ll (i,j)} E(|d_{i,j}\varphi|^2).$$

If we express $d_{i,j}\varphi$ as a linear combination of the functions $w_{n,m}$ then we obtain

$$\|\varphi - \varphi^{\nu}\|_2^2 \leqslant \sum_{H \leqslant (n,m)} |b_{n,m}|^2.$$

As we have seen in the proof of Theorem 4, $|H| \leqslant P(\nu \neq \infty)$, consequently,

$$P(\nu \neq \infty)^{-1/2-\alpha} \|\varphi - \varphi^{\nu}\|_2 \leqslant |H|^{-1/2-\alpha} \Big(\sum_{H \leqslant (n,m)} |b_{n,m}|^2\Big)^{1/2},$$

i.e., $\|\varphi\|_{\Lambda_2(\alpha)} \leqslant M$, which shows Theorem 6. ∎

Let us give the dual theorem to Theorem 4 also for $1 < p \leqslant 2$. If $p_n = O(1)$ and $q_n = O(1)$ then by Theorem 2 one has $H_p^- \sim L_0^p$ $(p > 1)$ and it is well known that their dual space is L_0^q $(1/p + 1/q = 1)$.

THEOREM 7. *If* $p_n = O(1)$, $q_n = O(1)$, $2 \leqslant q < \infty$ *and* $(b_{n,m};\ n, m \in \mathbf{P})$ *is a sequence such that*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |b_{n,m}|^q/(nm)^{2-q} < \infty,$$

then the Vilenkin polynomials

$$f_{N,M} := \sum_{n=1}^{N} \sum_{m=1}^{M} b_{n,m} w_{n,m}$$

converge in L_q *as* $\min(N, M) \to \infty$ *to a function* f *satisfying* $\hat{f}(n, m) = b_{n,m}$ $(n, m \in \mathbf{P})$ *and*

$$\|f\|_q \leqslant C_q \Big(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |b_{n,m}|^q/(nm)^{2-q}\Big)^{1/q}.$$

This theorem can be shown similarly to the case of the one-parameter trigonometric system (see [10], p. 193).

We have also proved in [27] that the dual space of $\mathscr{H}^{1,q}$ is $\mathscr{BMO}_{q'}$ $(1/q + 1/q' = 1,\ 1 < q < \infty)$ where $\mathscr{BMO}_q$ denotes the space of functions $\varphi \in L_0^q$ for which

$$\|\varphi\|_{\mathscr{BMO}_q} := \sup_{n,m} \|(E_{n,m}|\varphi - E_{n,\infty}\varphi - E_{\infty,m}\varphi + E_{n,m}\varphi|^q)^{1/q}\|_{\infty} < \infty.$$

Now we prove a result for $\mathscr{BMO}_q$ spaces analogous to Theorem 6.

THEOREM 8. *If* $2 \leqslant q < \infty$ *and* $(b_{n,m};\ n, m \in \mathbf{P})$ *is a sequence of complex numbers such that*

$$M := \sup_{n,m} (P_n Q_m)^{1/q} \Big(\sum_{(P_n, Q_m) \leqslant (k,l)} |b_{k,l}|^{q'}\Big)^{1/q'} < \infty$$

then there exists $\varphi \in \mathscr{BMO}_q$ *for which* $\hat{\varphi}(k, l) = b_{k,l}$ $(k, l \in \mathbf{P})$ *and* $\|\varphi\|_{\mathscr{BMO}_q} \leqslant M$.

Proof. By Theorem 1(ii) there exists $\varphi \in L_0^q$ such that $\hat{\varphi}(k, l) = b_{k,l}$ $(k, l \in \mathbf{P})$. Let $n, m, l, j \in \mathbf{N}$ and introduce the Vilenkin polynomials

$$P_{l,j}^{(n,m)} := \sum_{k=0}^{P_n - 1} \sum_{i=0}^{Q_m - 1} \hat{\varphi}(lP_n + k, jQ_m + i) w_{k,i}.$$

Then

$$\varphi - E_{n,\infty}\varphi - E_{\infty,m}\varphi + E_{n,m}\varphi = \sum_{l=1}^{\infty}\sum_{j=1}^{\infty} P_{l,j}^{(n,m)} w_{lP_n, jQ_m}.$$

As $P_{l,j}^{(n,m)}$ is $\mathscr{F}_{n,m}$ measurable, applying again Theorem 1(ii), we get the following inequalities $(1/q+1/q'=1)$:

$$(E_{n,m}|\varphi - E_{n,\infty}\varphi - E_{\infty,m}\varphi + E_{n,m}\varphi|^q)^{1/q} = (E_{n,m}|\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} P_{l,j}^{(n,m)} w_{lP_n, jQ_m}|^q)^{1/q}$$

$$\leqslant (\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} |P_{l,j}^{(n,m)}|^{q'})^{1/q'} \leqslant ((P_n Q_m)^{q'-1} \sum_{(P_n,Q_m)\leqslant(k,l)} |\hat{\varphi}(k,l)|^{q'})^{1/q'},$$

i.e., $\|\varphi\|_{\mathscr{BMO}_q} \leqslant M$. This completes the proof of Theorem 8. ∎

This theorem can be found for one-parameter martingales in [13], [22] and another version for nonlinear martingales is in [25].

6. Converse inequalities. In this section we extend Theorem 4 under certain conditions to the case $p>2$. Moreover, we prove the converse inequality. In the sequel we suppose that

(9) $$p_n = O(1), \qquad q_n = O(1).$$

If $f=(f_{n,m};\, n, m\in \mathbf{N})$ is a martingale and $b_{k,l} := \hat{f}(k,l)$ then it is obvious that

$$f_{n,m} = \sum_{k=1}^{P_n-1}\sum_{l=1}^{Q_m-1} b_{k,l} w_{k,l}$$

and, conversely, an arbitrary sequence $(b_{k,l};\, k, l\in\mathbf{P})$ defines a martingale. From now on we consider only those martingales for which

(10) $$b_{k,l}\to 0 \quad \text{as } \max(k,l)\to\infty,$$

(11) $$\Re(b_{k,l}-b_{k+1,l}-b_{k,l+1}+b_{k+1,l+1})\geqslant 0,$$
$$\Im(b_{k,l}-b_{k+1,l}-b_{k,l+1}+b_{k+1,l+1})\geqslant 0 \quad (k,l\in\mathbf{P}).$$

It follows from (10) and (11) that the sequences $(\Re b_{k,l})$ and $(\Im b_{k,l})$ are decreasing. Now we extend Theorem 4.

THEOREM 9. *Under the conditions* (9) *and* (11) *suppose that* $f\in\mathbf{H}_p$. *Then*

$$(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\hat{f}(n,m)|^p/(nm)^{2-p})^{1/p} \leqslant C_p\|f\|_{\mathbf{H}_p} \quad (0<p<\infty).$$

(Note that by Theorem 2 one has $H_p^- \sim \mathbf{H}_p$ for $0<p<\infty$.)

We are not going to prove Theorem 9 because the proof is similar to Móricz's proof for the one-parameter Walsh system (see [16]).

We show a sharper assertion than the converse inequality. If

$$\sigma_{n,m} := \sum_{k=1}^{n-1}\sum_{l=1}^{m-1} b_{k,l} w_{k,l}$$

then the following holds:

THEOREM 10. *Under the conditions* (9)–(11)

$$\|\sup_{n,m}|\sigma_{n,m}|\|_p \leqslant C_p(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|b_{n,m}|^p/(nm)^{2-p})^{1/p} \quad (0<p<\infty).$$

Proof. This theorem for the one-parameter Walsh system was also proved by Móricz for $p>1$ (see [17]). Since

$$|D_n(x)| := |\sum_{k=0}^{n-1} w_{k,0}(x,y)| \leqslant 2/x \quad (x\in[0,1),\ n\in\mathbf{N})$$

is also true for a bounded Vilenkin system (see [11]), applying two-parameter Abel rearrangement, similarly to the proof in [16] we get

$$|\sigma_{n,m}(x,y)| \leqslant C\sum_{k=1}^{i}\sum_{l=1}^{j}|b_{k,l}| \quad \text{for } \frac{1}{i+1}\leqslant x<\frac{1}{i} \text{ and } \frac{1}{j+1}\leqslant y<\frac{1}{j}.$$

Therefore

$$\|\sup_{n,m}|\sigma_{n,m}|\|_p^p = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{1/(i+1)}^{1/i}\int_{1/(j+1)}^{1/j}(\sup_{n,m}|\sigma_{n,m}(x,y)|)^p\,dx\,dy$$

$$\leqslant C_p\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i^2j^2}(\sum_{k=1}^{i}\sum_{l=1}^{j}|b_{k,l}|)^p.$$

Now a slightly modified version of the Hardy inequality is needed:

LEMMA ([19], Theorem 8). *If* $r>1$, $0<p<\infty$ *and* $(d_n,\, n\in\mathbf{P})$ *is a non-negative, nonincreasing sequence then*

$$\sum_{n=1}^{\infty} n^{-r}(\sum_{k=1}^{n} d_k)^p \leqslant C_p\sum_{n=1}^{\infty} d_n^p n^{p-r}.$$

Applying twice the Lemma we obtain the inequality

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i^2j^2}(\sum_{k=1}^{i}\sum_{l=1}^{j}|b_{k,l}|)^p \leqslant C_p\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|b_{k,l}|^p(kl)^{p-2}.$$

This completes the proof of Theorem 10. ∎

Finally, it is easy to see that the following corollary holds.

COROLLARY 1. *If* (9)–(11) *are satisfied and* $0 < p < \infty$ *is fixed then the following conditions are equivalent*:

$$\sup_{n,m} |\sigma_{n,m}| \in L^p; \quad f \in \mathbf{H}_p; \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |b_{n,m}|^p/(nm)^{2-p} < \infty.$$

Therefore

$$\|\sup_{n,m} |\sigma_{n,m}|\|_p \leqslant C_p \|f\|_{\mathbf{H}_p} \quad (0 < p < \infty). \tag{12}$$

As any martingale $f \in \mathbf{H}_p$ $(p \geqslant 1)$ can be identified with an L^1 function, the $\sigma_{n,m}$ are partial sums of the Vilenkin–Fourier series of the function corresponding to f. Vilenkin polynomials are dense in $\mathbf{H}_p$, consequently, by (12) and by Theorem 2 of Chapter 3.1 in [22], Corollary 2 follows immediately:

COROLLARY 2. *Let* $p_n = O(1)$ *and* $q_n = O(1)$. *If* $f \in L^p$ $(p > 1)$ *or* $f \in \mathbf{H}_1$ *such that* (10) *and* (11) *are satisfied then* $\sigma_{n,m} f \to f$ *a.e. and also in* L^p*-norm* $(p \geqslant 1)$ *as* $\min(n, m) \to \infty$.

Note that the fact that $\sigma_{n,m}$ converges a.e. has already been proved under the conditions (9)–(11) only (see [17]).

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin 1976.

[2] A. Bernard, *Espaces H^1 de martingales à deux indices. Dualité avec les martingales de type* «BMO», Bull. Sci. Math. 103 (1979), 297–303.

[3] J. Brossard, *Comparaison des "normes" L_p du processus croissant et de la variable maximale pour les martingales régulières à deux indices. Théorème local correspondant*, Ann. Probab. 8 (1980), 1183–1188.

[4] —, *Régularité des martingales à deux indices et inégalités de normes*, in: Processus Aléatoires à Deux Indices, Lecture Notes in Math. 863, Springer, Berlin 1981, 91–121.

[5] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1 (1973), 19–42.

[6] R. Cairoli, *Une inégalité pour martingales à indices multiples et ses applications*, in: Séminaire de Probabilités IV, Lecture Notes in Math. 124, Springer, Berlin 1970, 1–27.

[7] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. 134 (1975), 111–183.

[8] J. A. Chao, *Hardy spaces on regular martingales*, in: Martingale Theory in Harmonic Analysis and Banach Spaces, Lecture Notes in Math. 939, Springer, Berlin 1982, 18–28.

[9] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[10] R. E. Edwards, *Fourier Series. A Modern Introduction*, Vol. 2, Springer, Berlin 1982.

[11] S. Fridli and P. Simon, *On the Dirichlet kernels and a Hardy space with respect to the Vilenkin system*, Acta Math. Hungar. 45 (1985), 223–234.

[12] A. M. Garsia, *Martingale Inequalities. Seminar Notes on Recent Progress*, Math. Lecture Note Ser., Benjamin, New York 1973.

[13] N. R. Ladhawala, *Absolute summability of Walsh–Fourier series*, Pacific J. Math. 65 (1976), 103–108.

[14] C. Métraux, *Quelques inégalités pour martingales à paramètre bidimensionnel*, in: Séminaire de Probabilités XII, Lecture Notes in Math. 649, Springer, Berlin 1978, 170–179.

[15] F. Móricz, *On the maximum of the rectangular partial sums of double trigonometric series with non-negative coefficients*, Anal. Math. 15 (1989), 283–290.

[16] —, *On Walsh series with coefficients tending monotonically to zero*, Acta Math. Acad. Sci. Hungar. 38 (1981), 183–189.

[17] F. Móricz, F. Schipp and W. R. Wade, *On the integrability of double Walsh series with special coefficients*, Michigan Math. J. 37 (1990), 191–201.

[18] J. Neveu, *Discrete-Parameter Martingales*, North-Holland, 1975.

[19] N. M. Rivière and Y. Sagher, *Interpolation between L^∞ and H^1, the real method*, J. Funct. Anal. 14 (1973), 401–409.

[20] F. Schipp, *Multiple Walsh analysis*, in: Theory and Applications of Gibbs Derivatives, Proc. Kupari–Dubrovnik, P. L. Butzer and R. S. Stanković (eds.), Mathematical Institute, Beograd 1989, 73–90.

[21] —, *On Carleson's method*, in: Fourier Anal. and Approximation Theory, Proc. Budapest 1976, Colloq. Math. Soc. J. Bolyai 19, Vol. II, North-Holland, 1978, 679–695.

[22] F. Schipp, W. R. Wade, P. Simon and J. Pál, *Walsh Series. An Introduction to Dyadic Harmonic Analysis*, Akadémiai Kiadó, 1990.

[23] N. Ja. Vilenkin, *On a class of complete orthonormal systems*, Izv. Akad. Nauk SSSR Ser. Mat. 11 (1947), 363–400.

[24] F. Weisz, *Interpolation between two-parameter martingale Hardy spaces, the real method*, Bull. Sci. Math. 115 (1991), to appear.

[25] —, *Martingale Hardy spaces, BMO and VMO spaces with nonlinearly ordered stochastic basis*, Anal. Math. 16 (1990), 227–239.

[26] —, *Martingale Hardy spaces for $0 < p \leqslant 1$*, Probab. Theory Related Fields 84 (1990), 361–376.

[27] —, *On duality problems of two-parameter martingale Hardy spaces*, Bull. Sci. Math. 114 (1990), 395–410.

DEPARTMENT OF NUMERICAL ANALYSIS
EÖTVÖS L. UNIVERSITY
Bogdánfy u. 10/B, H-1117 Budapest, Hungary

Received April 6, 1990 (2671)