Inequalities relative to two-parameter Vilenkin–Fourier coefficients

by

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Abstract. The inequality

\[ \left( \sum_{n=1}^{n} \sum_{m=1}^{m} (nm)^{p-1} |f(n, m)|^{q} \right)^{1/q} \leq C_{p} \| f \|_{L_{p}} \quad (0 < p \leq 2) \]

and its dual inequality are proved for two-parameter Vilenkin–Fourier coefficients and for two-parameter martingale Hardy spaces \( H_{p}^{a} \) defined by means of the \( L_{p} \)-norm of the conditional quadratic variation. The inequality \( (*) \) is extended to bounded Vilenkin systems and monotone coefficients for all \( p \). The converse of the last inequality is also true for all \( p \). From this it follows easily that under the same conditions the two-parameter Vilenkin–Fourier series of an arbitrary \( L_{p} \) function \( (p > 1) \) converges a.e. to that function.

1. Introduction. Up to now inequality \( (*) \) has been known for one-parameter systems only. The proof for \( p = 1 \) is due to Hardy, and, for the trigonometric system, it can be found e.g. in Coifman–Weiss [9]. For the Walsh system it was proved first by Laduwawala [13] and for another proof see the book [22] written by Schipp, Wade, Simon and Pál. For Vilenkin systems it was proved by Fridli and Simon [11] but for another Hardy space. The inequality for \( 1 < p \leq 2 \) can be found in Edwards’s book [10].

First we establish the results of two-parameter martingale theory that will be used later. Our proof of \( (*) \) for \( 0 < p \leq 1 \) is based on the atomic description of \( H_{p}^{a} \) (see [27]) and for \( 1 < p \leq 2 \) it can be obtained by interpolation (see [24]).

In the next section a direct proof of the dual inequality to \( (*) \) is given. The analogue to this inequality for the BMO space and for the one-parameter Walsh system can be found in [13] and in [22].

Next \( (*) \) will be extended to bounded Vilenkin systems and monotone coefficients for all \( p > 2 \) (for the exact conditions see [10] and [11]). This proof is based on the proof for one-parameter systems given by Móricz in [16]. Under the above-mentioned conditions the converse of the last inequality is also true similarly to [16]; moreover, it is proved that the supremum of the absolute
values of the rectangle partial sums of a Vilenkin series is in $L^p$ if and only if the left side of (**) is finite $(0 < p < \infty)$. From this it follows easily that under the above conditions the two-parameter Vilenkin–Fourier series of an arbitrary $H^*_1$ or $L^p$ function $(p > 1)$ converges a.e. to that function.

2. Vilenkin orthonormal systems. In our paper $\Omega = [0, 1) \times [0, 1)$, $\mathcal{A}$ is the $\sigma$-algebra of Borel subsets of $\Omega$ and $P$ is Lebesgue measure. Let $(p_n, n \in \mathbb{N})$ and $(q_n, n \in \mathbb{N})$ be two sequences of natural numbers whose terms are at least 2. Introduce the notations $P_0 = Q_0 = 1$ and

$$P_{k+1} := \prod_{n=0}^{k} p_n, \quad Q_{k+1} := \prod_{n=0}^{k} q_n \quad (n \in \mathbb{N}).$$

Every $x \in [0, 1)$ can be uniquely written in the following way:

$$x = \sum_{k=0}^{\infty} x_k/p_{k+1}, \quad 0 \leq x_k < p_k, \quad x_k \in \mathbb{N}.$$

If there are two different forms, choose the one for which $\lim_{k \to \infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi i x}{p_n}, \quad r_n(y) := \exp \frac{2\pi i y}{q_n}$$

are called generalized Rademacher functions.

Let $\mathcal{A}_n$ and $\mathcal{A}_m$ be the $\sigma$-algebras generated by $\{r_0, \ldots, r_{n-1}\}$ and $\{r_0, \ldots, r_{m-1}\}$, respectively, and let $\mathcal{F}_{n,m}$ be the $\sigma$-algebra generated by $\mathcal{A}_n \times \mathcal{A}_m$. Let $\mathcal{G}_{n,m} := \mathcal{A}(\bigcup_{n=0}^{\infty} \mathcal{F}_{n,m})$.

It is easy to see that

$$\mathcal{F}_{n,m} := \sigma([kP_{k+1}^{1}, (k+1)P_{k+1}^{1}) \times [lQ_{l+1}^{1}, (l+1)Q_{l+1}^{1}]:$$

$$0 \leq k < P_m, 0 \leq l < Q_n.$$

The Kronecker product system of one-parameter Vilenkinsystems (see [23]) is called a two-parameter Vilenkin system $(w_{n,m}; n, m \in \mathbb{N})$.

$$w_{n,m}(x, y) := \prod_{k=0}^{n} r_k(x)^n \prod_{l=0}^{m} r_l(y)^m$$

where $n = \sum_{k=0}^{n} m_k p_k$, $m = \sum_{k=0}^{m} m_k q_k$, $0 \leq m_k < p_k$, $0 \leq n_k < q_k$, and $n_k, m_k \in \mathbb{N}$.

Denote by $E_{n,m}$ the conditional expectation operator relative to $\mathcal{G}_{n,m}$ $(n, m \in \mathbb{N})$. Instead of (complex) $L^p(\Omega, \mathcal{A}, P)$ we use the shorter notation $L^p$ and finally, for $0 < p < \infty$ let

$$L^p := \{f \in L^p: E_{0,n}f = E_{n,0}f = 0, n \in \mathbb{N}\}.$$

For $f \in L^1$ we shall denote by

$$\mathfrak{f}(n, m) := E(\mathfrak{f} w_{n,m}) \quad (n, m \in \mathbb{N})$$

the $(n, m)$th Vilenkin–Fourier coefficient of $f$. Similarly to the one-parameter case (see e.g. [10]) a partial generalization of the Parseval formula and the Riesz–Fischer theorem, the so-called Hausdorff–Young theorem can be proved for two parameters as well.

Theorem 1 (Hausdorff–Young). Suppose that $1 \leq p \leq 2$ and $1/p + 1/p' = 1$.

1. If $f \in L^p$, then

$$\|\mathfrak{f}\|_{L^p} \leq \|f\|_p := (\mathcal{E}(|f|^p))^{1/p'}.$$

2. If $a = (a_{n,m}; n, m \in \mathbb{N}) \in l^p$, then the sequence

$$a_{n,m} \sum_{k=0}^{\infty} a_{k,l} w_{k,l}$$

converges in $L^p$-norm as $\min(m, n) \to \infty$ to a function $f$, for which

$$\|f\|_p \leq \|a\|_p$$

where $f = (f(n, m); n, m \in \mathbb{N})$ and $l_p$ denotes the space of those sequences of numbers $a = (a_{n,m}; n, m \in \mathbb{N})$ for which $\|a\|_p := (\sum_{n,m} |a_{n,m}|^p)^{1/p} < \infty$.

3. Martingales. It is easy to see that the sequence of $\sigma$-algebras $(\mathcal{G}_{n,m})$ above satisfies the requirement that is usual in martingale theory. Namely, $(\mathcal{G}_{n,m})$ is clearly nondecreasing, i.e. if $(k, l) \leq (n, m)$ then $\mathcal{G}_{k,l} \subset \mathcal{G}_{n,m}$ (where $(k, l) \leq (n, m)$ means that $k \leq n$ and $l \leq m$). Moreover, $\mathcal{A} = \sigma(\mathcal{G}_{n,m}; n, m \in \mathbb{N})$ and the condition $F_2$ introduced by Cairoli and Walsh [7] is also satisfied: for an arbitrary pair $(n, m) \in \mathbb{N}^2 := \mathbb{N} \times \mathbb{N}$ the $\sigma$-algebras $\mathcal{F}_{n,m}$ and $\mathcal{F}_{n,m}$ are conditionally independent relative to $\mathcal{G}_{n,m}$. An integrable sequence $f = (f_{n,m}; n, m \in \mathbb{N})$ is said to be a martingale if

(i) it is adapted (i.e. $f_{n,m}$ is $\mathcal{G}_{n,m}$ measurable for all $n, m \in \mathbb{N}$),

(ii) $E_{k,l} f_{n,m} = f_{k,l}$ for all $(k, l) \leq (n, m)$.

For simplicity we always suppose that for a martingale $f$ we have $f_{0,0} = f_{0,0} = 0$ $(n \in \mathbb{N})$. Of course, the theorems that are to be proved later are true without this condition.

The following notations will be used for a martingale $f = (f_{n,m}; n, m \in \mathbb{N})$:

$$d_{n,m} f := f_{n,m} - f_{n-1,m} - f_{n-1,m-1} - f_{n-1,m-1}, \quad d_{0,0} f := 0,$$

$$f^* := \sup_{n,m} |f_{n,m}|, \quad \mathcal{S}(f) := \left(\sum_{n,m} |d_{n,m} f|^2\right)^{1/2},$$

$$s(f) := \left(\sum_{n,m} E_{n-1,m-1} |d_{n,m} f|^2\right)^{1/2}.$$

We now introduce Hardy spaces for $0 < p < \infty$ denote by $H^p_+$ and $H^p_-$ the spaces of martingales for which

$$\|\mathfrak{f}\|_{H^p_+} := \|s(f)\|_p < \infty, \quad \|f\|_{H^p_-} := \|s(f)\|_p < \infty$$

and $\|f\|_{H^p} := \|f^*\|_p < \infty$. 

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respectively. In martingale theory it is well known that if \( f \in H_p \) or \( f \in H_p \) then \( f_{m,n} \) converges a.e. and in \( L^p \)-norm as \( \min(n,m) \to \infty \) (\( p \geq 1 \), see [18]). Therefore, two of the Hardy spaces above can be identified with certain subspaces of \( L^p_k \) (\( p \geq 1 \)). Moreover, a sharper assertion can be shown:

**Theorem 2.** (i) For \( p \geq 1 \) one has \( H_p \sim I_0 \) where \( \sim \) denotes the equivalence of spaces and norms (see [6], [14], [18]).

(ii) \( \{ p_1 \} \) and \( \{ q_1 \} \) are bounded (i.e. \( p_1 = O(1) \) and \( q_1 = O(1) \)) then \( H_p \sim H_p \) for every \( 0 < p < \infty \) (see [3], [4], [24], [27]).

If either \( \{ p_1 \} \) or \( \{ q_1 \} \) is unbounded then the \( H_p \) space is different from all the other spaces introduced above \( (p \neq 2) \) though the following inequalities are true:

\[
1 \leq C_p \| f \|_{H_p} \leq 2 \quad (0 \leq p \leq 2),
\]

\[
1 \leq C_p \| f \|_{H_p} \leq 2 \quad (2 < p < \infty).
\]

These inequalities also hold for \( H_p \) instead of \( H_p \) (see [4], [24], [27]).

Let us extend the definition of Vilenkin-Fourier coefficients from \( L^1 \) functions to \( H_p \) martingales \( (0 < p < \infty) \) with the help of the previous two inequalities:

\[
f(n, m) := \lim_{\min(k,l) \to \infty} E(f_{k,l}, \hat{w}_{n,m}) \quad (n, m \in \mathbb{N})
\]

if \( f = (f_{k,l}; k, l \in \mathbb{N}) \in H_p \). It is easy to see that this limit exists for \( 0 < p < 1 \) as well.

Let us introduce the concept of a stopping time analogously to [21]. A mapping \( v \) which maps \( \Omega \) into the set of subspaces of \( \mathbb{N}^2 \cup \{ \infty \} \) is said to be a stopping time relative to \( (\mathcal{F}_n)_{n} \) if the elements of \( v(\omega) \) are incomparable (i.e. for distinct \( (k, l) \), \( (n, m) \in \mathbb{N}^2 \)), neither \( (k, l) \leq (n, m) \) nor \( (n, m) \leq (k, l) \); of course, \( (k, l) \leq \infty \) for all \( (k, l) \in \mathbb{N}^2 \) and if for every \( (n, m) \in \mathbb{N}^2 \)

\[\{\omega \in \Omega : (n, m) \in v(\omega)\} = \{(n, m) \in v(\omega) \} \in \mathcal{F}_n_{n,m} \cdot\]

We use the notation \( (k, l) \leq (n, m) \) if \( k < n \) and \( l > m \). For \( H \subseteq \mathbb{N}^2 \) we write \( H \leq (n, m) \) if there exists \( (k, l) \in H \) such that \( (k, l) < (n, m) \). So we immediately see that if \( v \) is a stopping time then

\[\{v \leq (n, m)\} \in \mathcal{F}_{n-1,m-1} \quad (n, m \in \mathbb{N}).\]

On the other hand, the converse of the previous assertion comes from the equality

\[\{v \leq (n + 1, m + 1)\} \cap \{v \leq (n + 1, m)\} \cap \{v \leq (n, m + 1)\} .\]

As in the one-parameter case, we can define a stopped martingale \( f_{n,m} \) for a martingale \( f \) and a stopping time \( v \):

\[f_{n,m} := \sum_{l \geq n} \sum_{m \geq l} \chi(v \leq (l, m)) d_l f .\]

where \( \chi(A) \) denotes the characteristic function of a set \( A \). Since \( \{v \leq (i, j)\} \in \mathcal{F}_{i-1,j-1} \), \( (f_{n,m}; n, m \in \mathbb{N}) \) is a martingale (see [27]).

The base of the following section will be the atomic description of \( H_p \) spaces. For this we first define an atom. A function \( a \in L^2 \) is said to be a \((p, q)\)-atom if there exists a stopping time \( v \) such that

(i) \( a_{n,m} := E_{n,m} a = 0 \) if \( v \neq (n, m) \),

(ii) \( \|a\|^p \leq P(v = \infty) \leq 1 \).

Now an atomic decomposition of \( H_p \) martingales can be formulated:

**Theorem 3.** [27]. A martingale \( f = (f_{n,m}; n, m \in \mathbb{N}) \) is an element of \( H_p \) \((0 < p \leq 1)\) if there exists a sequence of \((p, 2)\)-atoms \((a_k, k \in \mathbb{N})\) and a sequence of real numbers \( \mu_k = \mu_k (k \in \mathbb{N}) \) such that for all \( n, m \in \mathbb{N}\)

\[
\sum_{k=0}^\infty \mu_k E_{n,m} a_k = f_{n,m} \quad a.e. \quad \text{and} \quad \|a\| \leq \infty .
\]

Moreover, \( \|f\|_{H_p} \sim \inf \|a\|_{L^p} \) where the infimum is taken over all decompositions (2) of \( f \).

4. Hardy type inequalities. The following theorem, which is the main result of this paper, can be found in [9] and in [10] for the Fourier coefficients of the one-parameter trigonometric system for \( 1 \leq p \leq 2 \), furthermore, in [8], [11], [13] and in [22] for bounded one-parameter Vilenkin systems (i.e. \( p_1 = O(1) \)) for \( p = 1 \). In [11] a similar inequality is proved for \( p = 1 \) for one-parameter unbounded Vilenkin systems; the Hardy space used there is different from the ones above. Moreover, it is proved there that for an unbounded Vilenkin system there exists a function \( f \in H_1 \) such that

\[
\sum_{k=1}^\infty \|f\|_{H_p}^{k} / k = \infty .
\]

**Theorem 4.** For an arbitrary martingale \( f \in H_p \)

\[
(\sum_{n=1}^\infty \sum_{m=1}^\infty |f(n, m)|^p (nm)^{2-p} \leq C_p \|f\|_{H_p}^{p} \quad (0 \leq p \leq 2) .
\]

**Proof.** (i) First let \( 0 < p \leq 2 \). From the proof of Theorem 3 in [27] it follows that there exists a decomposition (2) of \( f \in H_p \) such that

\[\|a\|_{L^p} \leq C_p \|f\|_{H_p}^{p} \quad \text{and} \quad |f(n, m)| \leq \sum_{k=0}^\infty |\mu_k| |a_k(n, m)| .
\]

Having this the only thing we have to prove is that for an arbitrary \((p, 2)\)-atom \( a \)

\[
(\sum_{n=1}^\infty \sum_{m=1}^\infty |\mu_k| |a_k(n, m)| |(nm)^{2-p} \leq C_p .
\]
If \( v \) is the stopping time belonging to a fixed atom \( a \) then the support of \( a^* \) is obviously \( F := \{ v \neq \infty \} \). The rectangles in (1) are called the \( a_0 \)-atoms of the \( \sigma \)-algebra \( \mathcal{F}_{a_0} \). For the time being let \( m \) be fixed. To this \( m \) choose \( n \) such that there exists an atom \( A \in \mathcal{F}_{a_0} \) for which \( A \subset F \) but \( B \cap F^c \neq \emptyset \) for every atom \( B \in \mathcal{F}_{a_0 - m} \) \((F^c \) denotes the complement of \( F)\); denote this number by \( N(m) \). If there is no such \( n \) then let \( N(m) = \infty \). The sequence \( (N(m)) \) is obviously nonincreasing. Moreover, let
\[
 m_1 := \min \{ m : N(m) < \infty \}, \quad n_1 := N(m_1).
\]
We define a sequence \((n_k, m_k)\) recursively (if it does exist):
\[
 m_k := \min \{ m : N(m) < n_{k-1} \}, \quad n_k := N(m_k).
\]
Since \((n_k)\) is decreasing and \((m_k)\) is increasing, we have only finitely many pairs \((n_k, m_k)\); denote the number of these pairs by \( K \). Let
\[
 G := \{(n_k, m_k) : 1 \leq k \leq K \}, \quad H := \{(P_{m_k}, Q_{m_k}) : 1 \leq k \leq K \}.
\]
If \( G \not\subset (n, m) \) then it follows from the construction that there is no atom \( A \in \mathcal{F}_{a_0} \) such that \( A \subset F \), consequently, for all \( \omega \in \Omega \) we have \((n, m) \not\subset v(\omega)\). Thus for all \( \omega \)\n\[
 G \not\subset v(\omega).
\]
If \( G \not\subset (n, m) \) then for all \( \omega \) one has \( v(\omega) \not\subset (n, m) \). Consequently, using the definition of the atom we find that \( a_{n,m}(\omega) = 0 \) \( (\omega \in \Omega) \) if \( G \not\subset (n, m) \). Next, it is easy to show that \( a(n, m) = 0 \) if \( H \not\subset (n, m) \). So by Hölder's inequality
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |a(n, m)|^p/(nm)^{2-p} = \sum_{H \in (n,m)} |a(n, m)|^p/(nm)^{2-p} \leq \left( \sum_{H \in (n,m)} |a(n, m)|^2 \right)^{p/2} \left( \sum_{H \in (n,m)} 1/(nm)^2 \right)^{1-2/p}.
\]
It is obvious by the definition of the atom and from the Parseval inequality that
\[
\left( \sum_{H \in (n,m)} |a(n, m)|^2 \right)^{1/2} \leq \|a\|_2 \leq P(F)^{2-1/2}.
\]
We shall show that
\[
\sum_{H \in (n,m)} 1/(nm)^2 \leq 2P(F).
\]
Combining (4)–(6) yields (3), and the theorem follows.
Using the inequality
\[
\sum_{k=1}^{n} 1/k^2 \leq \int_{n-1}^{\infty} (1/x^2) dx \leq 2/n \quad (n \geq 2)
\]
we get immediately
\[
\frac{1}{2} \sum_{H \in (n,m)} 1/(nm)^2 \leq \sum_{k=1}^{K} Q_{m_k}^{-1}(P_{m_k}^{-1} - P_{n_k}^{-1}) =: |H|
\]
where \( P_{m_k}^{-1} := 0 \). By the construction of the set \( H \), for \( 1 \leq k \leq K \) there exists an atom \( A_k \in \mathcal{F}_{a_0,n_k} \) such that \( A_k \subset F \). Let \( A := \bigcup_{k=1}^{K} A_k \); then \( A \subset F \). We shall show that
\[
|H| \leq P(A).
\]
For \( 1 \leq k \leq K \) choose an atom \( B_k \in \mathcal{F}_{a_0,n_k} \) such that the intersection of any \( B_k \) and \( B_k \) is nonempty. Then it can easily be shown that the Lebesgue measure of \( B := \bigcup_{k=1}^{K} B_k \) is equal to \( |H| \). Let \( C := \bigcup_{k=1}^{K} C_k \) where \( C_k \in \mathcal{F}_{a_0,n_k} \) is an atom. By induction on \( K \) we show that \( C \) has minimal area if and only if the intersection of any \( C_k \) and \( C_k \) is nonempty. For \( k = 1 \) or \( k = 2 \) this is trivial. Let \( 1 \leq i < K \) be the minimal index for which \( C_{i+1} \cap C_i = \emptyset \). (If there is no such index then the intersection of any two sets \( C_k \) is nonempty.) It can be seen that
\[
P(C \cup \bigcup_{j=1}^{k} C_j) > P(B \cup \bigcup_{j=1}^{k} B_j)
\]
Moreover, by the induction hypothesis we have
\[
P(\bigcup_{j=1}^{K} C_j) \geq P(\bigcup_{j=1}^{K} B_j)
\]
consequently, \( P(C) > P(B) \). Thus we have proved (8), and hence (6), so the proof of the theorem is complete for \( 0 < p \leq 1 \).
(ii) Secondly, let \( 1 < p \leq 2 \). Denote by \( P \) the set of positive natural numbers and introduce on \( P^2 \) the measure \( \mu(n, m) = 1/(nm^2) \). If
\[
Tf(n, m) = nm\tilde{f}(n, m)
\]
then it follows by the Parseval formula and by the previous theorem for \( p = 1 \) that both
\[
T : L^2_{\tilde{p}} \to L^2(P^2, \mu) \quad \text{and} \quad H^* : L^2_{\tilde{p}} \to L^2(P^2, \mu)
\]
are bounded. (In contrast to the one-parameter case it is not true that \( T \) is of weak type \((1, 1)\). By a well known interpolation theorem (see [1]) the operator
\[
T : (H^*, L^2_{\tilde{p}}) \to (L^2(P^2, \mu), L^2(P^2, \mu))_{\theta, p} \quad (0 < \theta < 1)
\]
is bounded as well. However, on the one hand,
\[
(L^2(P^2, \mu), L^2(P^2, \mu))_{\theta, p} = L^p(\mathbb{F}^2, \mu)
\]
(see [1]), and, on the other hand, we have proved in [24] that
\[
(H^*, L^2_{\tilde{p}})_{0, p} = H^*_{\tilde{p}}
\]
where in both cases \( 0 < \theta < 1 \) and \( 1/p = (1-\theta) + \theta/2 \). This completes the proof of Theorem 4. \( \blacksquare \)
Of course, it can similarly be proved that the theorem is true even if we do not suppose for a martingale \( f \in H_\infty^p \) that \( f_{0,n} = f_{0,n} = 0 \).

In [27] we have introduced an atomic Hardy space \( \mathcal{H}^{1,q} \) generated by some special atoms \((1 < q \leq \infty)\). Applying Theorem 1(i) we can show a theorem similar to Theorem 4 for \( \mathcal{H}^{1,q} \) \((1 < q \leq \infty, p = 1)\):

**Theorem 5.** For an arbitrary martingale \( f \in \mathcal{H}^{1,q} \)
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |f(n, m)|/(nm) \leq \|f\|_{\mathcal{H}^{1,q}} \quad (1 < q \leq \infty).
\]

**5. Dual theorems.** In [27] we have shown that the dual of \( H_\infty^p \) is \( A_2(\alpha) \)
\((0 < p \leq 1, \alpha = 1/p - 1)\) where \( A_2(\alpha) \) denotes the space of functions \( \phi \in L_0^1 \) for which
\[
\|\phi\|_{A_2(\alpha)} := \sup_{\nu \ni \nu}(P(\nu \neq \infty)^{-1/2-\alpha} \|\phi - \phi^\vee\|_2) < \infty \quad (\alpha > 0)
\]

(the supremum is taken over all stopping times). As in the one-parameter case, \( A_2(0) \) is also denoted by \( \text{BMO}_2 \). Denote the set of sequences \((P_m, Q_m): 1 \leq k \leq K\) by \( \mathcal{H}^q \) where \( K \in \mathbb{N}, (n_k) \) is decreasing and \((m_n)\) is increasing. Now we give a direct proof for the dual inequality to Theorem 4 for \(0 < p \leq 1\).

**Theorem 6.** If \((b_{n,m}; n, m \in \mathbb{P})\) is a sequence of complex numbers such that
\[
M := \sup_{H \in \mathcal{H}^q} \|H\|^{-1/2-\alpha}(\sum_{H \in \mathcal{H}^q} |b_{n,m}|^2)^{1/2} < \infty
\]
then there exists \( \phi \in A_2(\alpha) \) for which \( \phi(n, m) = b_{n,m} \) \((n, m \in \mathbb{P})\) and \( \|\phi\|_{A_2(\alpha)} \leq M \). (The definition of \( |H| \) is also given in (7)).

**Proof.** It follows from the Riesz–Fischer theorem that there exists a function \( \phi \in L_0^1 \) such that \( \phi(n, m) = b_{n,m} \) \((n, m \in \mathbb{P})\). It is easy to show that for every stopping time \( \nu \)
\[
\|\phi - \phi^\vee\|_2^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E\left[|\chi(\nu < (i, j))|d_{i,j}|\phi|^2\right].
\]

Let us use again the sets \( G \) and \( H \) constructed for the stopping time \( \nu \) in the proof of Theorem 4. It is clear that
\[
\|\phi - \phi^\vee\|_2^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E\left[|\chi(\nu < (i, j))|d_{i,j}|\phi|^2\right] \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E(d_{i,j}|\phi|^2).
\]

If we express \( d_{i,j}|\phi \) as a linear combination of the functions \( w_{n,m} \) then we obtain
\[
\|\phi - \phi^\vee\|_2^2 \leq \sum_{H \in \mathcal{H}^q} |b_{n,m}|^2.
\]

As we have seen in the proof of Theorem 4, \( |H| \leq P(\nu \neq \infty) \), consequently,
\[
P(\nu \neq \infty)^{-1/2-\alpha} \|\phi - \phi^\vee\|_2 \leq |H|^{-1/2-\alpha}(\sum_{H \in \mathcal{H}^q} |b_{n,m}|^2)^{1/2},
\]
i.e., \( \|\phi\|_{A_2(\alpha)} \leq M \), which shows Theorem 6.

Let us give the dual theorem to Theorem 4 also for \(1 < p \leq 2\). If \( p_n = O(1) \) and \( q_n = O(1) \) then by Theorem 2 one has \( H_\infty^p \sim L_0^1 \) \((p > 1)\) and it is well known that their dual space is \( L_0^1 \) \((1/p + 1/q = 1)\).

**Theorem 7.** If \( p_n = O(1), q_n = O(1), \ 2 < q < \infty \) and \( (b_{n,m}; n, m \in \mathbb{P}) \) is a sequence such that
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |b_{n,m}|^2/(nm)^{2-\alpha} < \infty,
\]
then the Videnskii polynomials
\[
f_{N,M}(x) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{n,m}w_{n,m}
\]
converge in \( L_q \) as \( \min(N, M) \to \infty \) to a function \( f \) satisfying \( f(n, m) = b_{n,m} \) \((n, m \in \mathbb{P})\) and
\[
\|f\|_q \leq C_q(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |b_{n,m}|^2/(nm)^{2-\alpha})^{1/q}.
\]

This theorem can be shown similarly to the case of the one-parameter trigonometric system (see [10], p. 193).

We have also proved in [27] that the dual space of \( \mathcal{H}^{1,q} \) is \( \mathcal{BMO}_{q} \)
\((1/q + 1/q' = 1, 1 < q < \infty)\) where \( \mathcal{BMO}_{q} \) denotes the space of functions \( \phi \in L_0^1 \) for which
\[
\|\phi\|_{\mathcal{BMO}_{q}} := \sup_{\nu \ni \nu}(E_{n,m}[|\phi - \phi_{n,m} + \phi_{n,m}|^p])^{1/p} \leq \infty.
\]

Now we prove a result for \( \mathcal{BMO}_{q} \) spaces analogous to Theorem 6.

**Theorem 8.** If \(2 < q < \infty\) and \((b_{n,m}; n, m \in \mathbb{P})\) is a sequence of complex numbers such that
\[
M := \sup_{(n,m) \in \mathbb{P}} \|P_{n,m} \phi\|_{L_0^1}^{1/2}(\sum_{(n,m) \in \mathbb{P}} |b_{n,m}|^q)^{1/q} < \infty
\]
then there exists \( \phi \in \mathcal{BMO}_{q} \) for which \( \phi(k, l) = b_{k,l} \) \((k, l \in \mathbb{P})\) and \( \|\phi\|_{\mathcal{BMO}_{q}} \leq M \).

**Proof.** By Theorem 1(ii) there exists \( \phi \in L_0^1 \) such that \( \phi(k, l) = b_{k,l} \) \((k, l \in \mathbb{P})\). Let \( n, m, l \in \mathbb{N} \) and introduce the Videnskii polynomials
\[
P_{n,m} \phi := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \phi(k + l, n + m + l)w_{k,l}.
\]
Then
\[ \varphi - E_{n,m}\varphi - E_{n,m}\varphi + E_{n,m}\varphi = \sum_{i=1}^{\infty} \sum_{j=1}^{m} P_{n,m}^{[i,j]} w_{i,n} q_{n,m}. \]

As \( P_{n,m} \) is \( \mathcal{F}_{n,m} \) measurable, applying again Theorem 1(ii), we get the following inequalities \( (1/q + 1/q' = 1) \):

\[ (E_{n,m}\varphi - E_{n,m}\varphi - E_{n,m}\varphi + E_{n,m}\varphi)^{1/q} = (E_{n,m} \left[ \sum_{i=1}^{\infty} \sum_{j=1}^{m} P_{n,m}^{[i,j]} w_{i,n} q_{n,m} \right])^{1/q} \]
\[ \leq (\sum_{i=1}^{\infty} \sum_{j=1}^{m} |P_{n,m}^{[i,j]}| q_{n,m}^{1/q'}) \leq (P_{n,m} q_{n,m}^{m-1} \sum_{r=1}^{m} \sum_{n=1}^{\infty} \int |\psi(k, l)||q_{n,m}|^{1/q'}, \]

i.e., \( \|\varphi\|_{\mathcal{F}_{n,m}} \leq M \). This completes the proof of Theorem 8. \( \square \)

This theorem can be found for one-parameter martingales in [13], [22] and another version for nonlinear martingales is in [25].

6. Converse inequalities. In this section we extend Theorem 4 under certain conditions to the case \( p > 2 \). Moreover, we prove the converse inequality. In the sequel we suppose that

\( p_n = O(1), \quad q_n = O(1). \)

If \( f = (f_{n,m}; n, m \in \mathbb{N}) \) is a martingale and \( b_{k,l} := f(k, l) \) then it is obvious that

\[ (f_{n,m} w_{k,l}) \]
and, conversely, an arbitrary sequence \( (b_{k,l}; k, l \in \mathbb{P}) \) defines a martingale. From now on we consider only those martingales for which

\[ b_{k,l} \to 0 \quad \text{as} \max(k, l) \to \infty, \]

\[ \mathcal{H}(b_{k,l} - b_{k-1,l} - b_{k,l+1} + b_{k+1,l+1}) \geq 0, \quad (k, l \in \mathbb{P}). \]

It follows from (10) and (11) that the sequences \( (\mathcal{H} b_{k,l}) \) and \( (\mathcal{F} b_{k,l}) \) are decreasing. Now we extend Theorem 4.

Theorem 9. Under the conditions (9) and (11) suppose that \( f \in H_p \). Then

\[ (\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |f(n, m)|/nm^{2})^{1/p} \leq C_p \|f\|_{H_p}, \quad (0 < p < \infty). \]

(Note that by Theorem 2 one has \( H_p^{\infty} \sim H_p \), for \( 0 < p < \infty \).)

We are not going to prove Theorem 9 because the proof is similar to Móricz's proof for the one-parameter Walsh system (see [16]).

We show a sharper assertion than the converse inequality. If

\[ \sigma_{n,m} := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} b_{k,l} w_{k,l} \]

then the following holds:

Theorem 10. Under the conditions (9)–(11)

\[ \|\sup_{n,m} |\sigma_{n,m}|\|_p \leq C_p \left( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |b_{n,m}| / (nm^{2}) \right)^{1/p} \quad (0 < p < \infty). \]

Proof. This theorem for the one-parameter Walsh system was also proved by Móricz for \( p > 1 \) (see [17]). Since

\[ |D_n(x)| := \sum_{k=0}^{n-1} w_{k,0}(x, y) \leq 2/x \quad (x \in (0, 1), \quad n \in \mathbb{N}) \]

is also true for a bounded Vilenkin system (see [11]), applying two-parameter Abel rearrangement, similarly to the proof in [16] we get

\[ |\sigma_{n,m}(x, y)| \leq C \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{k,l}| \quad \text{for} \quad \frac{1}{i+1} \leq x < \frac{1}{i} \quad \text{and} \quad \frac{1}{j+1} \leq y < \frac{1}{j}. \]

Therefore

\[ \|\sup_{n,m} |\sigma_{n,m}|\|_p^p \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(i+j+1)^{p-1}} \left( \sup_{n,m} |\sigma_{n,m}(x, y)| \right)^p \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(i+j+1)^{p-1}} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{k,l}|^p \]

Now a slightly modified version of the Hardy inequality is needed:

Lemma ([19], Theorem 8). If \( r > 1 \), \( 0 < p < \infty \) and \( (d_n, n \in \mathbb{P}) \) is a non-negative, non-increasing sequence then

\[ \sum_{n=1}^{\infty} n^{r-1} \left( \sum_{k=1}^{\infty} d_k \right)^p \leq C_p \sum_{n=1}^{\infty} d_n n^{r-p}. \]

Applying twice the Lemma we obtain the inequality

\[ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(i+j+1)^{p-1}} \left( \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{k,l}|^p \right) \leq C_p \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(i+j+1)^{p-1}} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{k,l}|^p \]

This completes the proof of Theorem 10. \( \square \)
Finally, it is easy to see that the following corollary holds.

**Corollary 1.** If (9)–(11) are satisfied and $0 < p < \infty$ is fixed then the following conditions are equivalent:

$$\sup_{n,m} |\sigma_{n,m}| \in L^p; \ f \in H_p; \ \sum_{n=1}^{m} \sum_{m=1}^{\infty} |b_{n,m}|^p \frac{1}{p(nm)^{2-p}} < \infty.$$ 

Therefore

$$\|\sup_{n,m} |\sigma_{n,m}|\|_p \leq C_p \|f\|_{H_p} \quad (0 < p < \infty).$$

As any martingale $f \in H_p$ ($p > 1$) can be identified with an $L^1$ function, the $\sigma_{n,m}$ are partial sums of the Vilenkin–Fourier series of the function corresponding to $f$. Vilenkin polynomials are dense in $H_p$, consequently, by (12) and by Theorem 2 of Chapter 3.1 in [22], Corollary 2 follows immediately:

**Corollary 2.** Let $p_1 = O(1)$ and $q_1 = O(1)$. If $f \in L^p$ ($p > 1$) or $f \in H$, such that (10) and (11) are satisfied then $\sigma_{n,m} f \rightarrow f$ a.e. and also in $L^p$-norm ($p \geq 1$) as $\min(n,m) \rightarrow \infty$.

Note that the fact that $\sigma_{n,m}$ converges a.e. has already been proved under the conditions (9)–(11) only (see [17]).

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**References**


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Received April 6, 1990

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