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Bukhvalov type characterizations of Urysohn operators

by

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Abstract. The aim of this paper is to generalize to nonlinear operators the criteria of integral representability for linear operators due to A. V. Bukhvalov. We give Bukhvalov type criteria for recognizing the order bounded Urysohn operators acting between ideals of measurable functions.

Introduction. The present paper is devoted to obtaining criteria characterizing when a nonlinear operator has an integral representation as an Urysohn operator. Roughly speaking, an Urysohn operator T is defined by $(Tf)(x) := \int U(x, y, f(y))dy$, where the kernel U satisfies the Carathéodory conditions (i.e., the function $U(x, y, \cdot)$ is continuous in $\mathbf{R}$ for almost all (x, y) and the function $U(\cdot, \cdot, t)$ is measurable for all $t \in \mathbf{R}$).

Integral representation of operators have been of interest for many mathematicians. Recall the nowadays classical results about integral representability of continuous linear operators in L^p spaces obtained in the thirties by Dunford–Pettis and Kantorovich–Vulikh (see, for instance, [8]). In that time John von Neumann [17] raised the problem of finding a characterization of integral linear operators acting in L^2. This problem was solved by A. V. Bukhvalov in [2] in the context of ideals of measurable functions; an independent proof is due to A. R. Schep [22] (see also [3, 29]). Let E and F be ideals of measurable functions. Bukhvalov's theorem states that for a linear operator L: $E \to F$ a necessary and sufficient condition for L to be an integral operator is the following:

Given a sequence $(f_n)_{n=1}^{\infty}$ *in* E *such that* $0 \leqslant f_n \leqslant g$, $f_n \to 0$ $(*)$ *implies* $Lf_n(x) \to 0$ *a.e.*

On the other hand, a large representation theory for nonlinear functionals was developed in the late sixties [5, 6, 7, 9, 15, 16, 24, 27]. L. Drewnowski and W. Orlicz [6, 7] obtained criteria similar to Bukhvalov's for functionals. We remark that the functionals they consider need not be defined on the whole of an ideal of measurable functions. For the sake of convenience we shall not consider this more general case.

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We are interested in the following condition for a nonlinear operator $T: E \to F$ to be an Urysohn operator.

Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences in E such that $|f_n| \leq g$ and $|g_n| \leq g$. If $f_n - g_n \to 0$ $(*)$, then $Tf_n(x) - Tg_n(x) \to 0$ a.e.

To prove it, the methods of Bukhvalov–Schep and of Drewnowski–Orlicz are used. The proof of Bukhvalov's theorem is based on the lattice calculus of order bounded linear operators developed by Riesz, Kantorovich and Freudenthal. We outline briefly the proof. First one shows that the order bounded linear integral operators form a band. This band is identified with the band generated by the order σ-continuous operators of finite rank. Finally, one sees that the operators in this band are the only ones satisfying the Bukhvalov condition. To employ similar ideas to nonlinear operators a calculus for abstract Urysohn operators have been developed and applied to Urysohn operators in [14]. However, a difficulty arises now: Order bounded Urysohn operators do not form a band in the space of all abstract Urysohn operators (denoted by $\mathcal{U}(E, F)$) because of the continuity condition on the kernel [14, Example 5.4]. To overcome this difficulty we have to consider another class of integral operators without continuity conditions. These operators will be defined in a suitable sublattice of E. We shall need two steps: First of all, to represent restrictions of operators on the mentioned sublattice and then to extend them to the whole of the space E. So we shall divide our condition into the following two:

(a) *Given an order bounded sequence $(\mathbf{1}_{B_n})_{n=1}^{\infty}$ of characteristic functions in E, $\mathbf{1}_{B_n} \to 0$ $(*)$ implies $T(t\mathbf{1}_{B_n})(x) \to 0$ a.e. for all t.*

(b) *Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences in E such that $|f_n| \leq g$ and $|g_n| \leq g$. If $f_n(y) - g_n(y) \to 0$ a.e., then $Tf_n(x) - Tg_n(x) \to 0$ a.e.*

In Section 2, the subspace E_s of all simple functions in E with rational values is considered. We define a concept of integral operator on this sublattice and we show they form a band in $\mathcal{U}(E_s, F)$. Later this band is identified with the band in $\mathcal{U}(E_s, F)$ generated by the disjointly σ-continuous operators of finite rank. Then the condition (a) allows us to obtain a kernel for an operator on E_s. This kernel $U(x, y, t)$ is just defined for $t \in \mathbf{Q}$.

In Section 3 we use the condition (b) to extend the kernel $U(x, y, t)$ to a function $U'(x, y, t)$ defined for all $t \in \mathbf{R}$ and in such a way that the extension U' satisfies the Carathéodory conditions.

Section 4 contains the main results characterizing Urysohn operators.

Throughout this paper we assume that kernels satisfy the condition $U(x, y, 0) = 0$ for almost all (x, y) since it is not a restriction. Indeed, an operator T is an Urysohn operator if and only if so is the operator defined by $Sf := Tf - T0$ and when S has an integral representation its kernel satisfies the above property.

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1. Preliminaries. Methods used in this paper proceed from the theory of Riesz spaces. Standard monographs in this theory are [13, 21, 26, 29] to which we refer for terminology and basic results. In Section 2 the application of the lattice calculus of abstract Urysohn operators developed in [14] will be essential. Let E and F be Riesz spaces. The operator $T: E \to F$ is called *orthogonally additive* if $T(f+g) = Tf + Tg$ whenever $f, g \in E$ are disjoint ($f \perp g$ in symbols); T is called *order bounded* if it maps order bounded sets in E onto order bounded sets in F. An operator is said to be an *abstract Urysohn operator* if it is orthogonally additive and order bounded. We denote by $\mathcal{U}(E, F)$ the set of all abstract Urysohn operators from E into F. The vector space $\mathcal{U}(E, F)$ is partially ordered by the following relation: If $S, T \in \mathcal{U}(E, F)$, then $S \leq T$ means $Sf \leq Tf$ for all $f \in E$. With this order, a Kantorovich–Freudenthal type theorem is proved in [14, Theorem 3.2]. So, when the space F is Dedekind complete, we obtain a lattice calculus for abstract Urysohn operators.

We next consider a σ-finite and complete measure space (Y, Σ, ν). To simplify notations the σ-algebra Σ will not be indicated explicitly. We shall consider ν-measurable functions from Y into the extended system of real numbers. The characteristic function of a ν-measurable set B will be denoted by $\mathbf{1}_B$. We shall denote by $M(Y, \nu)$ the set of all ν-measurable and ν-almost everywhere finite functions on Y with the usual identification of ν-almost equal functions. Note that, since the measure is σ-finite, given $f \in M(y, \nu)$, the space Y can be decomposed in an increasing sequence of ν-measurable sets such that f is ν-integrable over each of them. It is enough to consider the sets $Y_n \cup \{y \in Y \mid |f(y)| \leq n\}$ for $n \in \mathbf{N}$, where $Y = \bigcup_{n=1}^{\infty} Y_n$ and $\nu(Y_n) < \infty$. Evidently, the sequence can also be chosen disjoint instead of increasing.

In our considerations, the space $M(y, \nu)$ will occur endowed with a metric and an order. We define the following metric [8, Section III.2]:

$$d_\nu(f, g) := \inf_{a>0} \arctan\big(a + \nu\{y \in Y \mid |f(y) - g(y)| > a\}\big)$$

for $f, g \in M(y, \nu)$. The convergence with respect to this metric is called *convergence in measure*. To characterize it, consider a sequence $(f_n)_{n=1}^{\infty}$ in $M(y, \nu)$ and $f \in M(y, \nu)$, and define the ν-measurable set

$$B_n^{\varepsilon} := \{y \in Y \mid |f_n(y) - f(y)| > \varepsilon\}$$

for $\varepsilon > 0$ and for $n \in \mathbf{N}$. Then $(f_n)_{n=1}^{\infty}$ converges in measure to f if and only if $\lim_{n\to\infty} \nu(B_n^{\varepsilon}) = 0$ for every $\varepsilon > 0$. In addition, $(f_n)_{n=1}^{\infty}$ converges in measure to

f on every $B \subset Y$ of finite measure if and only if each subsequence $(f_{n_k})_{k=1}^{\infty}$ contains a subsequence $(f_{n_{k_i}})_{i=1}^{\infty}$ such that $\lim_{i\to\infty} f_{n_{k_i}}(y) = f(y)$ ν-a.e. We shall often apply the dominated convergence theorem for convergence in measure [8, III.3, Theorem 7].

We introduce the following order in $M(Y, \nu)$: $f \leqslant g$ means $f(y) \leqslant g(y)$ ν-a.e. Recall that convergence with respect to this order is convergence ν-a.e. It is said that a sequence $(f_n)_{n=1}^{\infty}$ in $M(Y, \nu)$ $(*)$-*converges* to f (denoted by $f_n \to f$ $(*)$) if every subsequence $(f_{n_k})_{k=1}^{\infty}$ contains a subsequence $(f_{n_{k_i}})_{i=1}^{\infty}$ which converges in order to f. It follows that convergence in measure coincides with $(*)$-convergence on every $B \subset Y$ such that $\nu(B) < \infty$. With the considered order the set $M(Y, \nu)$ becomes a Dedekind complete and order separable Riesz space. For every $f \in M(Y, \nu)$, its *support* is defined as usual by $\operatorname{supp}(f) := \{y \in Y \mid f(y) \neq 0\}$. Evidently $f, g \in M(Y, \nu)$ are disjoint whenever $\operatorname{supp}(f) \cap \operatorname{supp}(g)$ is ν-null.

Let E be an order ideal in $M(Y, \nu)$. Consider the set $\{\mathbf{1}_{\operatorname{supp}(f)} \mid f \in E\}$ which is bounded above by $\mathbf{1}_Y$. So $g := \sup\{\mathbf{1}_{\operatorname{supp}(f)} \mid f \in E\}$ exists in $M(Y, \nu)$. The set $\operatorname{supp}(g)$ is said to be the *support* or *carrier* of the ideal E. Without restriction of generality we shall assume throughout this paper that the carrier of E is Y. Thus, given a ν-measurable set $B \subset Y$ with $\nu(B) > 0$ it is possible to find $B' \subset B$ with $\nu(B') > 0$ and $\mathbf{1}_{B'} \in E$. It follows that there exists an increasing sequence $(Y_n)_{n=1}^{\infty}$ of ν-measurable sets such that $Y = \bigcup_{n=1}^{\infty} Y_n$, $\nu(Y_n) < \infty$ and $\mathbf{1}_{Y_n} \in E$ for all $n \in \mathbf{N}$ [29, Theorem 86.2].

Now we introduce a set which will be of importance in our study. We shall denote by $P(Y, \nu)$ the set of all nonnegative ν-measurable functions on Y, identifying ν-almost equal functions. We remark that a function in $P(Y, \nu)$ may be infinite in a non-ν-null set. The set $P(Y, \nu)$ is ordered with the same relation as above: $f \leqslant g$ whenever $f(y) \leqslant g(y)$ ν-a.e. Then $P(Y, \nu)$ becomes a lattice satisfying the following property (see [22] or [29, Lemma 94.4]).

Proposition A. *Let $\{f_\alpha \mid \alpha \in \{\alpha\}\}$ be a set of nonnegative functions in $M(Y, \nu)$. Then $f := \sup_\alpha f_\alpha$ exists in $P(Y, \nu)$ and there is a sequence $\alpha_n \in \{\alpha\}$ such that $f = \sup_n f_{\alpha_n}$.*

Consider two σ-finite and complete measure spaces (X, μ) and (Y, ν). We define the product measure space as usual and denote by $(X \times Y, \mu \times \nu)$ its completion. We recall that every $\mu \times \nu$-measurable set can be approximated by finite unions of generalized rectangles [11, p. 56]. To be more precise, for every $\mu \times \nu$-measurable set Z, given $\varepsilon > 0$ one can find $\mu \times \nu$-measurable sets $A_k \times B_k$ for $k = 1, \ldots, n$ such that $\mu \times \nu\big(Z \triangle \big(\bigcup_{k=1}^{n} A_k \times B_k\big)\big) < \varepsilon$.

Let $U\colon X \times Y \times \mathbf{R} \to \mathbf{R}$ be a function satisfying the following conditions.

(C_0) $U(x, y, 0) = 0$ for $\mu \times \nu$-almost all $(x, y) \in X \times Y$.

(C_1) $U(\cdot, \cdot, t)$ is $\mu \times \nu$-measurable for all $t \in \mathbf{R}$.

(C_2) $U(x, y, \cdot)$ is continuous in $\mathbf{R}$ for $\mu \times \nu$-almost all $(x, y) \in X \times Y$.

By (C_1) and (C_2) (the *Carathéodory conditions*) the function $(x, y) \to U(x, y, f(y))$ is $\mu \times \nu$-measurable and $\mu \times \nu$-a.e. finite for all $f \in M(Y, \nu)$. Then the function $x \to \int_Y |U(x, y, f(y))|\,d\nu$ is well defined and μ-measurable by Fubini's theorem. We define

$$\operatorname{Dom}_Y(U) := \{f \in M(Y, \nu) \mid \text{the function } x \to \int_Y |U(x, y, f(y))|\,d\nu \text{ is } \mu\text{-a.e. finite}\}.$$

Throughout the paper, E and F will denote order ideals in $M(Y, \nu)$ and $M(X, \mu)$ respectively. An operator $T\colon E \to F$ is called an *Urysohn operator* with kernel U if

(1) $E \subset \operatorname{Dom}_Y(U)$,

(2) $(Tf)(x) = \int_Y U(x, y, f(y))\,d\nu$ μ-a.e. for all $f \in E$.

Note that the above operators are orthogonally additive by condition (C_0). A more detailed discussion and conditions for order boundedness of these operators are given in [14]. Conditions for continuity and compactness in L^p spaces are deeply studied in [12].

2. Integral operators on E_s. As before, let E and F be order ideals in $M(Y, \nu)$ and $M(X, \mu)$ respectively and assume that Y is the carrier of E. We denote by E_s the set of all simple functions in E with rational values. That is,

$$E_s := \{p = \sum_{i=1}^{n} t_i \mathbf{1}_{B_i} \in E \mid B_i \subset Y \ \nu\text{-measurable and } t_i \in \mathbf{Q} \text{ for } i = 1, \ldots, n\}.$$

We may assume, as usual, that the sets $(B_i)_{i=1}^{n}$ are pairwise disjoint.

Definition 2.1. An operator $T\colon E_s \to F$ is said to be an *integral operator* if there is a function $U\colon X \times Y \times \mathbf{Q} \to \mathbf{R}$ satisfying

(a) $U(x, y, 0) = 0$ for $\mu \times \nu$-almost all $(x, y) \in X \times Y$,

(b) $U(\cdot, \cdot, t)$ is $\mu \times \nu$-measurable for all $t \in \mathbf{Q}$

and such that for every $p \in E_s$

(1) $x \to \int_Y |U(x, y, p(y))|\,d\nu$ is μ-a.e. finite,

(2) $(Tp)(x) = \int_Y U(x, y, p(y))\,d\nu$ μ-a.e.

The condition (b) on the kernel U implies that T is well defined while orthogonal additivity of T is a consequence of (a).

Our goal in this section is to characterize integral operators on E_s as those belonging to the band in $\mathscr{U}(E_s, F)$ generated by operators of finite rank and by the following condition.

(†) *Given an order bounded sequence $(\mathbf{1}_{B_n})_{n=1}^{\infty}$ in E, $\mathbf{1}_{B_n} \to 0$ $(*)$ implies $T(t\mathbf{1}_{B_n})(x) \to 0$ μ-a.e. for all $t \in \mathbf{Q}$.*

To prove that, the steps of Bukhvalov's and Schep's proofs are followed and the lattice calculus developed in [14] is applied. Note that E_s is not a Riesz space since it is not a vector space over $\mathbf{R}$. However, it is a vector lattice over $\mathbf{Q}$. We point out that if abstract Urysohn operators acting from a vector lattice over $\mathbf{Q}$ into a Dedekind complete Riesz space are considered, then the abstract framework of [14], in particular Theorem 3.2, still holds. We state it for further reference.

THEOREM 2.2. *The partially ordered space $\mathcal{U}(E_s, F)$ is actually a Dedekind complete Riesz space and if $S, T \in \mathcal{U}(E_s, F)$, then for every $p \in E_s$*

$$(T \vee S)(p) = \sup\{Tq_1 + Sq_2 \mid p = q_1 + q_2,\ q_1 \perp q_2\},$$

$$(T \wedge S)(p) = \inf\{Tq_1 + Sq_2 \mid p = q_1 + q_2,\ q_1 \perp q_2\}.$$

In addition, if $T_\alpha \uparrow T$ in $\mathcal{U}(E_s, F)$, then $T_\alpha p \uparrow Tp$ for every $p \in E_s$.

2.1. *The band of order bounded integral operators.* To begin the proof of our goal we have to see that the order bounded integral operators form a band in $\mathcal{U}(E_s, F)$. This subsection is devoted to show this. First we shall prove that positive operators majorized by integral ones are integral. Two auxiliary results are needed. By fixing $t \in \mathbf{Q}$ the proofs are similar to those of the linear case (see [22] or [29, Chapter 14]) and will be omitted.

LEMMA 2.3. *Let T: $E_s \to F$ be an integral operator with kernel U. Then*

(1) $T \geqslant 0$ if and only if for all $t \in \mathbf{Q}$, $U(x, y, t) \geqslant 0$ $\mu \times \nu$-a.e.,
(2) $T = 0$ if and only if for all $t \in \mathbf{Q}$, $U(x, y, t) = 0$ $\mu \times \nu$-a.e.

LEMMA 2.4. *Let S: $E_s \to F$ be a positive orthogonally additive operator and let $t \in \mathbf{Q}$. Assume that $X' \times Y' \subset X \times Y$ is a $\mu \times \nu$-measurable set such that $\mathbf{1}_{Y'} \in E$ and $\int_{X'} S(t\mathbf{1}_{Y'})(x)d\mu$ is finite. If Γ denotes the semiring of all $\mu \times \nu$-measurable sets $A \times B \subset X' \times Y'$, then*

$$\lambda(A \times B, t) := \int_A S(t\mathbf{1}_B)(x)d\mu$$

is a finitely additive measure on Γ.

PROPOSITION 2.5. *Let T: $E_s \to F$ be a positive integral operator with kernel U. If S: $E_s \to F$ is an orthogonally additive operator such that $0 \leqslant S \leqslant T$, then there is a function V: $X \times Y \times \mathbf{Q} \to \mathbf{R}$ satisfying*

(1) $V(x, y, 0) = 0$ for $\mu \times \nu$-almost all $(x, y) \in X \times Y$,
(2) $V(\cdot, \cdot, t)$ is $\mu \times \nu$-measurable for all $t \in \mathbf{Q}$,
(3) $(Sp)(x) = \int_Y V(x, y, p(y))d\mu$ μ-a.e. for every $p \in E_s$,
(4) $0 \leqslant V(x, y, t) \leqslant U(x, y, t)$ $\mu \times \nu$-a.e. for all $t \in \mathbf{Q}$.

Proof. Since Y is the carrier of E, there is a sequence $(Y_n)_{n=1}^\infty$ of mutually disjoint subsets of Y such that $Y = \bigcup_{n=1}^\infty Y_n$ and $\mathbf{1}_{Y_n} \in E$ for all $n \in \mathbf{N}$. It is enough to define V on $X \times Y_n \times \mathbf{Q}$ for each $n \in \mathbf{N}$, so it is not a restriction to suppose that $\mathbf{1}_Y$ lies in E.

Fix $t \in \mathbf{Q}$. Before proving the statement we remark that the following claim holds, by the same reasoning as in the linear case [29, Theorem 94.2(i)].

Let $X' \subset X$ be a μ-measurable set such that $\int_{X'} T(t\mathbf{1}_Y)(x)d\mu < \infty$. Then there is a $\mu \times \nu$-measurable function $V(\cdot, \cdot, t)$ such that, for every $B \subset Y$, $S(t\mathbf{1}_B)(x) = \int_Y V(x, y, t\mathbf{1}_B(y))d\nu$ μ-a.e. on X'.

Note that if $t = 0$, then $V(x, y, 0) = 0$ for $\mu \times \nu$-almost all $(x, y) \in X' \times Y$.

Now since the measure μ is σ-finite, we may consider a disjoint sequence of μ-measurable sets $(A_k)_{k=1}^\infty$ such that $X = \bigcup_{k=1}^\infty A_k$ and $T(t\mathbf{1}_Y)$ is μ-integrable over each A_k. Applying the above claim to $A_k \times Y$ for each $k \in \mathbf{N}$, we get $\mu \times \nu$-measurable functions $V_k(\cdot, \cdot, t)$ such that for every $B \subset Y$ we have $S(t\mathbf{1}_B)(x) = \int_Y V_k(x, y, t\mathbf{1}_B(y))d\nu$ μ-a.e. on A_k. Set $V(x, y, t) := V_k(x, y, t)$ for $x \in A_k$ and for $y \in Y$. Then for every ν-measurable set B we obtain

$$S(t\mathbf{1}_B)(x) = \int_Y V(x, y, t\mathbf{1}_B(y))d\nu \quad \mu\text{-a.e. on } X.$$

It is evident that the above argument is valid for every $t \in \mathbf{Q}$. Consequently, we have defined a function V: $X \times Y \times \mathbf{Q} \to \mathbf{R}$ satisfying (1) and (2). Taking $p = \sum_{i=1}^n t_i \mathbf{1}_{B_i}$, the ν-measurable sets $(B_i)_{i=1}^n$ being pairwise disjoint, the following equalities hold μ-a.e.:

$$(Sp)(x) = \sum_{i=1}^n S(t_i\mathbf{1}_{B_i})(x) = \sum_{i=1}^n \int_Y V(x, y, t_i\mathbf{1}_{B_i}(y))d\nu = \int_Y V(x, y, p(y))d\nu.$$

Finally, (4) is a consequence of Lemma 2.3. ∎

The order bounded integral operators from E_s into F form a Riesz subspace of $\mathcal{U}(E_s, F)$. This follows from the following result which can be proved as a consequence of Theorem 2.2 and Proposition 2.5, just as in the linear case [29, Theorem 94.3].

PROPOSITION 2.6. *Let T: $E_s \to F$ be an integral operator with kernel U. If T is order bounded, then for every $p \in E_s$*

$$|T|p(x) = \int_Y |U(x, y, p(y))|d\nu \quad \mu\text{-a.e.}$$

Before showing that the order bounded integral operators form a band, we give a characterization of order boundedness.

PROPOSITION 2.7. *Let T: $E_s \to F$ be an integral operator with kernel U. Then T is order bounded if and only if for every $q \in E_s$ with $q \geqslant 0$, there is a $\mu \times \nu$-measurable and positive function M_q: $X \times Y \to \mathbf{R}$ satisfying*

(a) if $p \in [-q, q]$, then $|U(x, y, p(y))| \leqslant M_q(x, y)$ $\mu \times \nu$-a.e.,
(b) $M_q(x, \cdot)$ is ν-integrable for μ-almost all $x \in X$,
(c) $x \to \int_Y M_q(x, y)d\nu$ belongs to F.

In this case, for every function $p^* = \sum_{i=1}^n t_i \mathbf{1}_{Z_i}$ with $Z_i \subset X \times Y$ and $t_i \in \mathbf{Q}$ for $i = 1, \ldots, n$, if $|p^*| \leqslant q\mathbf{1}_X$, then $|U(x, y, p^*(x, y))| \leqslant M_q(x, y)$ $\mu \times \nu$-a.e.

Proof. It is straightforward that T is order bounded if the above condition holds.

Conversely, suppose that T is order bounded. By Theorem 2.2 the operator $|T|$ exists and is order bounded. Given $q \in E_s$, $q \geqslant 0$, it is possible to find $v \in F$ such that for every $p \in [-q, q]$, $|T|(p) \leqslant v$, and then by Proposition 2.6, $\int_Y |U(x, y, p(y))| dv \leqslant v(x)$ μ-a.e. For simplicity of notation define $V(x, y, t) := |U(x, y, tq(y))|$. Obviously $V(x, y, 0) = 0$ for $\mu \times \nu$-almost all $(x, y) \in X \times Y$ and $V(\cdot, \cdot, t)$ is $\mu \times \nu$-measurable for all $t \in \mathbf{Q}$. Furthermore, if $(B_i)_{i=1}^n$ are pairwise disjoint ν-measurable sets and $t_i \in \mathbf{Q}$, $|t_i| < 1$, for $i = 1, \ldots, n$, then $\int_Y V(x, y, \sum_{i=1}^n t_i \mathbf{1}_{B_i}(y)) dv \leqslant v(x)$ μ-a.e.

Let $E_s^* := \{\sum_{i=1}^n t_i \mathbf{1}_{Z_i} | Z_i \subset X \times Y \text{ and } t_i \in \mathbf{Q} \text{ for } i = 1, \ldots, n\}$. We may assume that for every function $\sum_{i=1}^n t_i \mathbf{1}_{Z_i}$ the sets $(Z_i)_{i=1}^n$ are mutually disjoint. Now define $M_q(x, y) := \sup\{V(x, y, p^*(x, y)) | p^* \in E_s^* \text{ and } |p^*| \leqslant \mathbf{1}_{X \times Y}\}$, the supremum being taken in $P(X \times Y, \mu \times \nu)$. Then there is a sequence $(u_n)_{n=1}^\infty$ in E_s^* such that $M_q(x, y) = \sup_n V(x, y, u_n(x, y))$. The operator defined by $p^* \to V(x, y, p^*(x, y))$ for $p^* \in E_s^*$ is projection commuting. Note that E_s^* is a sublattice of $M(X \times Y, \mu \times \nu)$ such that $P(E_s^*) \subset E_s^*$ for all order projections P of $M(X \times Y, \mu \times \nu)$. For each $n \in \mathbf{N}$ we may apply [7, Lemma 2.1] to the finite sequences $\{u_1, \ldots, u_n\}$ and $\{0, \ldots, 0\}$; consequently, there is $p_n^* \in E_s^*$ such that

(i) $|p_n^*| \leqslant \sup_{i=1,\ldots,n} |u_i|$,

(ii) $V(x, y, p_n^*(x, y)) = \sup_{i=1,\ldots,n} V(x, y, u_i(x, y))$.

Hence, $|p_n^*| \leqslant \mathbf{1}_{X \times Y}$ for all $n \in \mathbf{N}$ and the sequence $(V(x, y, p_n^*(x, y)))_{n=1}^\infty$ is increasing with supremum $M_q(x, y)$. By the monotone convergence theorem

$$\int_Y V(x, y, p_n^*(x, y)) dv \uparrow \int_Y M_q(x, y) dv \quad \mu\text{-a.e.}$$

Thus the result follows from the following claim. *If $p^* \in E_s^*$ with $|p^*| \leqslant \mathbf{1}_{X \times Y}$, then $\int_Y V(x, y, p^*(x, y)) dv \leqslant v(x)$ μ-a.e.*

Indeed, from it one deduces that $\int_Y M_q(x, y) dv \leqslant v(x)$ μ-a.e. and consequently $M_q(x, \cdot)$ is ν-integrable for μ-almost all $x \in X$ and $x \to \int_Y M_q(x, y) dv$ lies in F.

To show that M_q actually maps in $\mathbf{R}$, i.e., $M_q(x, y)$ is $\mu \times \nu$-a.e. finite, suppose the contrary. Then the $\mu \times \nu$-measurable set $Z := \{(x, y) \in X \times Y | M_q(x, y) = \infty\}$ has positive measure. Let $Z_x := \{y \in Y | (x, y) \in Z\}$ for all $x \in X$. Since $\mu \times \nu(Z) > 0$, there is a μ-measurable set A such that $\mu(A) > 0$ and $\nu(Z_x) > 0$ for all $x \in A$. So, for μ-almost all $x \in A$,

$$\infty = \int_{Z_x} M_q(x, y) dv \leqslant \int_Y M_q(x, y) dv \leqslant v(x).$$

Hence $v(x) = \infty$ in a set of positive measure, which is a contradiction.

Finally, for each $p \in E_s$ with $|p| \leqslant q$, let

$$p^*(x, y) := \frac{p(y)}{q(y)} \mathbf{1}_{X \times \operatorname{supp}(q)}(x, y).$$

Then $p^* \in E_s^*$, $|p^*| \leqslant \mathbf{1}_{X \times Y}$ and $|U(x, y, p(y))| = V(x, y, p^*(x, y)) \leqslant M_q(x, y)$ $\mu \times \nu$-a.e.

Our claim remains to be proved. Firstly, consider $p^* = \sum_{i=1}^n t_i \mathbf{1}_{Z_i}$ such that each Z_i is a finite union of generalized rectangles, that is, $Z_i = \bigcup_{k=1}^{m(i)} A_k^i \times B_k^i$ for $i = 1, \ldots, n$. Then one can write $p^* = \sum_{a=1}^r \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{A_a \times B_{ab}}$ with $|\alpha_{ab}| \leqslant 1$ for $b = 1, \ldots, l(a)$ and $a = 1, \ldots, r$, the μ-measurable sets $(A_a)_{a=1}^r$ being mutually disjoint and, for each $a = 1, \ldots, r$, the ν-measurable sets $(B_{ab})_{b=1}^{l(a)}$ being mutually disjoint too. For $a = 1, \ldots, r$ fixed we know that

$$\int_Y V(x, y, \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{B_{ab}}(y)) dv \leqslant v(x) \quad \mu\text{-a.e.}$$

So, $\int_Y V(x, y, \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{A_a \times B_{ab}}(x, y)) dv \leqslant \mathbf{1}_{A_a}(x) v(x)$ μ-a.e. for $a = 1, \ldots, r$ and consequently

$$\int_Y V(x, y, p^*(x, y)) dv \leqslant v(x) \quad \mu\text{-a.e.}$$

Now consider arbitrary $p^* \in E_s^*$ with $|p^*| \leqslant \mathbf{1}_{X \times Y}$; so $p^* = \sum_{i=1}^n t_i \mathbf{1}_{Z_i}$ with $t_i \in \mathbf{Q}$, $|t_i| \leqslant 1$ and the sets $(Z_i)_{i=1}^n$ pairwise disjoint. By the σ-finiteness of μ we get an increasing sequence $(X_m)_{m=1}^\infty$ of μ-measurable sets such that $X = \bigcup_{m=1}^\infty X_m$ and v is μ-integrable over each X_m. It is enough to see that

$$\int_Y V(x, y, p^*(x, y)) dv \leqslant v(x) \quad \mu\text{-a.e. in } X_m$$

for every $m \in \mathbf{N}$. Hence, there is no loss of generality in supposing that v is μ-integrable over X. Since $\int_Y V(x, y, t_i) dv \leqslant v(x)$ μ-a.e. for $i = 1, \ldots, n$, it follows that the functions $V(\cdot, \cdot, t_i)$ are $\mu \times \nu$-integrable. Consequently, for every $\varepsilon > 0$ there is $\delta > 0$ such that if $W \subset X \times Y$ is $\mu \times \nu$-measurable with $\mu \times \nu(W) \leqslant \delta$, then $\int_W V(x, y, t_i) d\mu \times \nu < \varepsilon/n$ for every $i = 1, \ldots, n$. Next each Z_i will be approximated by finite unions of generalized rectangles. Given $\delta > 0$ one may find a $\mu \times \nu$-measurable set W_1, a finite union of generalized rectangles, such that $\mu \times \nu(W_1 \triangle Z_1) \leqslant \delta/n$. Suppose that the sets $W_1, \ldots, W_k$ are defined in such a way that they are mutually disjoint, $\mu \times \nu(W_i \triangle Z_i) \leqslant i\delta/n$ and $\mu \times \nu(W_i \sim Z_i) \leqslant \delta/n$ for $i = 1, \ldots, k$. Let W_{k+1}^* be a finite union of generalized rectangles such that $\mu \times \nu(W_{k+1}^* \triangle Z_{k+1}) \leqslant \delta/n$ and let $W_{k+1} := W_{k+1}^* \sim \bigcup_{i=1}^k W_i$. Then

$$\mu \times \nu(W_{k+1} \triangle Z_{k+1}) \leqslant \mu \times \nu(W_{k+1}^* \triangle Z_{k+1}) + \sum_{i=1}^k \mu \times \nu(W_i \triangle Z_i)$$

$$\leqslant \delta/n + \sum_{i=1}^k \mu \times \nu(W_i \triangle Z_i) \leqslant (k+1)\delta/n$$

and

$$\mu \times \nu(W_{k+1} \sim Z_{k+1}) \leqslant \mu \times \nu(W_{k+1}^* \triangle Z_{k+1}) \leqslant \delta/n.$$

Thus, we have inductively defined a sequence $(W_i)_{i=1}^n$ of pairwise disjoint sets such that each W_i is a finite union of generalized rectangles and $\mu\times\nu(W_i\triangle Z_i)\leqslant\delta$. Consequently,

$$\int_{W_i\triangle Z_i} V(x, y, t_i)d\mu\times\nu<\varepsilon/n \quad \text{for } i=1,\ldots,n.$$

On the other hand, since the function $\sum_{i=1}^n t_i\mathbf{1}_{W_i}$ is of the kind considered above, we also have $\int_Y V(x, y, \sum_{i=1}^n t_i\mathbf{1}_{W_i}(x, y))d\nu\leqslant v(x)$ μ-a.e. So, for every μ-measurable set $A\subset X$,

$$\int_{A\times Y} V(x, y, \sum_{i=1}^n t_i\mathbf{1}_{W_i}(x, y))d\mu\times\nu\leqslant\int_A v(x)d\mu.$$

Therefore,

$$\int_{A\times Y} V(x, y, \sum_{i=1}^n t_i\mathbf{1}_{Z_i}(x, y))d\mu\times\nu$$
$$\leqslant\int_{A\times Y} V(x, y, \sum_{i=1}^n t_i\mathbf{1}_{W_i}(x, y))d\mu\times\nu+\sum_{i=1}^n\int_{W_i\triangle Z_i} V(x, y, t_i)d\mu\times\nu$$
$$\leqslant\int_A v(x)d\mu+\varepsilon.$$

Since ε and A are arbitrary, it follows that

$$\int_Y V(x, y, \sum_{i=1}^n t_i\mathbf{1}_{Z_i}(x, y))d\nu\leqslant v(x) \quad \mu\text{-a.e.}$$

This completes the proof. ∎

PROPOSITION 2.8. *The set of all order bounded integral operators from E_s into F is a band in $\mathscr{U}(E_s, F)$.*

Proof. It follows from Propositions 2.6 and 2.5 that it is an ideal. Consider a net $0\leqslant T_\alpha\uparrow T$ in $\mathscr{U}(E_s, F)$ and assume that each T_α is an integral operator with kernel U_α. Let us see that T is also an integral operator. If $t\in\mathbf{Q}$ is fixed, by Lemma 2.3, $U_\alpha(x, y, t)\uparrow$ in $M(X\times Y, \mu\times\nu)$. Let $U(x, y, t):=\sup_\alpha U_\alpha(x, y, t)$ in $P(X\times Y, \mu\times\nu)$. Then there is a sequence $(U_{nt}(x, y, t))_{n=1}^\infty$ such that $U_{nt}(x, y, t)\uparrow U(x, y, t)$ $\mu\times\nu$-a.e. Since $\mathbf{Q}$ is countable it is possible to get a sequence which does not depend on t.

Denote by T_n the integral operator generated by U_n, $n\in\mathbf{N}$. Fix $p\in E_s$. By Theorem 2.2 we know that $Tp=\sup_\alpha T_\alpha p$ and now it is not difficult to see, as in the linear case [29, Theorem 94.5], that

$$(Tp)(x)=\int_Y U(x, y, p(y))d\nu \quad \mu\text{-a.e.}$$

Finally, it remains to see that $U(x, y, t)$ is $\mu\times\nu$-a.e. finite for all $t\in\mathbf{Q}$. This is a consequence of the order boundedness of T. In fact, since Y is the carrier of E, there is a sequence $(Y_n)_{n=1}^\infty$ such that $Y=\bigcup_{n=1}^\infty Y_n$ and $\mathbf{1}_{Y_n}\in E$ for every $n\in\mathbf{N}$.

It is enough to show the $\mu\times\nu$-a.e. finiteness of $U(\cdot,\cdot,t)$ in each $X\times Y_n$, so we may suppose that $\mathbf{1}_Y\in E_s$. Now fix $t\in\mathbf{Q}$ and let $q:=|t|\mathbf{1}_Y$. By Proposition 2.7 there is a $\mu\times\nu$-measurable function M_q: $X\times Y\to\mathbf{R}$ such that if $|p|\leqslant q$, then $|U(x, y, p(y))|\leqslant M_q(x, y)$ $\mu\times\nu$-a.e. In particular, $|U(x, y, t\mathbf{1}_Y(y))|\leqslant M_q(x, y)$ $\mu\times\nu$-a.e. and consequently $U(x, y, t)$ is $\mu\times\nu$-a.e. finite for all $t\in\mathbf{Q}$. ∎

2.2. *Disjointly σ-continuous operators of finite rank.* Once it is proved that the order bounded integral operators form a band, the next step is to identify it with a band generated by operators of finite rank. We prove that in this subsection. We show that the suitable operators of finite rank are the disjointly σ-continuous ones (see [14, Definition 3.6 and Proposition 3.7]). An orthogonally additive operator T: $E_s\to F$ is *disjointly σ-continuous* if for each disjoint sequence $(p_n)_{n=1}^\infty$ in E_s, $\sum_{k=1}^n p_k\to p$ in order in E_s implies $\sum_{k=1}^n Tp_k\to Tp$ in order in F.

Recall that order convergence in $M(X, \mu)$ coincides with convergence μ-a.e. and consequently a sequence $(g_n)_{n=1}^\infty$ in F order converges to $g\in F$ if it is order bounded and $\lim_{n\to\infty} g_n(x)=g(x)$ μ-a.e. [26, Chap. III, § 9]. On the other hand, it is immediate that order convergence in E_s implies convergence ν-a.e.

PROPOSITION 2.9. (1) *Every integral operator from E_s into $M(X, \mu)$ is disjointly σ-continuous.*

(2) *Every order bounded integral operator from E_s into F is disjointly σ-continuous.*

Proof. It is enough to prove (1) since (2) is an easy consequence of it by the above remark and Proposition 2.7. But (1) follows from the countable additivity of μ. ∎

PROPOSITION 2.10. *Let Φ: $E_s\to\mathbf{R}$ be a disjointly σ-continuous orthogonally additive functional. Then there exists a function U: $Y\times\mathbf{Q}\to\mathbf{R}$ satisfying*

(a) *$U(y, 0)=0$ for ν-almost all $y\in Y$,*

(b) *$U(\cdot, t)$ is ν-measurable for all $t\in\mathbf{Q}$*

and such that $\Phi(p)=\int_Y U(y, p(y))d\nu$ for every $p\in E_s$.

Proof. Since Y is the carrier of E, there is a sequence $(Y_n)_{n=1}^\infty$ of pairwise disjoint ν-measurable sets such that $Y=\bigcup_{n=1}^\infty Y_n$ and $\mathbf{1}_{Y_n}\in E$ for all $n\in\mathbf{N}$. We may define U on each $Y_n\times\mathbf{Q}$, so there is no loss of generality in assuming that $\mathbf{1}_Y\in E_s$. Fix $t\in\mathbf{Q}$. For every ν-measurable set $B\subset Y$, define $\lambda(B, t):=\Phi(t\mathbf{1}_B)$. Since Φ is disjointly σ-continuous, $\lambda(\cdot, t)$ is a countably additive signed measure on the ν-measurable subsets of Y. Moreover, if $\nu(B)=0$, then $\mathbf{1}_B=0$ and so $\lambda(B, t)=\Phi(t\mathbf{1}_B)=0$. Thus $\lambda(\cdot, t)$ is absolutely continuous with respect to ν. By Radon–Nikodym's theorem there is a ν-measurable function $U(\cdot, t)$ such that $\Phi(t\mathbf{1}_B)=\lambda(B, t)=\int_B U(y, t)d\nu$. Since the above procedure is valid for all $t\in\mathbf{Q}$ there is a function U: $Y\times\mathbf{Q}\to\mathbf{R}$ satisfying (a) and (b) and such that

$$\Phi(t\mathbf{1}_B)=\int_B U(y, t)d\nu=\int_Y U(y, t\mathbf{1}_B(y))d\nu$$

for every $t \in \mathbf{Q}$ and for every ν-measurable set $B \subset Y$. Now it is straightforward that if $p \in E_s$, then $\Phi(p) = \int_Y U(y, p(y))d\nu$. ∎

We shall denote by E_s^u the set of all disjointly σ-continuous abstract Urysohn functionals defined on E_s. For every $\Phi \in E_s^u$ and for every $g \in F$, we shall denote by $\Phi \otimes g$: $E_s \to F$ the operator defined by $\Phi \otimes g(p) := \Phi(p)g$, $p \in E_s$. One deduces, by applying Proposition 2.10, that for every operator $\Phi \otimes g$ there is a function U: $Y \times \mathbf{Q} \to \mathbf{R}$ such that if $p \in E_s$, then $\Phi \otimes g(p) = \int_Y g(x)U(y, p(y))d\nu$ and obviously the function $x \to \int_Y |g(x)U(y, p(y))|d\nu = |g(x)| \int_Y |U(y, p(y))|d\nu$ is μ-a.e. finite.

Definition 2.11. An operator T: $E_s \to F$ is called an *integral operator of finite rank* if $T = \sum_{i=1}^n \Phi_i \otimes g_i$ where $\Phi_i \in E_s^u$ and $g_i \in F$ for $i = 1, \ldots, n$. We shall denote by $E_s^u \otimes F$ the set of all integral operators of finite rank.

Theorem 2.12. *The band of order bounded integral operators from E_s into F is the band in $\mathscr{U}(E_s, F)$ generated by $E_s^u \otimes F$.*

Proof. Since every element of $E_s^u \otimes F$ is an integral operator, by Theorem 2.8 every element of $(E_s^u \otimes F)^{dd}$ is so.

Conversely, assume that T: $E_s \to F$ is an order bounded integral operator and let U be its kernel. We have to show that $T \in (E_s^u \otimes F)^{dd}$ and to do this there is no restriction in assuming that T is positive.

Denote by A the carrier of F. Since $T(p) \in F$ for all $p \in E_s$, $T(p)\mathbf{1}_{X-A} = 0$ for all $p \in E_s$. Consequently, $U(x, y, t)\mathbf{1}_{X-A}(x) = 0$ $\mu \times \nu$-a.e. for all $t \in \mathbf{Q}$. Hence we may assume that X is the carrier of F. The carrier of E is Y by hypothesis, so there are increasing sequences $(X_n)_{n=1}^\infty$ and $(Y_n)_{n=1}^\infty$ such that $X = \bigcup_{n=1}^\infty X_n$, $Y = \bigcup_{n=1}^\infty Y_n$, $\mathbf{1}_{X_n} \in F$, $\mathbf{1}_{Y_n} \in E$ and $\nu(Y_n) < \infty$ for all $n \in \mathbf{N}$. For each $n \in \mathbf{N}$ we define the operator

$$(S_n p)(x) := \int_Y n(|p(y)| \wedge n)\mathbf{1}_{X_n \times Y_n}(x, y)d\nu \quad \text{for } p \in E_s.$$

It is immediate that $S_n \in E_s^u \otimes F$ for all $n \in \mathbf{N}$. Define $Z_n^t := \{(x, y) \in X_n \times Y_n | U(x, y, t) \leqslant n(|t| \wedge n\}$ for $n \in \mathbf{N}$ and $t \in \mathbf{Q}$. Let us see that $X \times Y = Z_n^t$ except for a $\mu \times \nu$-null set, keeping $t \in \mathbf{Q}$ fixed. Consider $(x, y) \in X \times Y$ such that $U(x, y, t)$ is finite. On the one hand, there is $n_1 \in \mathbf{N}$ such that $U(x, y, t) \leqslant n_1(|t| \wedge n_1)$; on the other hand, there is $n_2 \in \mathbf{N}$ such that $(x, y) \in X_{n_2} \times Y_{n_2}$ since $X \times Y = \bigcup_{n=1}^\infty X_n \times Y_n$. Hence, if $n = \max\{n_1, n_2\}$, then $(x, y) \in Z_n^t$.

Define $U_n(x, y, t) := U(x, y, t)\mathbf{1}_{Z_n^t}(x, y) + n(|t| \wedge n)\mathbf{1}_{X_n \times Y_n - Z_n^t}(x, y)$ for $(x, y, t) \in X \times Y \times \mathbf{Q}$ and $n \in \mathbf{N}$. It follows that

(a) $U_n(x, y, t) \uparrow U(x, y, t)$ $\mu \times \nu$-a.e. for all $t \in \mathbf{Q}$,

(b) $U_n(x, y, t) \leqslant n(|t| \wedge n)\mathbf{1}_{X_n \times Y_n}(x, y)$ for all $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$, let T_n: $E_s \to F$ be the operator defined by

$$(T_n p)(x) := \int_Y U_n(x, y, p(y))d\nu \quad \text{for } p \in E_s.$$

Applying (a) we deduce that $T_n \uparrow T$ in $\mathscr{U}(E_s, F)$. It follows from (b) that $0 \leqslant T_n \leqslant S_n$ and so $T_n \in (E_s^u \otimes F)^{dd}$ for all $n \in \mathbf{N}$. Therefore, $T \in (E_s^u \otimes F)^{dd}$ and the result is completely proved. ∎

2.3. *Characterizations of order bounded integral operators by sequences.* After Theorem 2.12, we are ready to prove the other main result of this section. Theorem 2.13 gives a Bukhvalov type criterion characterizing general order bounded integral operators acting on E_s. It is the first stage to obtain a condition for general Urysohn operators acting on E.

Theorem 2.13. *Let T: $E_s \to F$ be an abstract Urysohn operator. The following statements are equivalent.*

(1) *T is an integral operator.*

(2) *Let $(B_n)_{n=1}^\infty$ be a sequence of ν-measurable subsets of Y such that $\mathbf{1}_{B_n} \in E$ for all $n \in \mathbf{N}$. If the sequence $(\mathbf{1}_{B_n})_{n=1}^\infty$ is order bounded and $\mathbf{1}_{B_n} \to 0$ $(*)$, then $T(t\mathbf{1}_{B_n})(x) \to 0$ μ-a.e. for all $t \in \mathbf{Q}$.*

(3) (i) *Let $(B_n)_{n=1}^\infty$ be a sequence of ν-measurable subsets of Y such that $\nu(\bigcup_{n=1}^\infty B_n) < \infty$ and $\mathbf{1}_{B_n} \in E$ for all $n \in \mathbf{N}$. If $\nu(B_n) \to 0$, then $T(t\mathbf{1}_{B_n})(x) \to 0$ μ-a.e. for all $t \in \mathbf{Q}$.*

(ii) *T is disjointly σ-continuous.*

Proof. (1)$\Rightarrow$(2). Let U be the kernel of T. Consider an order bounded sequence $(\mathbf{1}_{B_n})_{n=1}^\infty$ of characteristic functions in E such that $\mathbf{1}_{B_n} \to 0$ $(*)$ and fix $t \in \mathbf{Q}$. Let $\mathbf{1}_B \in E$ such that $\mathbf{1}_{B_n} \leqslant \mathbf{1}_B$ for all $n \in \mathbf{N}$. Fix $x \in X$. It follows from $\mathbf{1}_{B_n} \to 0$ $(*)$ that $U(x, y, t\mathbf{1}_{B_n}(y)) = \mathbf{1}_{B_n}(y)U(x, y, t) \to 0$ $(*)$. On the other hand, $|U(x, \cdot, t\mathbf{1}_{B_n}(\cdot))| \leqslant |U(x, \cdot, t\mathbf{1}_B(\cdot))|$ for all $n \in \mathbf{N}$.

Define $A := \{x \in X | U(x, \cdot, t\mathbf{1}_B(\cdot))$ is not ν-integrable$\}$. Clearly $\mu(A) = 0$. By the dominated convergence theorem, for every $x \notin A$

$$T(t\mathbf{1}_{B_n})(x) = \int_Y U(x, y, t\mathbf{1}_{B_n}(y))d\nu \to 0.$$

(2)$\Rightarrow$(3). This is straightforward.

(3)$\Rightarrow$(1). Let $T \in \mathscr{U}(E_s, F)$. Then as a consequence of Theorem 2.2, $T = T_1 + T_2$ where $T_1 \in (E_s^u \otimes F)^d$ and $T_2 \in (E_s^u \otimes F)^{dd}$. Assume that T satisfies (3). The result follows from $T_1 = 0$ by applying Theorem 2.12. Since the disjointly σ-continuous operators form a band [14, Theorem 3.8], T_1 and T_2 are both disjointly σ-continuous. On the other hand, T_2 is an integral operator by Theorem 2.12 and so satisfies (3)(i). Hence so does $T_1 = T - T_2$. Thus, it is enough to prove that if $T \in (E_s^u \otimes F)^d$ and T satisfies (3), then $T = 0$. To do this, we shall see that $T(t\mathbf{1}_B) = 0$ for all ν-measurable sets $B \subset Y$ such that $\mathbf{1}_B \in E$ and for all $t \in \mathbf{Q}$. We may assume $0 < \nu(B) < \infty$ because of the disjoint σ-continuity of T.

Let A be a μ-measurable set such that $\mathbf{1}_A \in F$. Define the operator S: $E_s \to F$ by $(Sp)(x) := \mathbf{1}_A(x)\int_B |p(y)|d\nu$. Since $S \in E_s^u \otimes F$, one has $|T| \wedge S = 0$. Applying

Theorem 2.2 and following the steps of the linear case (see [3] and [29, Theorem 94.5]) yields that $T(t\mathbf{1}_B)(x)=0$ μ-a.e. on A. Since A is an arbitrary set such that $\mathbf{1}_A\in F$ it follows that $T(t\mathbf{1}_B)(x)=0$ μ-a.e. on the carrier of F. Now it is obvious that $T(t\mathbf{1}_B)(x)=0$ μ-a.e. on X since $T(t\mathbf{1}_B)\in F$. ∎

We remark that A. V. Bukhvalov has used in [4] another method to prove a similar result to Theorem 2.13. His smart approach characterizes nonlinear integral operators on the set of simple functions applying the corresponding characterization for linear operators. Unfortunately, this direct proof gives little information about the lattice structure of the order bounded integral operators and Theorem 2.12 cannot be derived.

3. Extension of kernels defined on E_s. In this section we consider an abstract Urysohn operator $T: E\to F$ such that the restriction $T|_{E_s}$ is an integral operator. That is, there is a function $U: X\times Y\times \mathbf{Q}\to\mathbf{R}$ satisfying

(a) $U(x, y, 0)=0$ for $\mu\times\nu$-almost all $(x, y)\in X\times Y$,
(b) $U(\cdot,\cdot,t)$ is $\mu\times\nu$-measurable for all $t\in\mathbf{Q}$,

and such that for every $p\in E_s$

(i) $x\to\int_Y|U(x, y, p(y))|d\nu$ is μ-a.e. finite,
(ii) $Tp(x)=\int_Y U(x, y, p(y))d\nu$ μ-a.e.

The purpose of this section is to extend U to a function U': $X\times Y\times\mathbf{R}\to\mathbf{R}$ satisfying the Carathéodory conditions. To be more precise, we shall show that $U(x, y, \cdot)$ is uniformly continuous on bounded subsets of $\mathbf{Q}$ for $\mu\times\nu$-almost all $(x, y)\in X\times Y$. This will be deduced from the following condition.

Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be order bounded sequences in E. If $f_n(y)-g_n(y)\to 0$ ν-a.e., then $Tf_n(x)-Tg_n(x)\to 0$ μ-a.e.

THEOREM 3.1. *Let T: $E\to F$ be an abstract Urysohn operator such that the restriction $T|_{E_s}$ is an integral operator with kernel U. Assume that for any order bounded sequences $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ in E, it follows from $f_n(y)-g_n(y)\to 0$ ν-a.e. that $Tf_n(x)-Tg_n(x)\to 0$ μ-a.e. Then, for $\mu\times\nu$-almost all $(x, y)\in X\times Y$, $U(x, y, \cdot)$ is uniformly continuous on bounded subsets of $\mathbf{Q}$.*

Proof. Since Y is the carrier of E and the measures are σ-finite, we may suppose without loss of generality that $\nu(Y)<\infty$, $\mathbf{1}_Y\in E$ and $\mu(X)<\infty$.

We shall see that given $s\in\mathbf{Q}$ with $s\geqslant 0$, a $\mu\times\nu$-null set Z_s can be found such that if $(x, y)\notin Z_s$, then $U(x, y, \cdot)$ is uniformly continuous on $[-s, s]$. The desired result follows by defining $Z:=\bigcup\{Z_s|s\in\mathbf{Q}\}$ since $\mu\times\nu(Z)=0$ and if $(x, y)\notin Z$, then $U(x, y, \cdot)$ is uniformly continuous on every bounded subset of $\mathbf{Q}$.

Fix $s\in\mathbf{Q}$ with $s\geqslant 0$. Since $T|_{E_s}$: $E_s\to F$ is order bounded and since $s\mathbf{1}_Y\in E_s$, by Proposition 2.7 there is a positive $\mu\times\nu$-measurable function M_s: $X\times Y\to\mathbf{R}$ satisfying

(i) if $p\in[-s\mathbf{1}_Y, s\mathbf{1}_Y]\cap E_s$, then $|U(x, y, p(y))|\leqslant M_s(x, y)$ $\mu\times\nu$-a.e.,
(ii) $M_s(x,\cdot)$ is ν-integrable for almost all $x\in X$,
(iii) $h(\cdot):=\int_Y M_s(\cdot, y)d\nu$ lies in F.

There is an increasing sequence $(X_m)_{m=1}^\infty$ such that $X=\bigcup_{m=1}^\infty X_m$ and h is μ-integrable over each X_m. It is enough to prove that for each $m\in\mathbf{N}$, $U(x, y, \cdot)$ is uniformly continuous on $[-s, s]$ for $\mu\times\nu$-almost all $(x, y)\in X_m\times Y$. So we shall assume that h is μ-integrable and consequently M_s is $\mu\times\nu$-integrable by Tonelli–Hobson's theorem.

The proof will be divided into several stages.

1. Consider the space E_s endowed with the norm

$$\Big\|\sum_{i=1}^n t_i\mathbf{1}_{B_i}\Big\|_\infty=\max_{1\leqslant i\leqslant n}|t_i|$$

and define a map Φ by

$$\Phi p:=\int_{X\times Y} U(x, y, p(y))d\mu\times\nu$$

for $p\in E_s$ with $|p|\leqslant s\mathbf{1}_Y$. This map is well defined since $|U(x, y, p(y))|\leqslant M_s(x, y)$ $\mu\times\nu$-a.e.

We shall show that Φ is uniformly continuous. To do this, we have to prove that for any sequences $(p_n)_{n=1}^\infty$ and $(q_n)_{n=1}^\infty$ in E_s with $|p_n|, |q_n|\leqslant s\mathbf{1}_Y$ for all $n\in\mathbf{N}$, it follows from $\lim_{n\to\infty}\|p_n-q_n\|_\infty=0$ that $\lim_{n\to\infty}(\Phi p_n-\Phi q_n)=0$.

Obviously, $|Tp_n(x)-Tq_n(x)|\leqslant 2h(x)$ μ-a.e. for all $n\in\mathbf{N}$. On the other hand, it is evident that $\|p_n-q_n\|_\infty\to 0$ implies $p_n(y)-q_n(y)\to 0$ ν-a.e. and consequently $Tp_n(x)-Tq_n(x)\to 0$ μ-a.e. By the dominated convergence theorem

$$\int_X[Tp_n(x)-Tq_n(x)]d\mu\to 0.$$

Hence, $\Phi p_n-\Phi q_n\to 0$.

2. Let $E_s^*:=\{\sum_{i=1}^n t_i\mathbf{1}_{Z_i}|Z_i\subset X\times Y \text{ and } t_i\in\mathbf{Q} \text{ for } i=1,\ldots,n\}$ and consider the norm

$$\Big\|\sum_{i=1}^n t_i\mathbf{1}_{Z_i}\Big\|_\infty=\max_{1\leqslant i\leqslant n}|t_i|$$

on it. Recall that in Proposition 2.7 it was proved that if $p^*\in E_s^*$ and $|p^*|\leqslant s\mathbf{1}_{X\times Y}$, then $|U(x, y, p^*(x, y))|\leqslant M_s(x, y)$ $\mu\times\nu$-a.e. So the map Φ may be extended to

$$\Phi^*p^*:=\int_{X\times Y} U(x, y, p^*(x, y))d\mu\times\nu$$

for $p^*\in E_s^*$ with $|p^*|\leqslant s\mathbf{1}_{X\times Y}$.

We shall follow the same procedure as in Proposition 2.7 to prove that Φ^* is uniformly continuous. By the above, for every $\varepsilon>0$ there is $\delta>0$ such that $p, q\in E_s$, $|p|, |q|\leqslant s\mathbf{1}_Y$ and $\|p-q\|_\infty<\delta$ imply $|\Phi p-\Phi q|\leqslant\varepsilon$. We shall see that if $p^*, q^*\in E_s^*$ with $|p^*|, |q^*|\leqslant s\mathbf{1}_{X\times Y}$ and $\|p^*-q^*\|_\infty<\delta$, then $|\Phi^*p^*-\Phi^*q^*|$

$\leqslant \varepsilon$. We may assume that $p^* = \sum_{i=1}^n t_i \mathbf{1}_{Z_i}$ and $q^* = \sum_{i=1}^n t_i' \mathbf{1}_{Z_i}$ with the $\mu\times\nu$-measurable sets $(Z_i)_{i=1}^n$ pairwise disjoint.

First assume that each Z_i is a finite union of generalized rectangles. Then $p^* = \sum_{a=1}^r \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{A_a\times B_{ab}}$ and $q^* = \sum_{a=1}^r \sum_{b=1}^{l(a)} \beta_{ab} \mathbf{1}_{A_a\times B_{ab}}$ with $|\alpha_{ab}|, |\beta_{ab}| \leqslant s$ for $b = 1, \ldots, l(a)$ and $a = 1, \ldots, r$, the μ-measurable sets $(A_a)_{a=1}^r$ being mutually disjoint and, for each $a = 1, \ldots, r$, the ν-measurable sets $(B_{ab})_{b=1}^{l(a)}$ being mutually disjoint. So,

$$\Big\| \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{B_{ab}} - \sum_{b=1}^{l(a)} \beta_{ab} \mathbf{1}_{B_{ab}} \Big\|_\infty < \delta$$

for every $a = 1, \ldots, r$. It follows that, for each a,

$$\Big| \int_{X\times Y} \Big[U\Big(x, y, \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{B_{ab}}(y)\Big) - U\Big(x, y, \sum_{b=1}^{l(a)} \beta_{ab} \mathbf{1}_{B_{ab}}(y)\Big)\Big]\, d\mu\times\nu \Big|$$
$$= \Big|\Phi\Big(\sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{B_{ab}}\Big) - \Phi\Big(\sum_{b=1}^{l(a)} \beta_{ab} \mathbf{1}_{B_{ab}}\Big)\Big| \leqslant \varepsilon.$$

Hence,

$$|\Phi^* p^* - \Phi^* q^*| = \Big| \int_{X\times Y} \Big[\sum_{a=1}^r \mathbf{1}_{A_a}(x)\Big]$$
$$\times \Big[U\Big(x, y, \sum_{b=1}^{l(a)} \alpha_{ab} \mathbf{1}_{B_{ab}}(y)\Big) - U\Big(x, y, \sum_{b=1}^{l(a)} \beta_{ab} \mathbf{1}_{B_{ab}}(y)\Big)\Big]\, d\mu\times\nu \Big| \leqslant \varepsilon.$$

Now consider arbitrary sets $(Z_i)_{i=1}^n$. Since M_s is $\mu\times\nu$-integrable, for every $\eta > 0$ there is $\zeta > 0$ such that if W is $\mu\times\nu$-measurable with $\mu\times\nu(W) < \zeta$, then $\int_W M_s(x, y)\, d\mu\times\nu < \eta/(2n)$. Given $\zeta > 0$, one may inductively define a finite sequence $(W_i)_{i=1}^n$ of pairwise disjoint sets such that each W_i is a finite union of generalized rectangles and $\mu\times\nu(W_i \triangle Z_i) < \zeta$. It follows that

$$\int_{W_i \triangle Z_i} U(x, y, t_i)\, d\mu\times\nu < \eta/(2n)$$

for $i = 1, \ldots, n$ and consequently $|\Phi^*(\sum_{i=1}^n t_i \mathbf{1}_{Z_i}) - \Phi^*(\sum_{i=1}^n t_i \mathbf{1}_{W_i})| \leqslant \eta/2$. Similarly $|\Phi^*(\sum_{i=1}^n t_i' \mathbf{1}_{Z_i}) - \Phi^*(\sum_{i=1}^n t_i' \mathbf{1}_{W_i})| \leqslant \eta/2$. On the other hand, we have already proved that $|\Phi^*(\sum_{i=1}^n t_i \mathbf{1}_{W_i}) - \Phi^*(\sum_{i=1}^n t_i' \mathbf{1}_{W_i})| \leqslant \varepsilon$. Thus, $|\Phi^* p^* - \Phi^* q^*| \leqslant \varepsilon + \eta$. Since $\eta > 0$ is arbitrary, $|\Phi^* p^* - \Phi^* q^*| \leqslant \varepsilon$ as desired.

3. For every $\mu\times\nu$-measurable set $Z \subset X\times Y$ and every $\delta > 0$ define

$$\omega(Z, \delta, s) := \sup\Big\{\int_Z |U(x, y, t) - U(x, y, t')|\, d\mu\times\nu \,\Big|\, t, t' \in \mathbf{Q},\ |t - t'| < \delta \text{ and } |t|, |t'| \leqslant s\Big\}$$

and for every $\delta > 0$ define

$$\omega(\delta, s) := \sup\Big\{\sum_{i=1}^n \omega(Z_i, \delta, s) \,\Big|\, Z_i \cap Z_j = \emptyset \text{ whenever } i \neq j, \text{ and } X\times Y = \bigcup_{i=1}^n Z_i\Big\}.$$

Now the proof can follow the arguments used by Drewnowski and Orlicz in the case of functionals. By [5, 2.1.2] $\lim_{\delta\to 0^+} \omega(\delta, s) = 0$ and then the required result follows from [5, 2.1.3]. ∎

4. Theorems on integral representation. In this section we show conditions for integral representation. We begin by establishing a condition for Urysohn operators in terms of sequences. In Theorem 4.2 we establish another characterization, which is not intrinsic with respect to the operator. We remark that, on account of [14, Proposition 5.3], the representation is essentially unique. We conclude by noting how to apply the main results to the problem of replacing nonmeasurable kernels by measurable ones.

THEOREM 4.1. *Let $T\colon E \to F$ be an abstract Urysohn operator. The following conditions are equivalent.*

(1) *T is an Urysohn operator.*

(2) *Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be order bounded sequences in E. If $f_n - g_n \to 0$ $(*)$, then $Tf_n(x) - Tg_n(x) \to 0$ μ-a.e.*

(3) (a) *For every order bounded sequence $(\mathbf{1}_{B_n})_{n=1}^\infty$ of characteristic functions in E, $\mathbf{1}_{B_n} \to 0$ $(*)$ implies $T(t\mathbf{1}_{B_n})(x) \to 0$ μ-a.e. for all $t \in \mathbf{Q}$.*

(b) *If $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ are order bounded sequences in E such that $f_n(y) - g_n(y) \to 0$ ν-a.e., then $Tf_n(x) - Tg_n(x) \to 0$ μ-a.e.*

Proof. (1)⇒(2). Assume that T is an Urysohn operator and let U be its kernel. Consider sequences $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ in $[-g, g] \subset E$ such that $f_n - g_n \to 0$ $(*)$. We shall take $0 \leqslant g(y) < \infty$ for all $y \in Y$. The operator T is order bounded, so given $g \in E^+$ and applying [14, Theorem 6.2] we can find a $\mu\times\nu$-measurable function $M_g\colon X\times Y \to \mathbf{R}$ satisfying

(i) if $f \in [-g, g]$, then $|U(x, y, f(y))| \leqslant M_g(x, y)$ $\mu\times\nu$-a.e.,
(ii) $M_g(x, \cdot)$ is ν-integrable for μ-almost all $x \in X$,
(iii) $x \to \int_Y M_g(x, y)\, d\nu$ belongs to F.

Define the following μ-null sets:

$$A := \{x \in X \mid M_g(x, \cdot) \text{ is not } \nu\text{-integrable}\},$$
$$A' := \{x \in X \mid \text{the set } \{y \in Y \mid U(x, y, \cdot) \text{ is not continuous in } \mathbf{R}\} \text{ has positive } \nu\text{-measure}\},$$
$$A_n := \{x \in X \mid \text{the set } \{y \in Y \mid \text{either } |U(x, y, f_n(y))| > M_g(x, y) \text{ or } |U(x, y, g_n(y))| > M_g(x, y)\} \text{ has positive } \nu\text{-measure}\}.$$

The result follows from

$$\lim_{n\to\infty} \int_Y \big(U(x, y, f_n(y)) - U(x, y, g_n(y))\big)\, d\nu = 0 \quad \text{for all } x \notin A \cup A' \cup \Big(\bigcup_{n=1}^\infty A_n\Big).$$

Fix such an x. Firstly we shall see that $U(x, \cdot, f_n(\cdot)) - U(x, \cdot, g_n(\cdot)) \to 0$ $(*)$; that is, $U(x,\cdot,f_n(\cdot))-U(x,\cdot,g_n(\cdot))\to 0$ in ν-measure on every $Y'\subset Y$ with $\nu(Y')<\infty$. Let $\varepsilon>0$ and consider $B_n^\varepsilon:=\{y\in Y' \mid |U(x,y,f_n(y))-U(x,y,g_n)| \geqslant \varepsilon\}$. We have to prove that $\lim_{n\to\infty}\nu(B_n^\varepsilon)=0$. For every $k\in\mathbf{N}$, put

$$Y_k:=\{y\in Y' \mid t,s\in[-g(y),g(y)],\ |t-s|<1/k \text{ imply } |U(x,y,t)-U(x,y,s)|<\varepsilon\}.$$

We have defined an increasing sequence $(Y_k)_{k=1}^\infty$ of ν-measurable sets such that $Y'=\bigcup_{k=1}^\infty Y_k$ modulo a ν-null set. Indeed, define $B:=\{y\in Y' \mid U(x,y,\cdot)$ is not continuous in $\mathbf{R}\}$. Since $x\notin A'$, B is ν-null. For every $y\in Y'\sim B$, $U(x,y,\cdot)$ is uniformly continuous on $[-g(y),g(y)]$. So, given $\varepsilon>0$ it is possible to find $\delta>0$ such that if $t,s\in[-g(y),g(y)]$, $|t-s|<\delta$, then $|U(x,y,t)-U(x,y,s)|<\varepsilon$. Let $k\in\mathbf{N}$ such that $1/k<\delta$; then $y\in Y_k$. Hence, $Y'\sim B=(\bigcup_{k=1}^\infty Y_k)\sim B$ and consequently $\nu(Y')=\lim_{k\to\infty}\nu(Y_k)$.

For every $\eta>0$ there is $k_0\in\mathbf{N}$ such that $\nu(Y_{k_0})>\nu(Y')-\eta/2$. For each $n\in\mathbf{N}$ let $B_n:=\{y\in Y' \mid |f_n(y)-g_n(y)|\geqslant 1/k_0\}$. It follows from $f_n-g_n\to 0$ in ν-measure that $\lim_{n\to\infty}\nu(B_n)=0$. So, there is $n_0\in\mathbf{N}$ such that $\nu(B_n)<\eta/2$ for all $n\geqslant n_0$. Since $y\in B_n^\varepsilon\cap Y_{k_0}$ implies $y\in B_n$, it follows that

$$\nu(B_n^\varepsilon)\leqslant \nu(Y'\sim Y_{k_0})+\nu(B_n)<\eta \quad \text{for all } n\geqslant n_0.$$

Thus, $\lim_{n\to\infty}\nu(B_n^\varepsilon)=0$ and $U(x,\cdot,f_n(\cdot))-U(x,\cdot,g_n(\cdot))\to 0$ in ν-measure on Y'. On the other hand, it follows from $f_n,g_n\in[-g,g]$ and $x\notin A_n$ that $|U(x,y,f_n(y))-U(x,y,g_n(y))|\leqslant M_g(x,y)$ ν-a.e. Since $x\notin A$, $M_g(x,\cdot)$ is ν-integrable. As a consequence of the dominated convergence theorem we get

$$\lim_{n\to\infty}\int_Y \big(U(x,y,f_n(y))-U(x,y,g_n(y))\big)d\nu=0.$$

(2)$\Rightarrow$(3). This is evident.

(3)$\Rightarrow$(1). Consider the restriction $T|_{E_s}$; by Theorem 2.13, it is an integral operator. Then there is a function $U\colon X\times Y\times\mathbf{Q}\to\mathbf{R}$ satisfying

(i) $U(x,y,0)=0$ for $\mu\times\nu$-almost all $(x,y)\in X\times Y$,
(ii) $U(\cdot,\cdot,t)$ is $\mu\times\nu$-measurable for all $t\in\mathbf{Q}$,

and such that for every $p\in E_s$

(iii) $x\to\int_Y|U(x,y,p(y))|d\nu$ is μ-a.e. finite,
(iv) $Tp(x)=\int_Y U(x,y,p(y))d\nu$ μ-a.e.

By (3)(b) and Theorem 3.1, for $\mu\times\nu$-almost all $(x,y)\in X\times Y$, $U(x,y,\cdot)$ is uniformly continuous on bounded subsets of $\mathbf{Q}$. So it may be extended to a function $U'(x,y,\cdot)$ which is defined and continuous in $\mathbf{R}$. Thus, an extension $U'\colon X\times Y\times\mathbf{R}\to\mathbf{R}$ has been defined satisfying (C_0) and (C_2). To see that (C_1) also holds let $t\in\mathbf{R}$. Then there is a sequence $(t_n)_{n=1}^\infty$ of rational numbers converging to t. Evidently $U(x,y,t_n)\to U'(x,y,t)$ $\mu\times\nu$-a.e. and it follows from the $\mu\times\nu$-measurability of $U(\cdot,\cdot,t_n)$ for all $n\in\mathbf{N}$ that $U'(\cdot,\cdot,t)$ is $\mu\times\nu$-measurable.

Next it will be seen that the function $x\to\int_Y|U'(x,y,f(y))|d\nu$ belongs to F for every $f\in E$. Since $T|_{E_s}$ is order bounded, $|T|_{E_s}|$ is an integral operator with kernel $|U|$ by Proposition 2.6. Note that $|T|_{E_s}|=|T||_{E_s}$ as a consequence of Theorem 2.2. On the other hand, $|T|$ is order bounded, so given $f\in E$ one can find $h\in F$ such that $|T|([-|f|,|f|])\subset[-h,h]$. Let $(p_n)_{n=1}^\infty$ be a sequence in $[-|f|,|f|]\cap E_s$ such that $\lim_{n\to\infty}p_n(y)=f(y)$ for ν-almost all $y\in Y$. Then

$$\lim_{n\to\infty}|U(x,y,p_n(y))|=|U'(x,y,f(y))| \quad \mu\times\nu\text{-a.e.},$$

and

$$\int_Y|U(x,y,p_n(y))|d\nu=|T|p_n(x)\leqslant h(x) \quad \mu\text{-a.e.}$$

By Fatou's lemma

$$\int_Y|U'(x,y,f(y))|d\nu\leqslant\liminf_{n\to\infty}\int_Y|U(x,y,p_n(y))|d\nu\leqslant h(x) \quad \mu\text{-a.e.}$$

Hence, the function $x\to\int_Y|U'(x,y,f(y))|d\nu$ lies in F.

Now define the operator $S\colon E\to F$ by $Sf(x):=\int_Y U'(x,y,f(y))d\nu$ for $f\in E$. By the above claim the operator S is well defined and order bounded and by the implications already proved it satisfies (3)(b). So T and S both satisfy (3)(b) and $T|_{E_s}=S|_{E_s}$. Therefore, $T=S$ and T is an Urysohn operator. ∎

We next characterize the class of order bounded integral operators in terms of operators of finite rank. We show a similar result to Theorem 2.12 but for operators defined on the whole of E. In the linear case an analogue was proved by Bukhvalov in [2] assuming a condition on the space of order continuous linear functionals. Recently, B. de Pagter [18] has obtained the result without additional hypothesis. Before setting the statement we need some notation. We shall denote by E^u the space of all disjointly σ-continuous abstract Urysohn functionals. Note that $\Phi\in E^u$ implies $\Phi|_{E_s}\in E_s^u$. An abstract Urysohn operator $T\colon E\to F$ is said to be a *disjointly σ-continuous operator of finite rank* if there are $\Phi_i\in E^u$ and $g_i\in F$ for $i=1,\ldots,n$ such that $Tf=\sum_{i=1}^n\Phi_i(f)g_i$ for every $f\in E$. The space of all disjointly σ-continuous operators of finite rank will be denoted by $E^u\otimes F$.

THEOREM 4.2. *Let $T\colon E\to F$ be an abstract Urysohn operator. The following assertions are equivalent.*

(1) *T is an Urysohn operator.*
(2) (a) *T lies in the band in $\mathscr{U}(E,F)$ generated by $E^u\otimes F$.*
(b) *If $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ are order bounded sequences in E, then $f_n(y)-g_n(y)\to 0$ ν-a.e. implies $Tf_n(x)-Tg_n(x)\to 0$ μ-a.e.*

Proof. (1)$\Rightarrow$(2). The argument of Theorem 2.12 shows that (a) holds. (b) is already proved in Theorem 4.1.

(2) ⇒ (1). Let T satisfy (2)(b). We need two simple facts.

(i) *If* $T \in E^u \otimes F$, *then* T *is an Urysohn operator.*

(ii) *If* $0 \leqslant T \leqslant S$ *in* $\mathcal{U}(E, F)$ *and* S *is an Urysohn operator, then so is* T.

Indeed, by Proposition 2.10 in (i) and by Proposition 2.5 in (ii), it is immediate that $T|_{E_s}$ is an integral operator in both cases. Thus T is an Urysohn operator by Theorem 4.1.

Now let $T \in (E^u \otimes F)^{dd}$. Since T is order bounded and hence is the difference of two positive operators, we may assume that $T > 0$. Then there is an increasing net (T_α) of positive operators in the ideal generated by $E^u \otimes F$ such that $T = \sup_\alpha T_\alpha$. As a consequence of (i) and (ii) each T_α is an Urysohn operator. On the other hand, $T_\alpha p \uparrow Tp$ for every $p \in E_s$ and so $T_\alpha|_{E_s} \uparrow T|_{E_s}$ in $\mathcal{U}(E_s, F)$. Since each $T_\alpha|_{E_s}$ is an integral operator, so is $T|_{E_s}$ by Proposition 2.8. Once again by Theorem 4.1, T is an Urysohn operator. ∎

Finally, we consider operators generated by nonmeasurable kernels. Yu. I. Gribanov in [10] proved that under the assumption of separability on the measure space, nonmeasurable kernels can be replaced in the case of linear operators by measurable ones. In [2], A. V. Bukhvalov obtained the full general theorem as an easy consequence of his criterion for integral representation. The same result has been proved with different techniques by W. Schachermayer in [20]. Following Bukhvalov, we might give similar theorems for Urysohn operators. We just state the result corresponding to integral functionals for the sake of convenience. We point out that for functionals the condition of Theorem 4.1 was improved in [7].

THEOREM 4.3. *Let* $\Phi\colon E \to \mathbf{R}$ *and let* $\Omega\colon Y \times \mathbf{R} \to \mathbf{R}$ *be a function with* $\Omega(y, 0) = 0$ *for* ν*-almost all* $y \in Y$ *(but probably not satisfying the Carathéodory conditions). Assume that*

(1) *for every* $f \in E$, $\Omega(\cdot, f(\cdot))$ *is* ν*-integrable,*

(2) *for every order bounded sequence* $(f_n)_{n=1}^\infty$ *in* E, $f_n \to f$ $(*)$ *implies* $\Omega(\cdot, f_n(\cdot)) \to \Omega(\cdot, f(\cdot))$ $(*)$.

(3) $\Phi f = \int_Y \Omega(y, f(y))\,d\nu$.

Then Φ *is an integral functional, that is, there is a function satisfying the Carathéodory conditions which is the kernel of* Φ.

Proof. Recall that every integral functional is order bounded (see [7, Lemma 2.2] or [14, Corollary 6.4]). We remark that in the proof of this fact the Carathéodory conditions are only used to make sure that (1) and (2) hold. So the same argument can be applied now. It follows that Φ is order bounded. Moreover, the proof shows that for every $g \in E^+$ there is a ν-integrable function $M_g\colon Y \to \mathbf{R}$ (actually, $M_g(y) = |\Omega(y, h(y))|$ ν-a.e. for some $h \in [-g, g]$) such that $f \in [-g, g]$ implies $|\Omega(y, f(y))| \leqslant M_g(y)$ ν-a.e.

Consider a sequence $(f_n)_{n=1}^\infty$ in E such that $|f_n| \leqslant g$ for all $n \in \mathbf{N}$ and $f_n \to f$ $(*)$. It follows from (2) that $\Omega(\cdot, f_n(\cdot)) \to \Omega(\cdot, f(\cdot))$ $(*)$. By the dominated convergence theorem

$$\lim_{n\to\infty} \int_Y \Omega(y, f_n(y))\,d\nu = \int_Y \Omega(y, f(y))\,d\nu$$

and so $\lim_{n\to\infty} \Phi f_n = \Phi f$. Hence, by [7, Theorem 3.2], Φ is an integral functional. ∎

We remark that, when $E = M(Y, \nu)$, the condition (2) in the above theorem is equivalent to the following:

(2′) *For every sequence* $(f_n)_{n=1}^\infty$ *in* $M(Y, \nu)$, $f_n \to f$ $(*)$ *implies* $\Omega(\cdot, f_n(\cdot)) \to \Omega(\cdot, f(\cdot))$ $(*)$.

The proof is easy by passing to subsequences and by keeping in mind that every sequence in $M(Y, \nu)$ which converges ν-a.e. is order bounded.

Theorem 4.3 has an important consequence. Recall that a function $N\colon Y \times \mathbf{R} \to \mathbf{R}$ defines a superposition or Nemytskiĭ operator $N\colon M(Y, \nu) \to M(Y, \nu)$ by $(Nf)(y) := N(y, f(y))$ ν-a.e. We note that Theorem 4.3 with the condition (2′) can be used to prove the following: If a Nemytskiĭ operator which is continuous in measure is generated by a nonmeasurable function, then it can also be generated by a function satisfying the Carathéodory conditions. This result was independently proved by several people. L. Drewnowski and W. Orlicz got it [7, Theorem 3.3] as a consequence of their theorem of integral representation. On the other hand, it was proved by I. Vrkoč in [25] for Nemytskiĭ operators acting on $M([0, 1])$ and by A. V. Ponosov in [19] for more general measure spaces. See also [1] for more details about this problem and its connection with the Nemytskiĭ conjecture, as well as for a different proof.

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Inequalities relative to two-parameter Vilenkin–Fourier coefficients

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Abstract. The inequality

$$(*)\qquad \Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(nm)^{p-2}|\hat{f}(n,m)|^p\Big)^{1/p}\leqslant C_p\|f\|_{H_p^-}\qquad (0<p\leqslant 2)$$

and its dual inequality are proved for two-parameter Vilenkin–Fourier coefficients and for two-parameter martingale Hardy spaces H_p^- defined by means of the L^p-norm of the conditional quadratic variation. The inequality $(*)$ is extended to bounded Vilenkin systems and monotone coefficients for all p. The converse of the last inequality is also true for all p. From this it follows easily that under the same conditions the two-parameter Vilenkin–Fourier series of an arbitrary L^p function $(p>1)$ converges a.e. to that function.

1. Introduction. Up to now inequality $(*)$ has been known for one-parameter systems only. The proof for $p=1$ is due to Hardy, and, for the trigonometric system, it can be found e.g. in Coifman–Weiss [9]. For the Walsh system it was proved first by Ladhawala [13] and for another proof see the book [22] written by Schipp, Wade, Simon and Pál. For Vilenkin systems it was proved by Fridli and Simon [11] but for another Hardy space. The inequality for $1<p\leqslant 2$ can be found in Edwards's book [10].

First we establish the results of two-parameter martingale theory that will be used later. Our proof of $(*)$ for $0<p\leqslant 1$ is based on the atomic description of H_p^- (see [27]) and for $1<p\leqslant 2$ it can be obtained by interpolation (see [24]).

In the next section a direct proof of the dual inequality to $(*)$ is given. The analogue to this inequality for the BMO space and for the one-parameter Walsh system can be found in [13] and in [22].

Next $(*)$ will be extended to bounded Vilenkin systems and monotone coefficients for all $p>2$ (for the exact conditions see (10) and (11)). This proof is based on the proof for one-parameter systems given by Móricz in [16]. Under the above-mentioned conditions the converse of the last inequality is also true similarly to [16]; moreover, it is proved that the supremum of the absolute

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