Contents of Volume 99, Number 3

STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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Published by PWN–Polish Scientific Publishers

ISBN 83-01-10521-6   ISSN 0039-3223

PRINTED IN POLAND

W R O C Ł A W S K A   D R U K A R N I A   N A U K O W A

Bounded sets in projective tensor products of hilbertizable locally convex spaces

by

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Abstract. The main results of this paper imply a positive solution of A. Grothendieck's problem of topologies for pairs (E, F) of hilbertizable Fréchet spaces or of complete hilbertizable locally convex spaces with continuous norms and with the property that the closed linear span of any bounded subset is a Fréchet space (w.r.t. the induced topology). Further results concern bounded and compact subsets of the quasi-completions of projective and inductive tensor products and an approximation of bilinear forms defined on $E\times F$ by bilinear forms defined on complemented Hilbert subspaces.

1. Introduction. The "problem of topologies" investigated by A. Grothendieck in [4] is whether the closed convex hulls of tensor products of bounded sets form a fundamental system of bounded subsets in the complete projective tensor product $E\tilde{\otimes}_\pi F$ of locally convex spaces E and F. We will prove here that this is so in the following two cases:

1. E and F are hilbertizable Fréchet spaces.

2. E and F are complete hilbertizable locally convex spaces with continuous norms and with the property that the closed linear span of any bounded subset is a Fréchet space.

In agreement with the terminology in [11, 15], locally convex spaces with the property that the closed linear span of any bounded subset is a Fréchet space (w.r.t. the induced topology) will be termed *quasi-Fréchet spaces* or *(QF)-spaces*. The concept of hilbertizable locally convex spaces is taken from [7]. It coincides with the concept of generalized Hilbert spaces used in [6].

Concerning the case of hilbertizable Fréchet spaces, A. Grothendieck published without proof a more general statement, which may be formulated as follows:

(*) *Suppose that E is a hilbertizable Fréchet space and that F is an arbitrary Fréchet space. Then each bounded subset of $E\tilde{\otimes}_\pi F$ is contained in the canonical image of a bounded subset of $E_W\tilde{\otimes}_\pi F$, where W is a suitably chosen closed bounded disc in E and E_W denotes the linear span of W endowed with the gauge functional of W as a norm.*

And in fact, a proof of (*) will be given here.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 46A05.

It was remarked by J.-P. Jurzak in [9] that the solution of the "problem of topologies" in the case $E = F$ is a domain of an Op*-algebra endowed with the graph topology (cf. [14] for the corresponding definitions) has applications to problems concerning the predual and the dual of an unbounded operator algebra. In fact J.-P. Jurzak used (*) as a theorem. A proof in the case of Fréchet domains of Op*-algebras was given by the present author in [12]. Another special case of (*), the case of E and F being separable hilbertizable Fréchet spaces, was proved by J. Taskinen in [16]. During the preparation of the final version of the present paper, J. Bonet, A. Defant, and A. Galbis [2] gave another proof in the case of E and F being general hilbertizable Fréchet spaces.

The results concerning Fréchet domains of Op*-algebras were generalized by the present author in [11, 13] to quasi-Fréchet domains. This generalization includes standard examples, having inductive limit topology, such as the Schwartz space of C^∞-functions with compact support, direct sums of Hilbert spaces (or, more generally, of quasi-Fréchet domains), or test function algebras of quantum field theory (cf. [3, 17] for the definition of test function algebra). This motivates the consideration of quasi-Fréchet locally convex spaces.

Note that the assumption concerning the existence of continuous norms cannot be omitted because of A. Grothendieck's counterexample

$$\Big(\sum_{n=1}^{\infty} \mathbf{C}\Big)\tilde{\otimes}_\pi\Big(\prod_{n=1}^{\infty} \mathbf{C}\Big).$$

The present paper is organized as follows. Section 2 contains the proof of (*). Section 3 contains the solution of the problem of topologies for pairs of complete hilbertizable (QF)-spaces with continuous norms and a somewhat more general result concerning bounded subsets in a certain space of nuclear operators. Section 4 contains three further applications concerning an approximation of bilinear forms on hilbertizable Fréchet of (QF)-spaces by bilinear forms which may be represented by operators acting between Hilbert spaces, bounded subsets of an inductive tensor product, and compact subsets of a projective tensor product.

2. Proof of Grothendieck's statement. In this section, we prove A. Grothendieck's statement (*) formulated in the introduction.

Since E is a hilbertizable Fréchet space, we can (and do) fix a sequence (E_n) of Hilbert spaces and a sequence (Q_n) of continuous linear operators $Q_n\colon E \to E_n$ such that the sequence of seminorms $(\|Q_n \cdot\|)$ is increasing and defines the topology of E. We also fix an increasing sequence (q_n) of seminorms generating the topology of F.

Let $\mathscr{M}$ be a bounded subset of $E\tilde{\otimes}_\pi F$. Then the values

$$\lambda_n := 1+\sup\{(\|\cdot\|_{E_n}\otimes q_n)((Q_n\otimes \mathrm{Id}_F)(s))\colon\ s\in\mathscr{M}\},$$

are finite. ($p\otimes q$ denotes the projective tensor product of seminorms, Id_F is the identity map on F.) We set

$$\varepsilon_n := 2^{-n}(\lambda_n)^{-1}.$$

Now we consider the linear subspace

$$H_0 := \{(\varepsilon_n Q_n\varphi)_{n=1}^\infty\colon\ \varphi\in E \text{ and } \sum_{n=1}^{\infty}\|\varepsilon_n Q_n\varphi\|^2<\infty\}$$

of the (complete) orthogonal sum of Hilbert spaces

$$H := \sum_{n=1}^{\infty}{}^{\oplus}\, E_n.$$

The following argument shows that H_0 is complete: Given a Cauchy sequence $((\varepsilon_n Q_n\varphi_k)_{n=1}^\infty)_{k=1}^\infty$ in H_0, the sequences $(Q_n\varphi_k)_{k=1}^\infty$ are Cauchy sequences in E_n. This means that (φ_k) is a Cauchy sequence in E. Denoting its limit by φ_0, we have

$$\sum_{n=1}^{\infty}\|\varepsilon_n Q_n(\varphi_k-\varphi_0)\|^2 \leqslant \lim_{l\to\infty}\sum_{n=1}^{\infty}\|\varepsilon_n Q_n(\varphi_k-\varphi_l)\|^2<\infty,$$

$$\lim_{k\to\infty}\sum_{n=1}^{\infty}\|\varepsilon_n Q_n(\varphi_k-\varphi_0)\|^2=0,$$

which shows that $(\varepsilon_n Q_n(\varphi_k-\varphi_0))_{n=1}^\infty$ and consequently also $(\varepsilon_n Q_n\varphi_0)_{n=1}^\infty$ are elements of H_0 and that $(\varepsilon_n Q_n\varphi_0)_{n=1}^\infty$ is the limit of the given Cauchy sequence.

Let P denote the orthogonal projection of H onto H_0, considered as an element of $\mathscr{L}(H, H_0)$ (the space of continuous linear operators mapping H into H_0). Let Y be the linear map of H_0 into E defined by $Y((\varepsilon_n Q_n\varphi)_{n=1}^\infty)=\varphi$. Since

$$\|Q_n\varphi\| \leqslant (\varepsilon_n)^{-1}\|(\varepsilon_n Q_n\varphi)_{n=1}^\infty\|_{H_0},$$

Y is an element of $\mathscr{L}(H_0, E)$. Further, we define the operators

$$J_m\colon\ E_m\ni f\to (f_n)_{n=1}^\infty\in H,$$

where $f_m=f$ and $f_n=0$ if $n\neq m$.

Given $s\in\mathscr{M}$,

$$(\|\cdot\|_H\otimes q_l)((J_nQ_n\otimes\mathrm{Id}_F)(s)) \leqslant (\|Q_n\cdot\|\otimes q_l)(s)\leqslant\lambda_{\max(n,l)}.$$

Consequently,

$$(\|\cdot\|_H\otimes q_l)\Big(\sum_{n=1}^{\infty}\varepsilon_n(J_nQ_n\otimes\mathrm{Id}_F)(s)\Big)\leqslant \lambda_l\sum_{n=1}^{l}\varepsilon_n+\sum_{n=l+1}^{\infty}\varepsilon_n\lambda_n<\infty.$$

This implies that the series $\sum\varepsilon_n(J_nQ_n\otimes\mathrm{Id}_F)(s)$ converges absolutely in $H\tilde{\otimes}_\pi F$ for all $s\in\mathscr{M}$ and that the set

$$\mathscr{R} := \Big\{\sum_{n=1}^{\infty}\varepsilon_n(PJ_nQ_n\otimes\mathrm{Id}_F)(s)\colon\ s\in\mathscr{M}\Big\}$$

is a bounded subset of $H_0\tilde{\otimes}_\pi F$.

Since Y is injective, $Y\otimes \mathrm{Id}_F$ is a topological linear isomorphism of $H_0\tilde{\otimes}_\pi F$ onto $E_W\tilde{\otimes}_\pi F$, where $W\subset E$ is the bounded disc $\{Yf\colon f\in H_0 \text{ and } \|f\|\leqslant 1\}$. Consequently, $(Y\otimes \mathrm{Id}_F)(\mathscr{R})$ is the canonical image of a bounded subset of $E_W\tilde{\otimes}_\pi F$.

Now it suffices to show that

$$(1)\qquad \sum_{n=1}^{\infty} \varepsilon_n(YPJ_nQ_n\otimes \mathrm{Id}_F)(s) = s$$

for all $s\in\mathscr{M}$ (which implies $\mathscr{M}=(Y\otimes \mathrm{Id}_F)(\mathscr{R})$). In order to do this, we use the fact that $E'\otimes F'$ is dense in $(E\tilde{\otimes}_\pi F)'$ w.r.t. the weak topology $\sigma((E\tilde{\otimes}_\pi F)', E\tilde{\otimes}_\pi F)$ because E has the approximation property (see [10], § 43.2, (12)).

Let $g\in E'$, $h\in F'$, and $s\in\mathscr{M}$ be given and let $s=\sum_{k=1}^{\infty}\psi_k\otimes\eta_k$ be a representation of s such that

$$\sum_{k=1}^{\infty}\|Q_n\psi_k\|\, q_l(\eta_k) < \infty$$

for all $n, l\in\mathbf{N}$. There exists $m\in\mathbf{N}$ such that $h(\eta)\leqslant mq_m(\eta)$ for all $\eta\in F$. Setting $\psi_0=\sum_{k=1}^{\infty}h(\eta_k)\psi_k$, we obtain

$$\|Q_n\psi_0\| \leqslant m(\|Q_n\cdot\|\otimes q_m)(s)\leqslant m\lambda_{\max(n,m)}.$$

Consequently, the series $\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_n\psi_0$ converges absolutely in H, and

$$\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_n\psi_0 = (\varepsilon_nQ_n\psi_0)_{n=1}^{\infty}\in H_0,$$

$$YP\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_n\psi_0=\psi_0.$$

Thus we obtain

$$\begin{aligned}(g\otimes h)\Big(\sum_{n=1}^{\infty}\varepsilon_n(YPJ_nQ_n\otimes \mathrm{Id}_F)(s)\Big) &= \sum_{n=1}^{\infty}\varepsilon_n(g\otimes h)\Big(\sum_{k=1}^{\infty}(YPJ_nQ_n\psi_k)\otimes\eta_k\Big)\\ &= \sum_{n=1}^{\infty}\varepsilon_n g(YPJ_nQ_n\psi_0)\\ &= g\Big(YP\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_n\psi_0\Big)=g(\psi_0)=(g\otimes h)(s).\end{aligned}$$

This implies (1) because $E'\otimes F'$ separates the points of $E\tilde{\otimes}_\pi F$. This completes the proof.

3. Bounded sets of nuclear operators. The main result of this section gives a characterization of bounded subsets in a certain locally convex space of nuclear operators which turns out to be a subspace of the complete projective tensor product of a pair of hilbertizable (QF)-spaces with continuous norms.

Before stating this result, we introduce some more notations. Given an absolutely convex 0-neighbourhood U in a locally convex space E, q_U denotes the gauge functional of U, $E_{(U)}$ denotes the quotient space $E/(q_U)^{-1}(0)$ endowed with the norm defined by q_U, and Q_U denotes the canonical quotient map $E\to E_{(U)}$. By a *Hilbert* (resp. *Banach*) *disc* in E we mean a closed bounded subset $M\subset E$ such that the linear span E_M of M admits a structure of a Hilbert (resp. Banach) space with unit ball M. The identical imbedding of the Hilbert (resp. Banach) space E_M into E will be denoted by J_M. The dual and the completion of E will be denoted by E' and $\tilde{E}$, resp. The strong dual, Mackey, and weak topologies will be denoted by β, μ, σ, resp. Given locally convex spaces E and F, $\mathscr{N}(E, F)$ denotes the linear space of all nuclear operators from E into F. The nuclear norm of a nuclear operator T acting between normed spaces will be denoted by $\nu(T)$. The transposed of an operator $T\in\mathscr{L}(E, F)$ will be denoted by T'.

THEOREM 3.1. *Suppose that E and F are hilbertizable (QF)-spaces with continuous norms. Let a subset $\mathscr{M}\subset\mathscr{L}(E'_\beta, F)$ be given such that the operators $Q_VS(Q_U)'$ $(S\in\mathscr{M})$ are nuclear and that*

$$\sup\{\nu(Q_VS(Q_U)')\colon S\in\mathscr{M}\}<\infty$$

for arbitrary absolutely convex 0-neighbourhoods $U\subset E$ and $V\subset F$. Then there exist Hilbert discs $M\subset E$ and $N\subset F$ such that $\mathscr{M}$ is contained in the set

$$(2)\qquad \{J_NR(J_M)'\colon R\in\mathscr{N}((E_M)', F_N) \text{ and } \nu(R)\leqslant 1\}.$$

Proof. Let $\mathfrak{U}$ and $\mathfrak{B}$ be 0-neighbourhood bases in E and F, resp., such that the mappings Q_U and Q_V are injective and that the completions of $E_{(U)}$ and $F_{(V)}$ are Hilbert spaces for all $U\in\mathfrak{U}$ and $V\in\mathfrak{B}$. Let $U_0\in\mathfrak{U}$ and $V_0\in\mathfrak{B}$ be fixed. Note that

$$\|Q_VS(Q_{U_0})'f\|\leqslant\|f\|\sup\{\nu(Q_VS(Q_{U_0})')\colon S\in\mathscr{M}\},$$

which implies that

$$N_0 := \{S(Q_{U_0})'f\colon S\in\mathscr{M},\ f\in(E_{(U_0)})' \text{ and } \|f\|\leqslant 1\}$$

is a bounded subset of F. The closed linear span G of N_0 is a Fréchet space. It follows from injectivity of Q_{U_0} and semireflexivity of E that the range of $(Q_{U_0})'$ is dense in E'_β. This implies further that the range of S is contained in G for all $S\in\mathscr{M}$. Similarly,

$$M_0 := \{S'(Q_{V_0})'g\colon S\in\mathscr{M},\ g\in(F_{(V_0)})' \text{ and } \|g\|\leqslant 1\}$$

is a bounded subset of E and its closed linear span D is a Fréchet space containing $S'(F')$ for all $S\in\mathscr{M}$.

Let (U_n) and (V_n) be sequences in $\mathfrak{U}$ and $\mathfrak{B}$, resp., such that $U_{n+1}\subset U_n$, $V_{n+1}\subset V_n$ and that $U_n\cap D$ and $V_n\cap G$ are 0-neighbourhood bases of D and G, resp. Let E_n and F_n be the completions of $E_{(U_n)}$ and $F_{(V_n)}$, resp. Consider the operators

$$Q_n\colon E\ni\varphi\to Q_{U_n}\varphi\in E_n,\qquad R_n\colon F\ni\psi\to Q_{V_n}\psi\in F_n.$$

Define the constants

$$\lambda_n:=1+\sup\{\nu(R_nS(Q_n)')\colon S\in\mathscr{M}\},\qquad \varepsilon_n:=2^{-n}(\lambda_n)^{-1}.$$

Now we consider the linear subspace

$$H_0:=\{(\varepsilon_nQ_n\varphi)_{n=1}^{\infty}\colon \varphi\in D \text{ and } \sum_{n=1}^{\infty}\|\varepsilon_nQ_n\varphi\|^2<\infty\}$$

of the (complete) orthogonal sum of Hilbert spaces

$$H:=\sum_{n=1}^{\infty}{}^{\oplus}E_n.$$

By the argument already used in Section 2, H_0 is a closed subspace of H. We denote by P the orthogonal projection of H onto H_0, considered as an element of $\mathscr{L}(H,H_0)$. Furthermore, we define the operator

$$Y\colon H_0\ni(\varepsilon_nQ_n\varphi)_{n=1}^{\infty}\to\varphi\in E,$$

which is easily seen to be continuous. It may be written as $Y=J_MY_0$, where M is the Hilbert disc

$$M=\{Yf\colon f\in H_0 \text{ and } \|f\|\leqslant 1\}$$

and Y_0 is an isometry of H_0 onto E_M. Similarly, the orthogonal projection of the Hilbert space

$$K:=\sum_{n=1}^{\infty}{}^{\oplus}F_n$$

onto its closed linear subspace

$$K_0:=\{(\varepsilon_nR_n\psi)_{n=1}^{\infty}\colon \psi\in G \text{ and } \sum_{n=1}^{\infty}\|\varepsilon_nR_n\psi\|^2<\infty\}$$

will be denoted by Q. The operator

$$Z\colon K_0\ni(\varepsilon_nR_n\psi)_{n=1}^{\infty}\to\psi\in F$$

belongs to $\mathscr{L}(K_0,F)$ and may be written as $Z=J_NZ_0$, where N is a Hilbert disc in F and Z_0 is an isometry of K_0 onto F_N. We also consider the operators

$$J_m\colon E_m\ni f\to(f_n)_{n=1}^{\infty}\in H,\qquad I_m\colon F_m\ni f\to(f_n)_{n=1}^{\infty}\in K,$$

where $f_n=0$ if $n\neq m$ and $f_m=f$.

Given $S\in\mathscr{M}$, the operator

$$T_S:=\sum_{n,m=1}^{\infty}\varepsilon_n\varepsilon_mZ_0QI_mR_mS(Q_n)'(J_n)'P'(Y_0)'$$

is nuclear and satisfies $\nu(T_S)\leqslant 1$ because

$$\sum_{n,m=1}^{\infty}\varepsilon_n\varepsilon_m\nu(R_mS(Q_n)')\leqslant\sum_{n,m=1}^{\infty}\varepsilon_n\varepsilon_m\lambda_{\max\{n,m\}}\leqslant\sum_{n,m=1}^{\infty}\varepsilon_n\varepsilon_m\lambda_n\lambda_m\leqslant 1.$$

It suffices now to show that $S=J_NT_S(J_M)'$. To do that, let $f\in E'$ and $h\in(F_1)'$ be given. Set $g=(R_1)'h$. Since the range of S is contained in G and

$$\sum_{m=1}^{\infty}\|\varepsilon_mR_mS(YPJ_nQ_n)'f\|^2\leqslant\sum_{m=1}^{\infty}(\varepsilon_m)^2\|R_mS(Q_n)'\|^2\|(J_M)'f\|^2$$
$$\leqslant\sum_{m=1}^{\infty}(\varepsilon_m)^2(\lambda_{\max\{m,n\}})^2\|(J_M)'f\|^2<\infty,$$

it follows that

$$(\varepsilon_mR_mS(YPJ_nQ_n)'f)_{m=1}^{\infty}=\sum_{m=1}^{\infty}\varepsilon_mI_mR_mS(YPJ_nQ_n)'f$$

belongs to K_0. This implies

$$ZQ\sum_{m=1}^{\infty}\varepsilon_mI_mR_mS(YPJ_nQ_n)'f=S(YPJ_nQ_n)'f.$$

Similarly,

$$\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_nS'g=\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_nS'(R_1)'h$$

belongs to H_0 and

$$YP\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_nS'g=S'g.$$

Thus we obtain

$$g(J_NT_S(J_M)'f)=g\Big(\sum_{n=1}^{\infty}\varepsilon_nZQ\sum_{m=1}^{\infty}\varepsilon_mI_mR_mS(YPJ_nQ_n)'f\Big)$$
$$=g\Big(\sum_{n=1}^{\infty}\varepsilon_nS(YPJ_nQ_n)'f\Big)$$
$$=f\Big(YP\sum_{n=1}^{\infty}\varepsilon_nJ_nQ_nS'g\Big)=f(S'g)=g(Sf).$$

Since this equality is satisfied for all f in E' and g in the range of $(R_1)'$ (which is dense in F'_β), it follows that $J_NT_S(J_M)'=S$. This completes the proof.

Remarks. 1. Theorem 3.1 is also valid for hilbertizable Fréchet spaces E and F (with or without continuous norms). This can be seen, e.g., by starting with decreasing sequences (U_n) and (V_n) of 0-neighbourhoods defining the topology of E and F, resp., such that the completions of $E_{(U_n)}$ and $F_{(V_n)}$ are Hilbert spaces and by applying then similar arguments to the proof of Theorem 3.1.

2. The seminorms

$$\mathcal{N}(E'_\beta, F) \ni S \to (Q_V S (Q_U)'),$$

where $U \subset E$ and $V \subset F$ are absolutely convex 0-neighbourhoods, define a locally convex topology on $\mathcal{N}(E'_\beta, F)$, which will be denoted by τ_π. Theorem 3.1 means in particular that the sets (2) form a fundamental system of bounded subsets in $\mathcal{N}(E'_\beta, F)[\tau_\pi]$.

3. In the special case when E has dimension one, one obtains the well-known result ([8], 6.5.6, [6], 1.9) that the Hilbert discs form a fundamental system of bounded subsets in F.

It is well-known that spaces of nuclear operators are isomorphic to subspaces of projective tensor products under suitable assumptions. Before stating a version of this fact sufficient for our purposes, we define a subspace $E\overline{\otimes}_\pi F$ of the complete projective tensor product by

$$E\overline{\otimes}_\pi F := \{\sum_{n=1}^{\infty} c_n \varphi_n \otimes \psi_n\colon (c_n) \in l_1 \text{ and } (\varphi_n), (\psi_n) \text{ are 0-sequences in } E \text{ and } F, \text{ resp.}\}.$$

PROPOSITION 3.2. *Suppose that the locally convex spaces E and F satisfy the following three conditions*:

(a) *E is semireflexive.*

(b) *Each 0-sequence in F is contained in a Banach disc.*

(c) *E or F is a projective limit of normed spaces with approximation property.*

Then the canonical map $E \otimes F \to \mathcal{L}(E'_\beta, F)$ extends to a topological linear isomorphism χ of $E\overline{\otimes}_\pi F$ onto $\mathcal{N}(E'_\beta, F)[\tau_\pi]$.

Proof. Assumption (c) implies that the canonical mapping $E\tilde{\otimes}_\pi F \to \mathcal{L}((\tilde{E})'_\beta, \tilde{F})$ is injective (see [7], 18.3.7 and 16.1.5). Assumptions (a) and (b) imply that the elements of $\mathcal{N}(E'_\beta, F)$ are exactly the operators

$$E' \ni f \to \sum_{n=1}^{\infty} c_n f(\varphi_n)\psi_n,$$

where $(c_n) \in l_1$ and (φ_n) and (ψ_n) are 0-sequences in E and F, resp. This establishes the algebraic isomorphism

$$\chi\colon E\overline{\otimes}_\pi F \to \mathcal{N}(E'_\beta, F)[\tau_\pi].$$

In order to show that χ is a topological isomorphism, it suffices to establish that

$$(q_U \otimes q_V)(t) = \nu(Q_V \chi(t)(Q_U)') \quad (t \in E \otimes F)$$

for absolutely convex 0-neighbourhoods $U \subset E$ and $V \subset F$ such that $E_{(U)}$ or $F_{(V)}$ has the approximation property. But this follows from 15.4.4 in [7] and from the isometry $E_{(U)}\tilde{\otimes}_\pi F_{(V)} \cong \mathcal{N}((E_{(U)})', \widetilde{F_{(V)}})$.

To finish this section, we consider the case of complete (QF)-spaces with continuous norms. It turns out that $\mathcal{N}(E'_\beta, F)$ is isomorphic to $E\tilde{\otimes}_\pi F$ and that A. Grothendieck's problem of topologies has a positive solution in this case.

PROPOSITION 3.3. *Suppose that E and F are complete hilbertizable (QF)-spaces with continuous norms. Then $\mathcal{N}(E'_\beta, F)[\tau_\pi]$ is complete.*

Proof. Let $(S_\delta)_{\delta\in\Delta}$ be a Cauchy net in $\mathcal{N}(E'_\beta, F)[\tau_\pi]$. Then, given $f \in E'$ and $g \in F'$, the nets $(S_\delta f)_{\delta\in\Delta}$ and $((S_\delta)'g)_{\delta\in\Delta}$ are Cauchy nets in F and E $(=(E'_\beta)')$, resp. Thus we can define linear operators $S\colon E' \to F$ and $T\colon F' \to E$ by setting

$$Sf = \lim_\delta S_\delta f, \quad Tg = \lim_\delta (S_\delta)' g.$$

These operators satisfy $g(Sf) = f(Tg)$ for all $f \in E'$ and $g \in F'$. Consequently, S is weakly continuous. Using semireflexivity of E and F, we conclude that $S \in \mathcal{L}(E'_\beta, F)$.

Also, given absolutely convex 0-neighbourhoods $U \subset E$ and $V \subset F$, the net $(Q_V S_\delta (Q_U)')_{\delta\in\Delta}$ is a Cauchy net w.r.t. the nuclear norm. We regard Q_V as an operator mapping F into the completion $\widetilde{F_{(V)}}$ of $F_{(V)}$. Thus there exists $S_{U,V} \in \mathcal{N}((E_{(U)})', \widetilde{F_{(V)}})$ such that

(3) $$\lim_\delta \nu(Q_V S_\delta (Q_U)' - S_{U,V}) = 0.$$

It is easy to see that $S_{U,V} = Q_V S (Q_U)'$. The set $\mathcal{M} = \{S\}$ satisfies, therefore, the assumptions of Theorem 3.1. Applying this theorem, we see that $S \in \mathcal{N}(E'_\beta, F)$. Finally, (3) implies that S is the limit of the given Cauchy net. This completes the proof.

Combining Propositions 3.2 and 3.3, we see that $E\tilde{\otimes}_\pi F$ is isomorphic to $\mathcal{N}(E'_\beta, F)[\tau_\pi]$ if E and F are complete hilbertizable (QF)-spaces with continuous norms. Applying Theorem 3.1 and Remark 2 after it, we obtain the following result.

COROLLARY 3.4. *Suppose that E and F are complete hilbertizable (QF)-spaces with continuous norms. Then the canonical images of unit balls of the spaces $E_M\tilde{\otimes}_\pi F_N$, where $M \subset E$ and $N \subset F$ are Hilbert discs, form a fundamental system of bounded subsets in $E\tilde{\otimes}_\pi F$.*

4. Some applications. Here we apply results of the previous section to obtain three further results. The first one concerns an approximation of elements of $\mathcal{L}(E, F'_\beta)$ (and, more generally, of bibounded bilinear forms on $E \times F$) by operators acting between Hilbert spaces. The second one concerns bounded subsets of the quasi-completion of the inductive tensor product

$E_\mu \otimes_\iota F_\mu$. The third one consists in a description of compact subsets of $E \overline{\otimes}_\pi F$.

Given locally convex spaces E and F, let $\mathscr{B}(E, F)$ denote the space of bilinear forms γ on $E \times F$ which satisfy

(4) $$p_{M,N}(\gamma) := \sup\{|\gamma(\varphi, \psi)|:\ \varphi \in M,\ \psi \in N\} < \infty$$

for all bounded subsets $M \subset E$ and $N \subset F$. The topology of bibounded convergence on $\mathscr{B}(E, F)$ (generated by the seminorms (4)) will be denoted by τ_b. $\mathscr{B}_{jc}(E, F)$ denotes the subspace of all jointly continuous bilinear forms. If we identify $T \in \mathscr{L}(E, F'_\beta)$ with the bilinear form γ_T: $(\varphi, \psi) \to (T\varphi)(\psi)$, $\mathscr{L}(E, F'_\beta)$ becomes a linear subspace of $\mathscr{B}(E, F)$ which contains $\mathscr{B}_{jc}(E, F)$ and τ_b induces on $\mathscr{L}(E, F'_\beta)$ the topology of uniform convergence on bounded sets. We will use notations such as $\gamma(A\cdot, B\cdot)$ for the bilinear form $(\varphi, \psi) \to \gamma(A\varphi, B\psi)$.

THEOREM 4.1. *Suppose that both E and F are hilbertizable (QF)-spaces with continuous norms or that both E and F are hilbertizable Fréchet spaces. Let a bounded subset $\mathscr{M} \subset \mathscr{B}(E, F)[\tau_b]$ and a τ_b-continuous seminorm p on $\mathscr{B}(E, F)$ be given. Then there are projections $P \in \mathscr{L}(E, E)$ and $Q \in \mathscr{L}(F, F)$ such that*

(5) $$\sup\{p(\gamma(\cdot,\cdot) - \gamma(P\cdot, Q\cdot)):\ \gamma \in \mathscr{M}\} \leqslant 1$$

and such that the ranges of P and Q are isomorphic (as topological linear spaces) to Hilbert spaces.

Proof. Let Hilbert discs $M \subset E$ and $N \subset F$ be chosen such that

$$p(\gamma) \leqslant \sup\{|\gamma(\varphi, \psi)|:\ \varphi \in M,\ \psi \in N\} = \|\gamma(J_M\cdot, J_N\cdot)\|.$$

Let D and G be the closed linear spans of M and N in E and F, resp. Since D and G are Fréchet spaces, the set of bilinear forms

$$D \times G \ni (\varphi, \psi) \to \gamma(\varphi, \psi) \quad (\gamma \in \mathscr{M})$$

is uniformly continuous, i.e., there are absolutely convex 0-neighbourhoods $U \subset E$ and $V \subset F$ such that

$$|\gamma(\varphi, \psi)| \leqslant \|Q_U \varphi\|\,\|Q_V \psi\|$$

for all $\gamma \in \mathscr{M}$, $\varphi \in D$, and $\psi \in G$. Moreover, U and V may be chosen such that the completions $\widetilde{E_{(U)}}$ and $\widetilde{F_{(V)}}$ of $E_{(U)}$ and $F_{(V)}$ are Hilbert spaces. We regard Q_U as an element of $\mathscr{L}(E, \widetilde{E_{(U)}})$. Then the operator $A := Q_U J_M$ acts between Hilbert spaces. We will use its polar decomposition $A = W|A|$ (where $|A| = (A^*A)^{1/2}$) and the spectral representation $|A| = \int \lambda\, dX_\lambda$. For $n \in \mathbf{N}$, we define the operator

$$P_n := J_M\Big(\int_{(1/n,\infty)} \lambda^{-1}\, dX_\lambda\Big) W^* Q_U.$$

Then $P_n \in \mathscr{L}(E, E)$. Using the fact that

$$W^* Q_U J_M = |A| = \int \lambda\, dX_\lambda,$$

we get

$$(P_n)^2 = J_M\Big(\int_{(1/n,\infty)} \lambda^{-1}\, dX_\lambda\Big) W^* Q_U J_M\Big(\int_{(1/n,\infty)} \lambda^{-1}\, dX_\lambda\Big) W^* Q_U = P_n,$$

$$P_n J_M = J_M\Big(\int_{(1/n,\infty)} dX_\lambda\Big).$$

Consequently, J_M maps the Hilbert space $\big(\int_{(1/n,\infty)} dX_\lambda\big)(E_M)$ onto the range of P_n and an inverse mapping is given by

$$P_n(E) \ni \varphi \to \Big(\int_{(1/n,\infty)} \lambda^{-1}\, dX_\lambda\Big) W^* Q_U \varphi.$$

Thus the range of P_n is isomorphic to a Hilbert space. Moreover, given $\gamma \in \mathscr{M}$, we obtain the estimate

$$\begin{aligned} p(\gamma(\cdot,\cdot) - \gamma(P_n\cdot,\cdot)) &\leqslant \|\gamma((\mathrm{Id} - P_n)J_M\cdot, J_N\cdot)\| \\ &= \Big\|\gamma\Big(J_M\Big(\int_{[0,1/n]} dX_\lambda\Big)\cdot, J_N\cdot\Big)\Big\| \\ &\leqslant \Big\|Q_U J_M\Big(\int_{[0,1/n]} dX_\lambda\Big)\Big\|\,\|Q_V J_N\| \leqslant (1/n)\|Q_V J_N\|. \end{aligned}$$

Constructing projections $Q_m \in \mathscr{L}(F, F)$ in a similar way, we obtain further

$$p(\gamma(P_n\cdot,\cdot) - \gamma(P_n\cdot, Q_m\cdot)) \leqslant (1/m)\|Q_U P_n J_M\| \leqslant (1/m)\|Q_U J_M\|.$$

Taking $P = P_n$ and $Q = Q_n$, where n is sufficiently large, we obtain (5). This completes the proof.

Remarks. 1. The bilinear forms $\gamma(P\cdot, Q\cdot)$ constructed in the preceding theorem are jointly continuous. Consequently, the completion of $\mathscr{B}_{jc}(E, F)[\tau_b]$ or of $\mathscr{L}(E, F'_\beta)[\tau_b]$ coincides with $\mathscr{B}(E, F)[\tau_b]$. Also $\gamma(P\cdot, Q\cdot)$ may be represented by an operator T_γ mapping the Hilbert space $P(E)$ into the dual of the Hilbert space $Q(F)$ such that

$$\gamma(P\varphi, Q\psi) = (T_\gamma P\varphi)(Q\psi).$$

2. Applied in the case when F has dimension one, Theorem 3.1 shows that there exists a net $(P_\delta)_{\delta\in\Delta}$ of projections $P_\delta \in \mathscr{L}(E, E)$ such that the spaces $P_\delta(E)$ are isomorphic to Hilbert spaces and such that $\lim_\delta P_\delta \varphi = \varphi$ w.r.t. the associated bornological topology of E and uniformly on bounded subsets of E (cf. [7], 13.1 and 11.2).

Next we consider bounded subsets of a certain inductive tensor product. We define a subspace $E_\mu \overline{\otimes}_\iota F_\mu$ of the complete inductive tensor product $E_\mu \tilde{\otimes}_\iota F_\mu$ of the associated Mackey spaces of E and F by

$$E_\mu \overline{\otimes}_\iota F_\mu = \Big\{\sum_{n=1}^{\infty} c_n \varphi_n \otimes \psi_n:\ (c_n) \in l_1 \text{ and } (\varphi_n) \text{ and } (\psi_n) \text{ are 0-sequences in } E \text{ and } F, \text{ resp.}\Big\}.$$

THEOREM 4.2. *Suppose that E and F are hilbertizable (QF)-spaces with continuous norms. Then the identity map on $E \otimes F$ extends to a linear isomorphism I of $E_\mu \tilde{\otimes}_\iota F_\mu$ onto $E \tilde{\otimes}_\pi F$ which is continuous and has the property that the restriction of I^{-1} to the closed linear span of an arbitrary bounded subset of $E \tilde{\otimes}_\pi F$ is continuous. Moreover, $E_\mu \tilde{\otimes}_\iota F_\mu$ and $E \tilde{\otimes}_\pi F$ are (QF)-spaces.*

Proof. Since the topology of the inductive tensor product is stronger than the topology of the projective tensor product, the identity map on $E \otimes F$ extends to a continuous linear map I of $E_\mu \tilde{\otimes}_\iota F_\mu$ onto $E \tilde{\otimes}_\pi F$.

Let a bounded subset $\mathscr{M}$ of $E \tilde{\otimes}_\pi F$ be given. Let bounded subsets $M \subset E$ and $N \subset F$ be chosen such that $\mathscr{M}$ is contained in the closed convex hull of $M \otimes N$. Let $D \subset E$ and $G \subset F$ be Fréchet subspaces containing M and N, resp. The identity imbedding of $D \otimes G$ into $E \otimes F$ extends to a continuous linear mapping $I_{D,G}$ of $D \tilde{\otimes}_\pi G$ into $E_\mu \tilde{\otimes}_\iota F_\mu$ because $D \tilde{\otimes}_\pi G = D_\mu \tilde{\otimes}_\iota G_\mu$. Furthermore, the composition $I \circ I_{D,G}$ is the continuous extension of the identity imbedding of $D \otimes_\pi G$ into $E \otimes_\pi F$. Since $D \otimes_\pi G$ is a topological subspace of $E \otimes_\pi F$ (by [6], Proposition 4.3), $I \circ I_{D,G}$ is a topological linear isomorphism of $D \tilde{\otimes}_\pi G$ onto its image. This implies in particular that $E \tilde{\otimes}_\pi F$ is a (QF)-space. Since

$$E_\mu \tilde{\otimes}_\iota F_\mu = \bigcup I_{D,G}(D \tilde{\otimes}_\pi G),$$

where the union is taken over all Fréchet subspaces $D \subset E$ and $G \subset F$, I is injective. Now it follows that the restriction of I^{-1} to the canonical image of $D \tilde{\otimes}_\pi G$ is continuous and that $E_\mu \tilde{\otimes}_\iota F_\mu$ is a (QF)-space. This completes the proof.

Remark. Suppose that the assumptions of Theorem 4.2 are satisfied. Then $E_\mu \tilde{\otimes}_\iota F_\mu$ and $E \tilde{\otimes}_\pi F$ are quasi-complete. Hence they are the quasi-completions of $E_\mu \otimes_\iota F_\mu$ and $E \otimes_\pi F$, resp.

Finally, we obtain a description of compact subsets of $E \tilde{\otimes}_\pi F$. We will use the following essentially known result applied in the more trivial case of Hilbert spaces.

LEMMA 4.3. *Suppose that G is a Banach space with bounded approximation property. Then, given a compact subset $M \subset G$, there exists a compact operator $T \in \mathscr{L}(G, G)$ such that*

$$M \subset \{T\varphi\colon \varphi \in G \text{ and } \|\varphi\| \leqslant 1\}.$$

Proof. Arguments of the proof of Theorem 2 in [1] based on the generalized Cohen factorization theorem for Banach modules ([5], Theorem 32.22) show that each compact operator A from a Banach space D into G admits a factorization $A = TS$, where $S \in \mathscr{L}(D, G)$ and $T \in \mathscr{L}(G, G)$ are compact operators. In order to establish the lemma, it suffices now to apply this factorization to the operator $J_N\colon G_N \to G$, where $N \subset G$ is a compact disc containing M.

PROPOSITION 4.4. *Suppose that E and F are hilbertizable (QF)-spaces with continuous norms. Let a precompact subset $\mathscr{M}$ of $E \tilde{\otimes}_\pi F$ be given. Then there exist Hilbert discs $M \subset E$ and $N \subset F$ and compact operators $X \in \mathscr{L}(E_M, E_M)$ and $Y \in \mathscr{L}(F_N, F_N)$ such that $\mathscr{M}$ is contained in the canonical image of the closed convex hull of $X(M) \otimes Y(N)$, taken in $E_M \tilde{\otimes}_\pi F_N$.*

Proof. Since $E \tilde{\otimes}_\pi F$ is a (QF)-space, there exists a 0-sequence (t_n) in $E \tilde{\otimes}_\pi F$ such that $\mathscr{M}$ is contained in the closed convex hull of $\{t_n\}_{n=1}^\infty$. Also there exists a 0-sequence (ε_n) of positive real numbers such that $((\varepsilon_n)^{-1} t_n)$ is still a 0-sequence. Applying the isomorphism $E \tilde{\otimes}_\pi F \cong \mathscr{N}(E'_\beta, F)[\tau_\pi]$ and Theorem 3.1, we find Hilbert discs $M \subset E$ and $N \subset F$ such that $\{(\varepsilon_n)^{-1} t_n\}_{n=1}^\infty$ is contained in the canonical image of the unit ball of $E_M \tilde{\otimes}_\pi F_N$. Consequently, $\{t_n\}_{n=1}^\infty$ is contained in the canonical image of some compact subset $\mathscr{R}$ of $E_M \tilde{\otimes}_\pi F_N$. The compact set $\mathscr{R}$ is contained in the closed convex hull of $M_0 \otimes N_0$ for some compact subsets $M_0 \subset E_M$ and $N_0 \subset F_n$ (e.g., by [10], § 41.4, (5)). By Lemma 4.3, there are compact operators $X \in \mathscr{L}(E_M, E_M)$ and $Y \in \mathscr{L}(F_N, F_N)$ such that $M_0 \subset X(M)$ and $N_0 \subset Y(N)$. Since the canonical image of the closed convex hull of $X(M) \otimes Y(N)$ is compact and contains $\{t_n\}_{n=1}^\infty$, it contains $\mathscr{M}$. This completes the proof.

Remark. Clearly the preceding proposition is valid also for hilbertizable Fréchet spaces E and F.

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Received September 22, 1988
Revised version August 7, 1990 (2481)

STUDIA MATHEMATICA 99 (3) (1991)

Bukhvalov type characterizations of Urysohn operators

by

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Abstract. The aim of this paper is to generalize to nonlinear operators the criteria of integral representability for linear operators due to A. V. Bukhvalov. We give Bukhvalov type criteria for recognizing the order bounded Urysohn operators acting between ideals of measurable functions.

Introduction. The present paper is devoted to obtaining criteria characterizing when a nonlinear operator has an integral representation as an Urysohn operator. Roughly speaking, an Urysohn operator T is defined by $(Tf)(x) := \int U(x, y, f(y))dy$, where the kernel U satisfies the Carathéodory conditions (i.e., the function $U(x, y, \cdot)$ is continuous in $\mathbf{R}$ for almost all (x, y) and the function $U(\cdot, \cdot, t)$ is measurable for all $t \in \mathbf{R}$).

Integral representation of operators have been of interest for many mathematicians. Recall the nowadays classical results about integral representability of continuous linear operators in L^p spaces obtained in the thirties by Dunford–Pettis and Kantorovich–Vulikh (see, for instance, [8]). In that time John von Neumann [17] raised the problem of finding a characterization of integral linear operators acting in L^2. This problem was solved by A. V. Bukhvalov in [2] in the context of ideals of measurable functions; an independent proof is due to A. R. Schep [22] (see also [3, 29]). Let E and F be ideals of measurable functions. Bukhvalov's theorem states that for a linear operator L: $E \to F$ a necessary and sufficient condition for L to be an integral operator is the following:

Given a sequence $(f_n)_{n=1}^{\infty}$ *in* E *such that* $0 \leqslant f_n \leqslant g$, $f_n \to 0$ $(*)$ *implies* $Lf_n(x) \to 0$ *a.e.*

On the other hand, a large representation theory for nonlinear functionals was developed in the late sixties [5, 6, 7, 9, 15, 16, 24, 27]. L. Drewnowski and W. Orlicz [6, 7] obtained criteria similar to Bukhvalov's for functionals. We remark that the functionals they consider need not be defined on the whole of an ideal of measurable functions. For the sake of convenience we shall not consider this more general case.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 47H99, 45P05; Secondary 47B38.