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Whitney's extension theorem for nonquasianalytic classes of ultradifferentiable functions

by

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Abstract. For a weight function ω let $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$ (resp. $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$) denote the nonquasianalytic class of ω-ultradifferentiable functions of Beurling type (resp. Roumieu type) on $\mathbf{R}^N$. Recently, Meise and Taylor (resp. Bonet, Meise, and Taylor) have characterized those weight functions ω for which the analogue of E. Borel's theorem holds for $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$ (resp. $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$). In the present note it is shown that for these weight functions and arbitrary compact sets K in $\mathbf{R}^N$ even the analogue of Whitney's extension theorem holds. In the Roumieu case, the proof is a modification of the one given by Bruna [5]. However, the existence of appropriate cut-off functions is now reduced—by Hörmander's solution of the $\bar\partial$-problem—to the existence of subharmonic functions with very special properties. The Beurling case can be reduced to the Roumieu case.

Various versions of Whitney's extension theorem and of E. Borel's theorem for different classes of ultradifferentiable functions have been presented by many authors. We only mention Carleson [6], Ehrenpreis [8], Komatsu [14], Bruna [5], Meise and Taylor [16], Petzsche [18], and Bonet, Meise, and Taylor [3], since they influenced our research.

In the present paper we use the classes $\mathscr{E}_{(\omega)}$ and $\mathscr{E}_{\{\omega\}}$ introduced by Beurling [1] and by Petzsche and Vogt [19], where we assume that ω is a weight function in the sense of Braun, Meise, and Taylor [4]. This means that ω: $[0, \infty[\to [0, \infty[$ is a continuous function which satisfies

$$(\alpha)\ \omega(2t) = O(\omega(t)), \qquad (\gamma)\ \log t = o(\omega(t)),$$
$$(\beta)\ \int_1^\infty (\omega(t)/t^2)\,dt < \infty, \qquad (\delta)\ \varphi\colon t \mapsto \omega(e^t) \text{ is convex.}$$

Let φ^* denote the Young conjugate of φ. Then for open sets $\Omega \neq \emptyset$ in $\mathbf{R}^N$ one defines the spaces

$$\mathscr{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega)\colon \text{for each } K \subset \Omega \text{ compact there is } m \in \mathbf{N} \text{ with } \sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N} |f^{(\alpha)}(x)| \exp(-m^{-1}\varphi^*(m|\alpha|)) < \infty\},$$

$$\mathscr{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega)\colon \text{for each } K \subset \Omega \text{ compact and each } m \in \mathbf{N}\ \sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N} |f^{(\alpha)}(x)| \exp(-m\varphi^*(|\alpha|/m)) < \infty\}.$$

They are nonquasianalytic for each weight function ω.

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The main result of the present paper is to characterize for which of these classes the analogue of Whitney's extension theorem holds. Extending previous results of Meise and Taylor [16] and Bonet, Meise and Taylor [3], we show that the following assertions are equivalent:

(1) There exists $C > 1$ so that for all $y > 0$

$$\int_1^\infty \frac{\omega(yt)}{t^2}\,dt \leqslant C\omega(y)+C.$$

(2) For each closed set A in $\mathbf{R}^N$ and each Whitney jet F of type $\mathscr{E}_{\{\omega\}}$ on A there exists $f \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ so that F is the restriction of f to A.

(3) For each closed set A in $\mathbf{R}^N$ and each Whitney jet F of type $\mathscr{E}_{(\omega)}$ on A there exists $g \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ so that F is the restriction of g to A.

For the precise definition of a Whitney jet F of type $\mathscr{E}_{\{\omega\}}$ (resp. $\mathscr{E}_{(\omega)}$) we refer to Definition 3.2 (resp. 4.1).

The basic idea for the proof of this result in the case $\mathscr{E}_{\{\omega\}}$ goes back to Bruna [5], who indicated that the analogue of Whitney's extension theorem holds in a class of nonquasianalytic functions if it holds for a point and if the class contains cut-off functions satisfying certain estimates. Since it had been shown in [16] and [3] that Whitney's extension theorem for a point holds in $\mathscr{E}_{(\omega)}$ and $\mathscr{E}_{\{\omega\}}$ if and only if ω satisfies condition (1), the main step in the proof is to construct these special cut-off functions whenever ω satisfies (1). This is done in Section 2 of the present paper, using Hörmander's $\bar\partial$-method. In order to apply it, we show that (1) implies the following: There exists $A \in \mathbf{N}$ so that for each $k \in \mathbf{N}$ there exists $r_0 > 0$ such that for each $0 < r < r_0$ there exist a subharmonic function $u_{k,r}$ and $B(k, r) > 0$ so that for all $z \in \mathbf{C}$ we have

$$r|\mathrm{Im}\,z| - \frac{A}{k}\omega(|z|) - B(k, r) \leqslant u_{k,r}(z) \leqslant r|\mathrm{Im}\,z| - \frac{1}{k}\omega(|z|)$$

where $B(k, r)$ can be estimated from above in a certain sense (see 2.9). Then the case $\mathscr{E}_{\{\omega\}}$ is treated in Section 3 in the same way as Bruna [5] proved his version of Whitney's extension theorem. The case $\mathscr{E}_{(\omega)}$ is reduced to the case $\mathscr{E}_{\{\omega\}}$ in Section 4.

It should be noted that our main result implies that Whitney's extension theorem holds for the Carleman classes $\mathscr{E}^{\{M_p\}}$ and $\mathscr{E}^{(M_p)}$ (see Komatsu [13]) whenever $(M_p)_{p\in\mathbf{N}_0}$ satisfies the conditions (M1), (M2) and (M3) (see 3.11). Hence it extends the results of Bruna [5], Kantor [12] (see 4.8), and Chung and Kim [7].

The results of the present paper were used by Kaballo [11] to derive estimates for the distribution of the eigenvalues of integral operators with ultradifferentiable kernels.

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1. Preliminaries. Following Braun, Meise and Taylor [4], we introduce the classes of nonquasianalytic functions which we are going to use in the sequel. To do this we recall some basic definitions and facts concerning weight functions.

1.1. Definition. A continuous increasing function ω: $[0, \infty[\to [0, \infty[$ is called a *weight function* if it satisfies

(α) there exists $K \geqslant 1$ with $\omega(2t) \leqslant K(\omega(t)+1)$ for all $t \geqslant 0$,

(β) $\displaystyle\int_0^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty$,

(γ) $\displaystyle\lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0$,

(δ) φ: $t \mapsto \omega(e^t)$ is convex.

The *Young conjugate* φ^*: $[0, \infty[\to \mathbf{R}$ of φ is defined by

$$\varphi^*(y) := \sup\{xy - \varphi(x)\colon x \geqslant 0\}.$$

1.2. Remark. (a) For each weight function ω we have $\lim_{t\to\infty}\omega(t)/t = 0$ by the remark following 1.3 of Meise, Taylor and Vogt [17].

(b) For each weight function ω there exists a weight function σ satisfying $\sigma(t) = \omega(t)$ for all large $t > 0$ and $\sigma|[0, 1] \equiv 0$. Since the subsequent definitions involving ω do not change if ω is replaced by σ, we sometimes will assume tacitly that $\omega|[0, 1] \equiv 0$ in the sequel. Then φ^* has only nonnegative values and $\varphi^{**} = \varphi$.

1.3. Definition. Let ω be a weight function.

(a) For a compact set K in $\mathbf{R}^N$ and $\mu > 0$ let

$$\mathscr{E}_\omega(K, \mu) := \{f \in C^\infty(K)\colon \|f\|_{K,\mu} := \sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N} |f^{(\alpha)}(x)|\exp(-\mu\varphi^*(|\alpha|/\mu)) < \infty\}.$$

(b) For an open set $\Omega \subset \mathbf{R}^N$ define

$$\mathscr{E}_{\{\omega\}}(\Omega) := \{f \in C^\infty(\Omega)\colon \text{for each } K \Subset \Omega \text{ there is } m \in \mathbf{N} \text{ with } \|f\|_{K,1/m} < \infty\}$$
$$= \operatorname*{proj}_{K\Subset\Omega}\ \operatorname*{ind}_{m\to\infty} \mathscr{E}_\omega(K, 1/m),$$

$$\mathscr{E}_{(\omega)}(\Omega) := \{f \in C^\infty(\Omega)\colon \text{for each } K \Subset \Omega \text{ and each } m \in \mathbf{N}\colon \|f\|_{K,m} < \infty\}$$
$$= \operatorname*{proj}_{K\Subset\Omega}\ \operatorname*{proj}_{m\to\infty} \mathscr{E}_\omega(K, m).$$

The elements of $\mathscr{E}_{\{\omega\}}(\Omega)$ (resp. $\mathscr{E}_{(\omega)}(\Omega)$) are called *$\omega$-ultradifferentiable functions of Roumieu* (resp. *Beurling*) *type* on Ω. Sometimes we write $\mathscr{E}_*$, where $*$ can be replaced by $\{\omega\}$ or (ω).

(c) For a compact set K in $\mathbf{R}^N$ we let

$$\mathscr{D}_*(K) := \{f \in \mathscr{E}_*(\mathbf{R}^N): \operatorname{supp} f \subset K\}$$

and we endow $\mathscr{D}_*(K)$ with the induced topology. For an open set Ω in $\mathbf{R}^N$ we define

$$\mathscr{D}_*(\Omega) := \operatorname{ind}_{K \Subset \Omega} \mathscr{D}_*(K).$$

1.4. Remark. In Braun, Meise and Taylor [4] it is shown that for each weight function ω the spaces $\mathscr{D}_*(\mathbf{R}^N)$ are nontrivial, i.e. that the classes $\mathscr{E}_*$ are nonquasianalytic. By [4], 4.9, $\mathscr{E}_{(\omega)}(\Omega)$ is a nuclear Fréchet space, while $\mathscr{E}_{\{\omega\}}(\Omega)$ is complete, nuclear and reflexive for each open set $\Omega \neq \emptyset$ in $\mathbf{R}^N$.

1.5. Example. The following functions $\omega: [0, \infty[\to [0, \infty[$ are examples of weight functions:

(1) $\omega(t) := t^\alpha, \quad 0 < \alpha < 1,$
(2) $\omega(t) := (\log(1+t))^\beta, \quad \beta > 1,$
(3) $\omega(t) := t(\log(e+t))^{-\beta}, \quad \beta > 1,$
(4) $\omega(t) := \exp(\beta(\log(1+t))^\alpha), \quad 0 < \alpha < 1, \beta > 0.$

Note that for $\omega(t) = t^\alpha$, $0 < \alpha < 1$, the space $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ is the classical Gevrey class $\Gamma^{(d)}(\mathbf{R}^N)$ for $d := \alpha^{-1}$.

1.6. Definition. (a) Let $u: \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2}\, dt < \infty.$$

Then we define its harmonic extension $P_u: \mathbf{C} \to \mathbf{R}$ by

$$P_u(x+iy) := \begin{cases} \dfrac{|y|}{\pi} \displaystyle\int_{-\infty}^{\infty} \frac{u(t)}{(t-x)^2+y^2}\, dt & \text{if } |y| > 0, \\ u(x) & \text{if } y = 0. \end{cases}$$

(b) For a weight function ω we extend ω to $\mathbf{C}$ by the definition $z \mapsto \omega(|z|)$. By P_ω we denote the harmonic extension of $t \mapsto \omega(|t|)$.

Note that for u as in 1.6(a), the function P_u is continuous on $\mathbf{C}$ and harmonic in the open upper and lower half plane. Moreover, we have $\omega \leqslant P_\omega$ for each weight function ω.

From Meise and Taylor [16], 3.10, and Bonet, Meise and Taylor [3], 3.8, we recall:

1.7. Theorem. *For a weight function ω the following assertions are equivalent*:

(1) *There exists $C > 0$ with* $\int_1^\infty \frac{\omega(yt)}{t^2}\, dt \leqslant C\omega(y) + C$ *for all* $y > 0$.

(2) $\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \varepsilon\omega(t)/\omega(\varepsilon t) = 0.$

(3) *There exists $K > 1$ with* $\limsup_{t \to \infty} \omega(Kt)/\omega(t) < K$.

(4) *There exists $D > 0$ so that $P_\omega(z) \leqslant D\omega(z) + D$ for all $z \in \mathbf{C}$.*

(5) *For each positive integer N and each family $(a_\alpha)_{\alpha \in \mathbf{N}_0^N}$ of complex numbers satisfying $\sup_{\alpha \in \mathbf{N}_0^N} |a_\alpha| \exp(-m\varphi^*(|\alpha|/m)) < \infty$ for all $m \in \mathbf{N}$, there exists $f \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ such that $f^{(\alpha)}(0) = a_\alpha$ for all $\alpha \in \mathbf{N}_0^N$.*

(6) *For each positive integer N and each family $(a_\alpha)_{\alpha \in \mathbf{N}_0^N}$ of complex numbers satisfying $\sup_{\alpha \in \mathbf{N}_0^N} |a_\alpha| \exp(-m^{-1}\varphi^*(|\alpha| m)) < \infty$ for some $m \in \mathbf{N}$, there exists $f \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ such that $f^{(\alpha)}(0) = a_\alpha$ for all $\alpha \in \mathbf{N}_0^N$.*

1.8. Definition. A weight function ω is called a *strong weight function* if it satisfies one of the equivalent conditions in 1.7.

1.9. Remark. (a) For each strong weight function ω there exists a strong weight function κ which is concave and satisfies $\kappa(0) = 0$, and such that there exists $A \geqslant 1$ such that

$$A^{-1}\kappa(t) - A \leqslant \omega(t) \leqslant A\kappa(t) + A \quad \text{for all } t \geqslant 0.$$

This holds by [16], 1.3, and implies $\mathscr{E}_{(\omega)} = \mathscr{E}_{(\kappa)}$ and $\mathscr{E}_{\{\omega\}} = \mathscr{E}_{\{\kappa\}}$.

(b) Note that by 1.1(δ) each concave weight function is differentiable on $]0, \infty[$.

In [16], 3.10, and [3], 3.8, it was also shown that strong weight functions are characterized by the fact that for compact sets K in $\mathbf{R}^N$ with $\mathring{K} \neq \emptyset$ which are convex (or $K = \overline{G}$, where G is open and ∂G is real-analytic) one can describe the image of the restriction map $\varrho_K: \mathscr{E}_*(\mathbf{R}^N) \to C(K)$ in a certain way. Subsequently we want to extend this result to arbitrary compact sets in $\mathbf{R}^N$.

2. Existence of optimal cut-off functions in $\mathscr{E}_{\{\omega\}}(\mathbf{R})$. Bruna [5] has noticed that Whitney's extension theorem for nonquasianalytic classes of Roumieu type holds on arbitrary compact sets if it holds for points and if there exist cut-off functions in the class which satisfy certain estimates. In a different setting and with a different proof we show in this section that such cut-off functions can be constructed in $\mathscr{E}_{\{\omega\}}$ whenever ω is a strong weight function. To give a precise statement, we first introduce the Young conjugate ω^* of ω, which was used already by Petzsche and Vogt [19] in a related context.

2.1. Definition. For a weight function ω its *Young conjugate* ω^*: $]0, \infty[\to]0, \infty[$ is defined by

$$\omega^*(s) := \sup_{t \geqslant 0} (\omega(t) - ts).$$

Note that $\omega^*(s)$ is finite by 1.2(b) and that ω^* is decreasing and convex.

2.2. PROPOSITION. *Let ω be a strong weight function which is concave and satisfies $\omega(0) = 0$. Then for each $n \in \mathbf{N}$ there exist $m \in \mathbf{N}$, $M > 0$ and $0 < r_0 < 1/2$ such that for each $0 < r < r_0$ there exists $f_{n,r} \in C^\infty(\mathbf{R})$ which has the following properties*:

(1) $\quad 0 \leqslant f_{n,r} \leqslant 1, \quad \operatorname{supp} f_{n,r} \subset [-9r/8, 9r/8], \quad f_{n,r}|[-r, r] \equiv 1,$

(2) $$\sup_{x \in \mathbf{R}} \sup_{j \in \mathbf{N}_0} |f_{n,r}^{(j)}(x)| \exp\left(-\frac{1}{m}\varphi^*(mj)\right) \leqslant M \exp\left(\frac{1}{n}\omega^*(nr)\right).$$

The proof of Proposition 2.2 requires several steps and will be given at the end of this section. The underlying idea is to use Hörmander's L^2-method to construct first certain entire functions $F_{n,r}$, and to use then the theorem of Paley–Wiener to get the desired functions $f_{n,r}$ from the functions $F_{n,r}$.

2.3. LEMMA. *Let ω be a weight function. Then there exists $A > 0$ so that for each $0 < r \leqslant 1$, each $k \in \mathbf{N}$ and each subharmonic function u on $\mathbf{C}$ which satisfies*

$$(*) \qquad u(z) \leqslant r|\operatorname{Im} z| - \frac{1}{k}\omega(z) \quad \textit{for all } z \in \mathbf{C}$$

there exists an entire function F on $\mathbf{C}$ which satisfies $F(0) = 1$ and

$$|F(z)| \leqslant A \exp\left(r|\operatorname{Im} z| - \frac{1}{k}\omega(z) + 3\log(1+|z|^2)\right) \sup_{|w| \leqslant 1} \exp(-u(w)) \quad \textit{for all } z \in \mathbf{C}.$$

Proof. Choose $\chi \in C^\infty(\mathbf{R}^2)$ which satisfies $0 \leqslant \chi \leqslant 1$, $\chi(z) = 1$ for $|z| \leqslant 1/2$ and $\chi(z) = 0$ for $|z| \geqslant 1$ and let $B := \sup_{z \in \mathbf{C}} |\bar\partial\chi(z)|$. Then we have ($\lambda$ the Lebesgue measure on $\mathbf{R}^2 = \mathbf{C}$)

$$\int_{\mathbf{C}} \left|\frac{1}{z}\bar\partial\chi(z)\right|^2 \exp(-2u(z))\, d\lambda(z) \leqslant B^2 \sup_{|z| \leqslant 1} \exp(-2u(z)) \int_{1/2 \leqslant |z| \leqslant 1} |1/z^2|\, d\lambda(z)$$
$$= B^2 2\pi \log 2 \sup_{|w| \leqslant 1} \exp(-2u(w)).$$

By Hörmander [10], 4.4.2, this implies the existence of $v \in C^\infty(\mathbf{R}^2)$ which satisfies $\bar\partial v(z) = (1/z)\bar\partial\chi(z)$ for $z \in \mathbf{C}$ and

$$\int_{\mathbf{C}} \frac{|v(z)|^2 \exp(-2u(z))}{(1+|z|^2)^2}\, d\lambda(z) \leqslant B^2 \pi \log 2 \sup_{|w| \leqslant 1} \exp(-2u(w)).$$

Then the function

$$F\colon z \mapsto \chi(z) - zv(z)$$

is in $C^\infty(\mathbf{R}^2)$ and satisfies $\bar\partial F = \bar\partial\chi - z\bar\partial v = 0$ as well as $F(0) = \chi(0) = 1$.

Moreover, with $A := \left(\max(\pi B^2, \pi\exp(\omega(1)))\right)^{1/2}$ we get from $(*)$

$$\left(\int_{\mathbf{C}} \left|F(z)\exp\left(-r|\operatorname{Im} z| + \frac{1}{k}\omega(z) - 3\log(1+|z|^2)\right)\right|^2 d\lambda(z)\right)^{1/2}$$
$$\leqslant (B^2\pi)^{1/2} \sup_{|w| \leqslant 1} \exp(-u(w)) + \left(\pi\exp\left(\frac{1}{k}\omega(1)\right)\right)^{1/2}$$
$$\leqslant A \sup_{|w| \leqslant 1} \exp(-u(w)),$$

since $u(0) \leqslant 0$. By the properties of ω, this estimate and standard arguments imply the desired estimate for F.

2.4. LEMMA. *For each weight function ω there exists $L \in \mathbf{N}$ so that for each $k \in \mathbf{N}$ there exists $B > 0$ so that for each $0 < r < 1/2$ the following holds: If there is an entire function F with $F(0) = 1$ which satisfies for some $M > 0$ the estimate*

$$(*) \qquad |F(z)| \leqslant M \exp\left(r|\operatorname{Im} z| - \frac{1}{k}\omega(z)\right) \quad \textit{for all } z \in \mathbf{C}\ ,$$

then there exists $\psi \in \mathscr{E}_{(\omega)}(\mathbf{R})$ satisfying

(1) $\quad 0 \leqslant \psi \leqslant 1, \quad \psi(x) = 0$ *for* $x \leqslant -r$, $\quad \psi(x) = 1$ *for* $x \geqslant r$,

(2) $$\sup_{x \in \mathbf{R}} \sup_{j \in \mathbf{N}_0} |\psi^{(j)}(x)| \exp\left(-\frac{1}{2Lk}\varphi^*(2Lkj)\right) \leqslant BM^2.$$

Proof. By [4], 1.3, we can choose $L \in \mathbf{N}$ and $y_0 > 0$ so that

$$\varphi^*(y) - y \geqslant L\varphi^*(y/L) - L \quad \text{for all } y \geqslant y_0.$$

Now fix $k \in \mathbf{N}$ and $0 < r < 1/2$ and an entire function F which satisfies $(*)$ and $F(0) = 1$. By the theorem of Paley–Wiener there exists $f \in C^\infty(\mathbf{R})$ with $\operatorname{supp} f \subset [-r, r]$ so that

$$F(z) = \hat f(z) := \int_{-\infty}^{\infty} f(t) e^{-izt}\, dt.$$

Since ω satisfies 1.1(γ), we can choose D, depending on ω and k, so that

$$\log(1+t^2) \leqslant \frac{1}{2k}\omega(t) + D \quad \text{for all } t \geqslant 0.$$

Then [4], 3.3(a), in connection with $(*)$ implies

$$\|f\|_{2k} := \sup_{x \in \mathbf{R}} \sup_{j \in \mathbf{N}_0} |f^{(j)}(x)| \exp\left(-\frac{1}{2k}\varphi^*(2kj)\right) \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat f(t)| \exp\left(\frac{1}{2k}\omega(t)\right) dt$$
$$\leqslant \frac{M}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2k}\omega(t)\right) dt \leqslant \frac{M}{2\pi} e^D \int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \frac{M}{2} e^D.$$

Next note that $g:=\operatorname{Re} f$ is in $\mathcal{D}_{\{\omega\}}(\mathbf{R})$ and satisfies $\operatorname{supp} g \subset [-r, r]$, $\int_{-r}^{r} g(x)\,dx = 1$ and $\|g\|_{2k} \leqslant (M/2)e^{D}$. Therefore $h:=g^2$ is in $\mathcal{D}_{\{\omega\}}(\mathbf{R})$ by [4], 4.4, and the proof of [4], 4.4, shows that there exists E, depending on L and k, so that

$$\|h\|_{2Lk} \leqslant E\|g\|_{2k}^2 = \tfrac{1}{4}M^2Ee^{2D}.$$

Now $0<r<1/2$ implies

$$1 = \int_{-r}^{r} g(t)\,dt \leqslant \Big(\int_{-r}^{r} g^2(t)\,dt\Big)^{1/2}\Big(\int_{-r}^{r} 1\,dt\Big)^{1/2} \leqslant \int_{-\infty}^{\infty} h(t)\,dt.$$

Next we define $\psi\colon \mathbf{R}\to\mathbf{R}$ by

$$\psi(x) := \Big(\int_{-\infty}^{\infty} h(t)\,dt\Big)^{-1}\int_{-\infty}^{x} h(s)\,ds.$$

Then ψ satisfies

$$0 \leqslant \psi \leqslant 1, \qquad \operatorname{supp}\psi \subset [-r, \infty[, \qquad \psi|[r,\infty[\equiv 1,$$

$$|\psi(x)| \leqslant \Big|\int_{-r}^{x} h(s)\,ds\Big| \leqslant 2r \max_{x\in[-r,r]}|h(x)| \leqslant \max_{x\in[-r,r]}|h(x)|,$$

$$|\psi^{(j)}(x)| \leqslant |h^{(j-1)}(x)| \quad \text{ for } j\in\mathbf{N}.$$

Since $\varphi^*(mj)/m \geqslant \varphi^*((m(j-1))/m$ for all $j\in\mathbf{N}$, $m\in\mathbf{N}$, we get from this with $B:=\frac{1}{4}Ee^{2D}$

$$\|\psi\|_{2Lk} \leqslant \|h\|_{2Lk} \leqslant \tfrac{1}{4}M^2Ee^{2D} = BM^2.$$

2.5. LEMMA. *Let ω be a strong weight function which is concave and satisfies $\omega(0)=0$. For $T>1$ we define $\omega_T\colon \mathbf{R}\to[0,\infty[$ by*

$$\omega_T\colon t\mapsto \begin{cases} \omega(t) & \text{if } |t|\geqslant T,\\ \dfrac{\omega'(T)}{2T}t^2 - \dfrac{\omega'(T)}{2}T + \omega(T) & \text{if } |t|\leqslant T.\end{cases}$$

Then there exists $D>0$ so that for all $T>1$ we have

$$\sup_{x\in\mathbf{R}} \frac{\partial}{\partial y}P_{\omega_T}(x+i) \leqslant D\frac{\omega(T)}{T}.$$

Proof. Note that

$$\pi\frac{\partial}{\partial y}P_{\omega_T}(x+i) = \int_{-\infty}^{\infty}\frac{t}{t^2+1}\omega_T'(x+t)\,dt = \int_{-\infty}^{\infty}\frac{s-x}{(s-x)^2+1}\omega_T'(s)\,ds.$$

Let C be the constant from 1.7(1). Depending on the relation between x and T, we will cut the real line into three or four intervals and give different estimates for each one. Luckily, the first two of these intervals can be treated in the same way for all x. Since ω is an even function, it suffices to consider $x>0$.

$$\int_{-\infty}^{-T}\frac{s-x}{(s-x)^2+1}\omega_T'(s)\,ds = \int_{T}^{\infty}\frac{s+x}{(s+x)^2+1}\omega'(s)\,ds \leqslant \int_{T}^{\infty}\frac{\omega'(s)}{s+x}ds \leqslant \int_{T}^{\infty}\frac{\omega'(s)}{s}ds$$
$$= \frac{\omega(s)}{s}\Big|_T^{\infty} + \int_{T}^{\infty}\frac{\omega(s)}{s^2}ds \leqslant C\frac{\omega(T)+1}{T} \leqslant C_1\frac{\omega(T)}{T},$$

where $C_1 = C(1+1/\omega(1))$.

The second interval that has to be considered is $[-T, T]$. Note that $\omega'(T)\leqslant\omega(T)/T$ since ω is concave.

$$\int_{-T}^{T}\frac{s-x}{(s-x)^2+1}\omega_T'(s)\,ds = \int_{-T-x}^{T-x}\frac{t}{t^2+1}\frac{\omega'(T)}{T}(t+x)\,dt = \frac{\omega'(T)}{T}\int_{-T-x}^{T-x}\frac{t^2+tx}{t^2+1}dt$$
$$= \frac{\omega'(T)}{T}\Big(t+\frac{x}{2}\log(t^2+1) - \arctan t\Big)\Big|_{-T-x}^{T-x}$$
$$= \frac{\omega'(T)}{T}\Big(2T + \frac{x}{2}\log\frac{(T-x)^2+1}{(T+x)^2+1} - \arctan(T-x) - \arctan(T+x)\Big)$$
$$= 2\omega'(T) + \frac{x}{2T}\omega'(T)\log\frac{(T-x)^2+1}{(T+x)^2+1} - \frac{\omega'(T)}{T}(\arctan(T-x) - \arctan(T+x)).$$

Thus we have

$$\int_{-T}^{T}\frac{s-x}{(s-x)^2+1}\omega_T'(s)\,ds \leqslant (2+\pi)\frac{\omega(T)}{T} + \frac{x}{2T}\omega'(T)\log\frac{(T-x)^2+1}{(T+x)^2+1}. \tag{1}$$

The first term on the right hand side of (1) has correct size. The second one is negative and will be used to cancel another term in certain cases.

With $a:=\max(T, 2x)$ we have

$$\int_{a}^{\infty}\frac{s-x}{(s-x)^2+1}\omega_T'(s)\,ds \leqslant \int_{a}^{\infty}\frac{\omega'(s)}{s-x}ds \leqslant 2\int_{a}^{\infty}\frac{\omega'(s)}{s}ds \leqslant 2C\frac{\omega(a)+1}{a} \leqslant 2C_1\frac{\omega(T)}{T}. \tag{2}$$

Case $2x\leqslant T$: In this case $a=T$ and (2) completes the proof.

Case $2x\geqslant T$: Then (2) gives the right estimate for

$$\int_{2x}^{\infty}\frac{s-x}{(s-x)^2+1}\omega_T'(s)\,ds.$$

To treat the remaining interval, we have to consider two subcases.

Subcase $x<T$:

$$\int_{T}^{2x}\frac{s-x}{(s-x)^2+1}\omega'(s)\,ds \leqslant \omega'(T)\int_{T}^{2x}\frac{s-x}{(s-x)^2+1}ds = \frac{\omega'(T)}{2}\log\frac{x^2+1}{(T-x)^2+1}.$$

Together with (1) this gives

$$\int_{-T}^{2x} \frac{s-x}{(s-x)^2+1}\,\omega'(s)\,ds$$

$$\leqslant 6\frac{\omega(T)}{T}+\frac{\omega'(T)}{2}\left(\log\frac{x^2+1}{(T+x)^2+1}-\left(1-\frac{x}{T}\right)\log\frac{(T-x)^2+1}{(T+x)^2+1}\right)$$

$$\leqslant 6\frac{\omega(T)}{T}+\frac{\omega'(T)}{2}\left(\left(\frac{x}{T}-1\right)\log\frac{(1-x/T)^2+1/T^2}{(1+x/T)^2+1/T^2}\right).$$

Note that for $a, b, c > 0$ the inequality $b > a$ implies $(a+c)(b+c)^{-1} > a/b$. This implies

$$\frac{(1-x/T)^2+1/T^2}{(1+x/T)^2+1/T^2} \leqslant \frac{1-x/T}{1+x/T}.$$

Thus we have with $C_2 := \sup_{0\leqslant\xi\leqslant 1/2}(-\xi\log\xi)$

$$\int_{-T}^{2x} \frac{s-x}{(s-x)^2+1}\,\omega'(s)\,ds \leqslant 6\frac{\omega(T)}{T}+\frac{\omega'(T)}{2}\left(\frac{x}{T}-1\right)\log\frac{1-x/T}{1+x/T}$$

$$\leqslant 6\frac{\omega(T)}{T}+\frac{\omega'(T)}{2}\left(\left(\frac{x}{T}-1\right)\log\left(1-\frac{x}{T}\right)+\left(1-\frac{x}{T}\right)\log\left(1+\frac{x}{T}\right)\right)$$

$$\leqslant 6\frac{\omega(T)}{2}+\frac{\omega'(T)}{2}(C_2+\log 2).$$

Subcase $T \leqslant x$: As ω is concave, ω' is decreasing, hence

$$\int_{T}^{2x} \frac{s-x}{(s-x)^2+1}\,\omega_T'(s)\,ds = \int_{T}^{x} \frac{s-x}{(s-x)^2+1}\,\omega'(s)\,ds+\int_{x}^{2x} \frac{s-x}{(s-x)^2+1}\,\omega'(s)\,ds$$

$$\leqslant \omega'(x)\int_{T}^{x} \frac{s-x}{(s-x)^2+1}\,ds+\omega'(x)\int_{x}^{2x} \frac{s-x}{(s-x)^2+1}\,ds$$

$$= \omega'(x)\int_{T}^{2x} \frac{s-x}{(s-x)^2+1}\,ds = \frac{\omega'(x)}{2}\log\frac{x^2+1}{(T-x)^2+1}$$

$$\leqslant \frac{\omega'(T)}{2}\log\frac{x^2+1}{(T-x)^2+1}.$$

If $T \leqslant x/2$, then $x-T \geqslant x/2$ and $(x^2+1)/((T-x)^2+1) \leqslant 4$, which completes the proof in this case. If, on the other hand, we have $x/2 \leqslant T \leqslant x$, then we get

using (1)

$$\int_{-T}^{2x} \frac{s-x}{(s-x)^2+1}\,\omega_T'(s)\,ds$$

$$\leqslant 6\frac{\omega(T)}{T}+\frac{\omega'(T)}{2}\left(\frac{x}{2T}\log\frac{(T-x)^2+1}{(T+x)^2+1}+\log\frac{x^2+1}{(T-x)^2+1}\right)$$

$$\leqslant 6\frac{\omega(T)}{T}+\frac{\omega'(T)}{2}\left(\log\frac{x^2+1}{(T+x)^2+1}+\left(\frac{x}{T}-1\right)\log\frac{(T-x)^2+1}{(T+x)^2+1}\right)$$

$$\leqslant 6\frac{\omega(T)}{T}.$$

This concludes the proof of the lemma.

2.6. Definition. For ω as in 2.5 and $T > 1$ let ω_T be defined as in 2.5. Then we define $h_T\colon \mathbf{C}\to\mathbf{R}$ by

$$h_T(z) := \begin{cases} P_{\omega_T}(z+i) & \text{if } \operatorname{Im} z \geqslant 0,\\ P_{\omega_T}(z-i) & \text{if } \operatorname{Im} z < 0.\end{cases}$$

Note that by the symmetry properties of ω_T and of the Poisson kernel, h_T is continuous on $\mathbf{C}$.

2.7. Lemma. *For ω as in 2.5 there exist $E, F, G > 0$ so that for all $T > 1$ and all $z\in\mathbf{C}$ we have*

$$E^{-1}h_T(z)-F\omega(T) \leqslant \omega(z) \leqslant h_T(z)+G.$$

Proof. First we note that

(1) $$\omega_T(t)-\omega(T) \leqslant \omega(t) \leqslant \omega_T(t) \quad \text{for all } t\in\mathbf{R} \text{ and all } T > 1.$$

The first inequality follows from the definition of ω_T and $\omega'(T) \leqslant \omega(T)/T$. The second one is a consequence of the fact that the convex function $\omega_T|[0, T]$ and the concave function ω have the same derivative at T.

Next denote by h the function which we get if we replace ω_T in 2.6 by ω. Then (1) and the properties of the Poisson kernel imply

(2) $$h_T(z)-\omega(T) \leqslant h(z) \leqslant h_T(z) \quad \text{for all } z\in\mathbf{C} \text{ and all } T > 1.$$

Now note that $\omega(0) = 0$ and the concavity of ω imply that ω is subadditive on $\mathbf{R}$ (see e.g. Björck [2], 1.2.1) and hence $|\omega(x+s)-\omega(x)| \leqslant \omega(s)$ for all $x\in\mathbf{R}$, $s\in\mathbf{R}$. Consequently, we have

(3) $$|P_\omega(z+s)-P_\omega(z)| \leqslant \omega(1) \quad \text{for all } z\in\mathbf{C},\ s\in[-1, 1].$$

Furthermore, we get for $0<y\leqslant 1$

$$|P_\omega(x+iy)-P_\omega(x)| = \left|\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\omega(t)}{(t-x)^2+y^2}\,dt-\omega(x)\right|$$
$$\leqslant \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{|\omega(t)-\omega(x)|}{(t-x)^2+y^2}\,dt \leqslant \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\omega(t-x)}{(t-x)^2+y^2}\,dt$$
$$= P_\omega(iy) \leqslant \max_{0\leqslant y\leqslant 1} P_\omega(iy) =: Q.$$

Since $P_\omega(z+iy)-P_\omega(z)$ is the harmonic extension of $x\mapsto P_\omega(x+iy)-P_\omega(x)$, this implies

(4) $$|P_\omega(z+iy)-P_\omega(z)|\leqslant 2Q \quad \text{for all } z\in\mathbf{C} \text{ and } y\in[-1,1].$$

Now (4) and (3) imply that for $G:=2Q+\omega(1)$ we have

(5) $$|P_\omega(z+w)-P_\omega(w)|\leqslant G \quad \text{for all } z,w\in\mathbf{C} \text{ with } |w|\leqslant 1.$$

Next we recall from 1.7 that there exists $D>1$ so that

(6) $$\omega(z)\leqslant P_\omega(z)\leqslant D(\omega(z)+1) \quad \text{for all } z\in\mathbf{C}.$$

From (6), (5) and (2) we now get for $z\in\mathbf{C}$ with $\operatorname{Im} z\geqslant 0$

$$\omega(z)\leqslant P_\omega(z)\leqslant P_\omega(z+i)+G = h(z)+G\leqslant h_T(z)+G.$$

Since the same arguments apply to $\operatorname{Im} z<0$, we have

(7) $$\omega(z)\leqslant h_T(z)+G.$$

This proves the second inequality in our claim. On the other hand, (6), (2) and (5) give for all $z\in\mathbf{C}$

$$1+\omega(z)\geqslant D^{-1}P_\omega(z)\geqslant D^{-1}(h(z)-G)\geqslant D^{-1}(h_T(z)-\omega(T)-G).$$

Therefore we can find F, depending only on ω, so that

$$D^{-1}h_T(z)-F\omega(T)\leqslant\omega(z) \quad \text{for all } z\in\mathbf{C}.$$

2.8. Lemma. *Let ω be a strong weight function which is concave and satisfies $\omega(0)=0$. Then for each $\nu\in\mathbf{N}$ there exist $m\in\mathbf{N}$, $M>0$ and $0<r_0<1/2$ such that for each $0<r<r_0$ there exists $g_{\nu,r}\in C^\infty(\mathbf{R})$ which has the following properties*:

(1) $$0\leqslant g_{\nu,r}\leqslant 1,\quad g_{\nu,r}(x)=0 \text{ for } x\leqslant -r,\quad g_{\nu,r}(x)=1 \text{ for } x\geqslant r,$$

(2) $$\sup_{x\in\mathbf{R}}\sup_{j\in\mathbf{N}_0}|g_{\nu,r}^{(j)}(x)|\exp\left(-\frac{1}{m}\varphi^*(mj)\right)\leqslant M\exp\left(\frac{1}{\nu}\omega^*(\nu r)\right).$$

Proof. Denote by A, L, D, E, F, G the constants from Lemmata 2.3, 2.4, 2.5 and 2.7. Without restriction we can assume that L, D, E and F are natural numbers. Then for $\nu\in\mathbf{N}$ let $k:=(2EF+D)\nu\in\mathbf{N}$ and $m:=4Lk$ $=4L(2EF+D)\nu$. Next choose $0<r_0<1/2$ so that the equation $\omega(t)/t=r_0k/D$ has a solution $t>1$. Then fix $0<r<r_0$ and choose $T=T(k,r)>1$ so that

(3) $$\omega(T)=T\frac{rk}{D}.$$

Now define $u_{\nu,r}\colon \mathbf{C}\to\mathbf{R}$ by

$$u_{\nu,r}(z):=r|\operatorname{Im} z|-\frac{1}{k}h_T(z)-\frac{G}{k}.$$

Note that Lemma 2.7 implies for all $z\in\mathbf{C}$

(4) $$u_{\nu,r}(z)\leqslant r|\operatorname{Im} z|-\frac{1}{k}\omega(z),$$
$$-u_{\nu,r}(z)\leqslant -r|\operatorname{Im} z|+\frac{E}{k}\omega(z)+\frac{EF}{k}\omega(T)+\frac{G}{k}.$$

Next observe that $u_{\nu,r}$ is subharmonic in the open upper and lower half plane and that by the definition of h_T and Lemma 2.5 we get from (3)

$$-\frac{1}{k}\frac{\partial}{\partial y}h_T(x) = -\frac{1}{k}\frac{\partial}{\partial y}P_{\omega_T}(x+i)\geqslant -\frac{D}{k}\frac{\omega(T)}{T} = -r \quad \text{for all } x\in\mathbf{R}.$$

Therefore we have for each $g\in\mathcal{D}(\mathbf{C})$ which satisfies $g\geqslant 0$

$$\int_{\mathbf{C}} u_{\nu,r}(z)\Delta g(z)\,d\lambda(z) = 2\int_{-\infty}^{\infty}\left(r-\frac{1}{k}\frac{\partial}{\partial y}h_T(x)\right)g(x)\,dx\geqslant 0,$$

which proves that $u_{\nu,r}$ is subharmonic on $\mathbf{C}$. From Lemma 2.3 and (4) we get then the existence of an entire function $F_{\nu,r}$ which satisfies $F_{\nu,r}(0)=1$ and

$$|F_{\nu,r}(z)|\leqslant A\exp\left(r|\operatorname{Im} z|-\frac{1}{k}\omega(z)+3\log(1+|z|^2)\right)\sup_{|w|\leqslant 1}\exp(-u_{\nu,r}(w))$$

for all $z\in\mathbf{C}$. From this, (4) and 1.1(γ) we get the existence of $C=C(\nu)$ such that for all $z\in\mathbf{C}$

$$|F_{\nu,r}(z)|\leqslant C(\nu)\exp\left(r|\operatorname{Im} z|-\frac{1}{2k}\omega(z)\right)\exp\left(\frac{EF}{k}\omega(T)\right).$$

Hence Lemma 2.4 implies the existence of $B(\nu)>0$ and $g_{\nu,r}\in C^\infty(\mathbf{R})$ which satisfies (1) and

(5) $$\sup_{x\in\mathbf{R}}\sup_{j\in\mathbf{N}_0}|g_{\nu,r}^{(j)}(x)|\exp\left(-\frac{1}{4Lk}\varphi^*(4Lkj)\right)\leqslant B(\nu)C(\nu)^2\exp\left(\frac{2EF}{k}\omega(t)\right).$$

Now note that the definitions of ω^* and of $T=T(k,r)$ give

$$\omega^*(\nu r)\geqslant\omega(T)-\nu rT=\omega(T)-\frac{\nu rD\omega(T)}{rk}=\omega(T)\left(1-\frac{\nu D}{k}\right).$$

Hence our choice of k implies

$$(6)\qquad \frac{1}{\nu}\omega^*(\nu r) \geqslant \omega(T)\left(\frac{1}{\gamma}-\frac{D}{k}\right) = \omega(T)\frac{(2EF+D)\nu-D\nu}{\nu k} = \frac{2EF}{k}\omega(T).$$

From this and (5) together with our choice of m we get (2) if we let $M = B(\nu)C(\nu)^2$.

Proof of Proposition 2.2. For $n\in\mathbf{N}$ let $\nu := 16n$. For this ν choose $m\in\mathbf{N}$, $M>0$ and $0<r_0<1/2$ according to Lemma 2.8. Then fix $0<r<r_0$, define $s:=r/16$ and choose $g_{\nu,r}$ so that the conditions 2.8(1) and 2.8(2) hold with r replaced by s. Then define

$$f_{n,r}(x) := \begin{cases} g_{\nu,s}(x+\frac{17}{16}r) & \text{for } x\leqslant 0,\\ g_{\nu,s}(-x-\frac{17}{16}r) & \text{for } x\geqslant 0,\end{cases}$$

and note that by 2.8(2) we have

$$\sup_{x\in\mathbf{R}}\sup_{j\in\mathbf{N}}|f_{n,r}^{(j)}(x)|\exp\left(-\frac{1}{m}\varphi^*(mj)\right) \leqslant M\exp\left(\frac{1}{\nu}\omega^*(\nu s)\right)$$
$$\leqslant M\exp\left(\frac{16}{\nu}\omega^*\left(\frac{\nu s}{16}\right)\right) = M\exp\left(\frac{1}{n}\omega^*(nr)\right).$$

Hence 2.2(2) is satisfied. From 2.8(1) it follows easily that also 2.2(1) holds.

2.9. Remark. Note that the following is implied by combining 2.8(4) with 2.8(6): There exist $A, E, G\in\mathbf{N}$, depending only on ω, so that for each $\nu\in\mathbf{N}$ there exists $0<r_0<1/2$ so that for each $0<r<r_0$ there exists a subharmonic function $u_{\nu,r}$ on $\mathbf{C}$ so that we have for all $z\in\mathbf{C}$

$$r|\operatorname{Im} z| - \frac{E}{\nu}\omega(z) - \frac{1}{\nu}\omega^*(\nu r) - \frac{G}{\nu} \leqslant u_{\nu,r}(z) \leqslant r|\operatorname{Im} z| - \frac{1}{A\nu}\omega(z).$$

Because of $\omega(0)=0$ this gives

$$-\frac{1}{\nu}\omega^*(\nu r) - \frac{G}{\nu} \leqslant u_{\nu,r}(0)$$

and this lower bound is essential for our purposes. The following consideration shows that this lower bound is optimal to a certain extent: Whenever $u_{\nu,r}$ satisfies the given upper estimate, then the subaveraging property of $u_{\nu,r}$ implies for each $\varrho>0$

$$u_{\nu,r}(0) \leqslant \frac{1}{2\pi}\int_0^{2\pi} u_{\nu,r}(\varrho e^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi}\varrho r|\sin t|\,dt - \frac{1}{k}\omega(\varrho) = \frac{2}{\pi}\varrho r - \frac{1}{k}\omega(\varrho).$$

Taking the infimum over $\varrho>0$, this and $\nu = Ak$ give

$$u_{\nu,r}(0) \leqslant -\frac{1}{k}\omega^*\left(\frac{2kr}{\pi}\right) = -\frac{1}{A\nu}\omega^*\left(\frac{2A}{\pi}\nu r\right).$$

3. The extension theorem for the classes $\mathcal{E}_{\{\omega\}}$. In this section, we characterize those Whitney jets on closed subsets of $\mathbf{R}^N$ which are restrictions of functions in $\mathcal{E}_{\{\omega\}}(\mathbf{R}^N)$, ω a strong weight function. To do this, we use the idea of proof of Bruna [5] together with the main result of the previous section.

3.1. Definition. Let $A\neq\emptyset$ be a closed subset of $\mathbf{R}^N$. A *jet* on A is a family $F=(f^\alpha)_{\alpha\in\mathbf{N}_0^N}\in C(A)^{\mathbf{N}_0^N}$. For a jet F on A and for $x, y\in A$, $z\in\mathbf{R}^N$, $m\in\mathbf{N}_0$ and $\alpha\in\mathbf{N}_0^N$ with $|\alpha|\leqslant m$ define

$$(T_x^m F)(z) := \sum_{|\beta|\leqslant m}\frac{1}{\beta!}(z-x)^\beta f^\beta(x),$$

$$(R_x^m F)_\alpha(y) := f^\alpha(y) - \sum_{|\alpha+\beta|\leqslant m}\frac{1}{\beta!}f^{\alpha+\beta}(x)(y-x)^\beta.$$

F is called a *Whitney jet* if it satisfies

$$|(R_x^m F)_\alpha(y)| = o(|x-y|^{m-|\alpha|})\quad\text{for all } m\in\mathbf{N}_0 \text{ and } |\alpha|\leqslant m \text{ as } |x-y|\to 0.$$

3.2. Definition. Let ω be a weight function and let $A\neq\emptyset$ be a closed subset of $\mathbf{R}^N$. A Whitney jet $F=(f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ on A is called an *ω-Whitney jet of Roumieu type* if the following holds: For each compact set $K\subset A$ there exist $m\in\mathbf{N}$ and $M>0$ such that

$$(1)\qquad \sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N}|f^\alpha(x)|\exp\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right)\leqslant M,$$

(2) for each $l\in\mathbf{N}_0$, each $\alpha\in\mathbf{N}_0^N$ with $|\alpha|\leqslant l$ and all $x, y\in K$

$$|(R_x^l F)_\alpha(y)| \leqslant M\frac{|x-y|^{l+1-|\alpha|}}{(l+1-|\alpha|)!}\exp\left(\frac{1}{m}\varphi^*(m(l+1))\right).$$

By $\mathcal{E}_{\{\omega\}}(A)$ we denote the linear space of all ω-Whitney jets of Roumieu type on A.

3.3. Remark. For each weight function ω and each closed set $A\subset\mathbf{R}^N$ the restriction map $\varrho_A\colon f\mapsto (f^{(\alpha)}|A)_{\alpha\in\mathbf{N}_0^N}$ maps $\mathcal{E}_{\{\omega\}}(\mathbf{R}^N)$ into $\mathcal{E}_{\{\omega\}}(A)$.

For a singleton A, condition 3.2(2) is empty. Moreover, Bonet, Meise and Taylor [3], 3.8 (see 1.7), shows that in this case ϱ_A is surjective if and only if ω is a strong weight function. To show that for each strong weight function ω, the map $\varrho_A\colon \mathcal{E}_{\{\omega\}}(\mathbf{R}^N)\to\mathcal{E}_{\{\omega\}}(A)$ is surjective for each closed set A in $\mathbf{R}^N$, we need some preparation.

3.4. Lemma. *Let ω be a weight function. Then for each $m\in\mathbf{N}$ there exists $0<s_0<1$ such that for each $0<s<s_0$ we have*

$$\sup_{j\in\mathbf{N}_0}\frac{j!}{s^j}\exp\left(-\frac{1}{m}\varphi^*(mj)\right)\geqslant\exp\left(\frac{1}{m}\omega^*(3ms)\right).$$

Proof. First note that we can find $a \geqslant 1$ and a weight function κ with $\kappa|[0,1] \equiv 0$ so that $\kappa|[a,\infty[= \omega|[a,\infty[$. This implies the existence of $s_a > 0$ so that for all $0 < s < s_a$ we have $\omega^*(s) = \kappa^*(s)$. Hence we can assume that ω satisfies $\omega|[0,1] \equiv 0$, which implies $\varphi^{**} = \varphi$ by 1.2(b). Now fix $m \in \mathbf{N}$ and choose $0 < s_0 < (3e^2)^{-1}$ so that

$$\omega^*(ms) = \sup_{t \geqslant 1}(\omega(t) - mts) \quad \text{for all } 0 < s < s_0.$$

Then we get for $0 < s < s_0$

$$\begin{aligned}
\frac{1}{m}\omega^*(ms) &= \sup_{t\geqslant 1}\left(\frac{1}{m}\varphi(\log t) - ts\right) = \sup_{t\geqslant 1}\left(\frac{1}{m}\varphi^{**}(\log t) - ts\right)\\
&= \sup_{t\geqslant 1}\left(\frac{1}{m}\sup_{y>0}(y\log t - \varphi^*(y)) - ts\right)\\
&= \sup_{y>0}\left(\sup_{t\geqslant 1}\left(\frac{y}{m}\log t - ts\right) - \frac{1}{m}\varphi^*(y)\right)\\
&= \sup_{y>0}\left(\frac{y}{m}\log\frac{y}{ems} - \frac{1}{m}\varphi^*(y)\right) = \sup_{z>0}\left(z\log\frac{z}{es} - \frac{1}{m}\varphi^*(mz)\right).
\end{aligned}$$

Since $n! \geqslant (n/3)^n$ for all $n \in \mathbf{N}$, we get from this for $0 < s < s_0$

$$\begin{aligned}
\sup_{j\in\mathbf{N}_0}\frac{j!}{s^j}\exp\left(-\frac{1}{m}\varphi^*(mj)\right) &= \exp\sup_{j\in\mathbf{N}_0}\left(\log(j!) - j\log s - \frac{1}{m}\varphi^*(mj)\right)\\
&= \exp\sup_{j\in\mathbf{N}_0}\left(\log(j+1)! - \log(j+1) - j\log s - \frac{1}{m}\varphi^*(mj)\right)\\
&\geqslant \exp\sup_{j\in\mathbf{N}_0}\left(\log\left(\frac{j+1}{3}\right)^{j+1} - j\log se - \frac{1}{m}\varphi^*(mj)\right)\\
&\geqslant \exp\sup_{z>0}\left(\log\left(\frac{z}{3}\right)^z - z\log se - \frac{1}{m}\varphi^*(mz)\right)\\
&= \exp\sup_{z>0}\left(z\log\frac{z}{3se} - \frac{1}{m}\varphi^*(mz)\right)\\
&= \exp 3\sup_{z>0}\left(\frac{z}{3}\log\frac{z}{3}\frac{1}{se} - \frac{1}{3m}\varphi^*\left(3m\frac{z}{3}\right)\right) = \exp\left(\frac{1}{m}\omega^*(3ms)\right).
\end{aligned}$$

3.5. Lemma. *For each weight function ω and all $a, b, c, d > 0$ there is $n \in \mathbf{N}$ with*

$$\lim_{y\downarrow 0}\left(\frac{a}{n}\omega^*\left(\frac{n}{b}y\right) - c\omega^*(dy)\right) = -\infty.$$

Proof. Let $n > \max(bd, a/c)$. Then

$$\lim_{y\downarrow 0}\left(\frac{a}{n}\omega^*\left(\frac{n}{b}y\right) - c\omega^*(dy)\right) \leqslant \lim_{y\downarrow 0}\left(\frac{a}{n}\omega^*(dy) - c\omega^*(dy)\right) = -\infty.$$

Next we state Bruna's version of Whitney's cover of the complement of a compact set in $\mathbf{R}^N$, which is convenient also for our purposes (see Bruna [5], 3.2, or Stein [20], Chapter VI).

3.6. Lemma. *Let $K \neq \varnothing$ be a compact subset of $\mathbf{R}^N$. Then there exists a collection of closed cubes $(Q_j)_{j\in\mathbf{N}}$ with sides parallel to the axes such that*

(a) $\mathbf{R}^N \setminus K = \bigcup_{j\in\mathbf{N}} Q_j$,
(b) $\mathring{Q}_i \cap \mathring{Q}_j = \varnothing$ *for* $i \neq j$,
(c) $\operatorname{diam} Q_j \leqslant \operatorname{dist}(Q_j, K) \leqslant 4\operatorname{diam} Q_j$ *for all* $j \in \mathbf{N}$,
(d) *let Q_j^* denote the cube which has the same center as Q_j, expanded by $9/8$; then there exist $m_0, M_0 > 0$ so that*

$$m_0 \operatorname{diam} Q_j \leqslant \operatorname{dist}(z, K) \leqslant M_0 \operatorname{diam} Q_j \quad \text{for all } z \in Q_j^*,$$

(e) $\sharp\{j \in \mathbf{N}: Q_i^* \cap Q_j^* \neq \varnothing\} \leqslant 12^{2N}$ *for each* $i \in \mathbf{N}$,
(f) *there exist $m_1, M_1 > 0$ so that for $i, j \in \mathbf{N}$ with $Q_i^* \cap Q_j^* \neq \varnothing$ we have*

$$m_1 \operatorname{diam} Q_i \leqslant \operatorname{diam} Q_j \leqslant M_1 \operatorname{diam} Q_i.$$

Now Proposition 2.2, Lemma 3.6, [4], 4.4, [16], 3.3(4), and the arguments of the proof of Bruna [5], 3.3, show that the following holds:

3.7. Lemma. *Let ω be a strong weight function which is concave and satisfies $\omega(0) = 0$. Furthermore, let $K \neq \varnothing$ be a compact subset of $\mathbf{R}^N$ and let $(Q_j)_{j\in\mathbf{N}}$ and $(Q_j^*)_{j\in\mathbf{N}}$ be as in Lemma 3.6. Then we have, in the notation of Lemma 3.6: For each $n \in \mathbf{N}$ there exist $p \in \mathbf{N}$, $0 < r_0 < 1/2$, $C > 0$ and a sequence $(\Phi_j)_{j\in\mathbf{N}}$ in $\mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$ which satisfy*

(a) $\Phi_j \geqslant 0$ *for all* $j \in \mathbf{N}$,
(b) $\operatorname{supp}\Phi_j \subset Q_j^*$ *for all* $j \in \mathbf{N}$,
(c) $\sum_{j\in\mathbf{N}}\Phi_j(x) = 1$ *for all* $x \in \mathbf{R}^N \setminus K$,
(d) *if* $\operatorname{dist}(Q_j, K) < r_0 M_1^{-1}$ *then*

$$\sup_{x\in\mathbf{R}^N}\sup_{\alpha\in\mathbf{N}_0^N}|\Phi_j^{(\alpha)}(x)|\exp\left(-\frac{1}{p}\varphi^*(p|\alpha|)\right) \leqslant C\exp\left(\frac{N12^{2N}}{n}\omega^*\left(\frac{n}{N12^{2N}}\frac{m_1}{2\sqrt{N}}\operatorname{diam} Q_j\right)\right).$$

3.8. Lemma. *Let ω be a strong weight function, let $K \neq \varnothing$ be a compact subset of $\mathbf{R}^N$ and let $F = (f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ be in $\mathscr{E}_{\{\omega\}}(K)$. Then there exist a compact cube H with $K \subset \mathring{H}$, $j \in \mathbf{N}$ and $A > 0$ so that for each $x \in K$ there is $f_x \in \mathscr{D}_{\{\omega\}}(H)$ such that*

$$f_x^{(\alpha)}(x) = f^\alpha(x) \quad \text{for all } \alpha \in \mathbf{N}_0^N \text{ and all } x \in K, \tag{1}$$

$$\sup_{x\in K}\sup_{y\in H}\sup_{\alpha\in\mathbf{N}_0^N}|f_x^{(\alpha)}(y)|\exp\left(-\frac{1}{j}\varphi^*(j|\alpha|)\right) \leqslant A. \tag{2}$$

Moreover, there exist $k\in\mathbf{N}$, $B>0$ and $d_0>0$ so that for all $\alpha\in\mathbf{N}_0^N$ and all $x, y\in K$, $z\in\mathbf{R}^N$ with $|z-x|+|z-y|<d_0$ we have

$$(3)\qquad |(f_x-f_y)^{(\alpha)}(z)|\leqslant B\exp\Big(\frac{1}{k}\varphi^*(k|\alpha|)\Big)\exp\Big(-\frac{1}{k}\omega^*\big(k(|z-x|+|z-y|)\big)\Big).$$

Proof. Condition 3.2(1) implies that $\{(f^\alpha(x))_{\alpha\in\mathbf{N}_0^N}: x\in K\}$ is a bounded subset of $\mathscr{E}_{\{\omega\}}(\{0\})$ if we endow $\mathscr{E}_{\{\omega\}}(\{0\})$ with its natural inductive limit topology. Since ω is a strong weight function, the map $\varrho_{\{0\}}$: $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)\to\mathscr{E}_{\{\omega\}}(\{0\})$ is surjective by Bonet, Meise and Taylor [3], 3.8 (see 1.7). Consequently, for a fixed cube H_0 with $0\in\mathring{H}_0$, the restriction map

$$\varrho\colon \mathscr{D}_{\{\omega\}}(H_0)\to\mathscr{E}_{\{\omega\}}(\{0\}),\qquad \varrho(f):=(f^{(\alpha)}(0))_{\alpha\in\mathbf{N}_0^N},$$

is surjective, too. Since both spaces are (DFN)-spaces (see [4], 3.6) each bounded set in $\mathscr{E}_{\{\omega\}}(\{0\})$ is the image of a bounded set in $\mathscr{D}_{\{\omega\}}(H_0)$. From this we get the existence of a compact cube H in $\mathbf{R}^N$ with $K\subset\mathring{H}$ so that there exists a bounded set Q in $\mathscr{D}_{\{\omega\}}(H)$ such that for each $x\in K$ there exists $f_x\in Q$ satisfying $f_x^{(\alpha)}(x)=f^\alpha(x)$ for all $\alpha\in\mathbf{N}_0^N$. Obviously, this implies the first part of the assertion.

To prove the second one, fix $x, y\in K$, $l\in\mathbf{N}$ and $\alpha\in\mathbf{N}_0^N$ with $|\alpha|\leqslant l$. Then note that by Malgrange [15], p. 3, we have

$$T_x^l F(z)-T_y^l F(z)=\sum_{|\beta|\leqslant l}\frac{1}{\beta!}(z-x)^\beta(R_y^l F)_\beta(x).$$

Since F satisfies condition 3.2(2), there exist $m\in\mathbf{N}$ and $A>0$ so that for all $x, y\in K$ and $z\in\mathbf{R}^N$ we have

$$|(T_x^l F-T_y^l F)^{(\alpha)}(z)|\leqslant A\exp\Big(\frac{1}{m}\varphi^*(m(l+1))\Big)\frac{(|z-x|+|x-y|)^{l+1-|\alpha|}}{(l+1-|\alpha|)!}$$
$$\leqslant A\exp\Big(\frac{1}{m}\varphi^*(m(l+1))\Big)2^{l+1-|\alpha|}\frac{(|z-x|+|z-y|)^{l+1-|\alpha|}}{(l+1-|\alpha|)!}.$$

Now note that by [4], 1.4, there exist $L>1$ and $D>0$ such that

$$\varphi^*(t)-t\geqslant L\varphi^*(t/L)-D\qquad\text{for all } t>0.$$

Evaluating at $t=Lm(l+1)$, we get

$$(4)\quad |(T_x^l F-T_y^l F)^{(\alpha)}(z)|\leqslant Ae^D\frac{(|z-x|+|z-y|)^{l+1-|\alpha|}}{(l+1-|\alpha|)!}\exp\Big(\frac{1}{Lm}\varphi^*(Lm(l+1))\Big).$$

On the other hand, Taylor's formula and (2) imply the existence of $A_0>0$ so that for each $l\in\mathbf{N}_0$, $\alpha\in\mathbf{N}_0^N$ with $|\alpha|\leqslant l$ and all $x\in K$, $z\in\mathbf{R}^N$ we have

$$(5)\qquad |(f_x-T_x^l F)^{(\alpha)}(z)|\leqslant A_0\frac{|z-x|^{l+1-|\alpha|}}{(l+1-|\alpha|)!}\exp\Big(\frac{1}{j}\varphi^*(j(l+1))\Big).$$

Now choose $B>3\max(A_0, Ae^D)$ and $\nu\in\mathbf{N}$, $\nu>2\max(j, Lm)$; then (4) and (5) give for all $l\in\mathbf{N}_0$, $\alpha\in\mathbf{N}_0^N$ with $|\alpha|\leqslant l$ and all $x, y\in K$, $z\in\mathbf{R}^N$

$$|(f_x-f_y)^{(\alpha)}(z)|\leqslant B\frac{(|z-x|+|z-y|)^{l+1-|\alpha|}}{(l+1-|\alpha|)!}\exp\Big(\frac{2}{\nu}\varphi^*\Big(\frac{\nu}{2}(l+1)\Big)\Big)$$
$$\leqslant B\frac{(|z-x|+|z-y|)^{l+1-|\alpha|}}{(l+1-|\alpha|)!}\exp\Big(\frac{1}{\nu}\varphi^*(\nu|\alpha|)\Big)\exp\Big(\frac{1}{\nu}\varphi^*(\nu(l+1-|\alpha|))\Big)$$

because of the convexity of $t\mapsto(2/\nu)\varphi^*(t)$.

Now observe that this and Lemma 3.4 imply (3) by taking the infimum over l.

3.9. THEOREM. *For each strong weight function ω and each compact set $K\neq\emptyset$ in $\mathbf{R}^N$ the restriction map ϱ_K: $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)\to\mathscr{E}_{\{\omega\}}(K)$ is surjective.*

Proof. Because of Remark 1.9 we can assume that ω is concave and satisfies $\omega(0)=0$. Fix $F=(f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ in $\mathscr{E}_{\{\omega\}}(K)$. Then there exist numbers k, j, A, B, and d_0 as in 3.8. We let m_0, M_0, and m_1 be as in 3.6 and assume $m_0\leqslant 1$. Because of 3.5, there is $n\in\mathbf{N}$ such that

$$(1)\quad g(y):=\frac{N12^{2N}}{n}\omega^*\Big(\frac{n}{N12^{2N}}\frac{m_1}{2\sqrt{N}}\frac{1}{M_0}y\Big)-\frac{1}{k}\omega^*\Big(\frac{7k}{m_0}y\Big)\to-\infty\qquad\text{as } y\to 0.$$

We apply Lemma 3.7 with this n to get numbers p, r_0, and C and a sequence $(\Phi_j)_{j\in\mathbf{N}}$ in $\mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$. We may assume $p\geqslant k$. For each $i\in\mathbf{N}$, choose $x_i\in K$ so that $\operatorname{dist}(x_i, Q_i)=\operatorname{dist}(K, Q_i)$. Then for all $z\in Q_i^*$ we have by Lemma 3.6

$$|z-x_i|\leqslant\operatorname{dist}(x_i, Q_i)+\operatorname{diam}Q_i^*=\operatorname{dist}(K, Q_i)+\tfrac{9}{8}\operatorname{diam}Q_i\leqslant 6\operatorname{diam}Q_i.$$

On the other hand, $\operatorname{dist}(z, K)\geqslant m_0\operatorname{diam}Q_i$ for $z\in Q_i^*$, hence,

$$(2)\qquad \operatorname{dist}(z, K)\leqslant|z-x_i|\leqslant\frac{6}{m_0}\operatorname{dist}(z, K)\qquad\text{for all } z\in Q_i^*.$$

Now we define

$$\tilde{f}(z)=\begin{cases} f^0(z) & \text{for } z\in K,\\ \sum_{i=1}^{\infty}\Phi_i(z)f_{x_i}(z) & \text{for } z\in\mathbf{R}^N\setminus K.\end{cases}$$

We establish the following claim: There are $P\geqslant k$ and $C_1, C_2>0$ such that for all $x\in K$ and $z\notin K$ with $|z-x|<\min(m_0d_0/7, m_0r_0/(4M_1))$ we have for all $\gamma\in\mathbf{N}_0^N$

$$(3)\qquad |(\tilde{f}-f_x)^{(\gamma)}(z)|\exp\Big(-\frac{1}{P}\varphi^*(P|\gamma|)\Big)\leqslant C_1\exp\Big(-\frac{1}{k}\omega^*\Big(\frac{7k}{m_0}|x-z|\Big)\Big)$$
$$+C_2\exp\big(g(\operatorname{dist}(z, K))\big).$$

To prove this claim, fix $x\in K$, $z\notin K$ satisfying the above inequality and let $\gamma\in\mathbf{N}_0^N$. Then

$$(4)\qquad (\tilde f-f_x)^{(\gamma)}(z)=\sum_{\beta\leqslant\gamma}\binom{\gamma}{\beta}\sum_{i=1}^{\infty}\Phi_i^{(\beta)}(z)(f_{x_i}-f_x)^{(\gamma-\beta)}(z).$$

First we estimate the term with $\beta=0$. If $z\in\operatorname{supp}\Phi_i\subset Q_i^*$, then (2) and $m_0\leqslant 1$ imply

$$|x-z|+|x_i-z|\leqslant|x-z|+6m_0^{-1}\operatorname{dist}(z,K)\leqslant 7m_0^{-1}|x-z|\leqslant d_0.$$

Therefore 3.8(3) shows that the term with $\beta=0$, multiplied by $\exp(-(1/P)\varphi^*(P|\gamma|))$, is estimated by the first term on the right hand side of (3) with $C_1=B$.

Next fix $\beta\neq 0$ and choose $\bar z\in K$ with $|z-\bar z|=\operatorname{dist}(z,K)$. Then $\sum_{i=1}^{\infty}\Phi_i^{(\beta)}(z)=0$ implies

$$\sum_{i=1}^{\infty}\Phi_i^{(\beta)}(z)(f_{x_i}-f_x)^{(\gamma-\beta)}(z)=\sum_{i=1}^{\infty}\Phi_i^{(\beta)}(z)\big((f_{x_i}-f_{\bar z})^{(\gamma-\beta)}(z)+(f_{\bar z}-f_x)^{(\gamma-\beta)}(z)\big)$$
$$=\sum_{i=1}^{\infty}\Phi_i^{(\beta)}(z)(f_{x_i}-f_{\bar z})^{(\gamma-\beta)}(z).$$

Because of this and $|z-x_i|+|z-\bar z|\leqslant(6m_0^{-1}+1)\operatorname{dist}(z,K)\leqslant 7m_0^{-1}|z-x|\leqslant d_0$, we can apply 3.8(3) to get

$$(5)\qquad |(f_{x_i}-f_{\bar z})^{(\gamma-\beta)}(z)|\leqslant B\exp\Big(\frac1k\varphi^*(k(|\gamma|-|\beta|))\Big)\exp\Big(-\frac1k\omega^*\Big(\frac{7k}{m_0}\operatorname{dist}(z,K)\Big)\Big).$$

On the other hand, if $z\in\operatorname{supp}\Phi_i$, we have

$$\operatorname{dist}(Q_i,K)\leqslant 4\operatorname{diam}Q_i\leqslant\frac{4}{m_0}\operatorname{dist}(z,K)\leqslant\frac{4}{m_0}|x-z|<\frac{r_0}{M_1},$$

and because of $\operatorname{diam}Q_i\geqslant M_0^{-1}\operatorname{dist}(z,K)$, Proposition 3.7(d) yields

$$(6)\qquad |\Phi_i^{(\beta)}(z)|\leqslant C\exp\Big(\frac1p\varphi^*(p|\beta|)\Big)\exp\Big(\frac{12^{2N}N}{n}\omega^*\Big(\frac{n}{12^{2N}N}\frac{m_1}{2\sqrt N}\frac{1}{M_0}\operatorname{dist}(z,K)\Big)\Big).$$

Taking into account that by 3.6(e) no more than 12^{2N} of the $\Phi_i(z)$ are nonzero, we get estimates for every term in (4) if we combine (5) and (6). We have $\sum_{\beta\leqslant\gamma}\binom{\gamma}{\beta}=2^{|\gamma|}$. Thus we have shown (3) because, as in the proof of 3.8(4), there are $P>p$ and $C_2>0$ with

$$C2^{|\gamma|}\exp\Big(\frac1p\varphi^*(p|\gamma|)\Big)\leqslant C_2\exp\Big(\frac1P\varphi^*(P|\gamma|)\Big)\quad\text{ for all }\gamma.$$

Now, given $\alpha\in\mathbf{N}_0^N$, we define

$$\tilde f_\alpha(z)=\begin{cases}\tilde f^{(\alpha)}(z) & \text{for } z\in\mathbf{R}^N\setminus K,\\ f^{\alpha}(z) & \text{for } z\in K.\end{cases}$$

We want to show that $\tilde f$ is a C^∞-function with $\tilde f^{(\alpha)}=\tilde f_\alpha$ for all $\alpha\in\mathbf{N}_0^N$. By Hestenes [9], Lemma 1, it suffices to show that all $\tilde f_\alpha$ are continuous. This is clear near points $x\notin K$, so let x be in K. Then by 3.1 we have for $z\in K$

$$|\tilde f_\alpha(z)-\tilde f_\alpha(x)|=|(R_x^{|\alpha|}F)_\alpha(z)|=o(1).$$

If $z\notin K$ and $|x-z|$ is small enough, then

$$|\tilde f_\alpha(z)-\tilde f_\alpha(x)|=|\tilde f^{(\alpha)}(z)-f^\alpha(x)|$$
$$\leqslant|\tilde f^{(\alpha)}(z)-f_x^{(\alpha)}(z)|+|f_x^{(\alpha)}(z)-f_x^{(\alpha)}(x)|\to 0\quad\text{ as } z\to x,$$

where we apply (3), (1), and $\lim_{t\to 0}\omega^*(t)=\infty$ to estimate the first term and the continuity of $f_x^{(\alpha)}$ for the other.

To see that $\tilde f$ is in $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$, let $n\in\mathbf{N}$ be as in (1), find $\varepsilon_0>0$ so that $g(y)\leqslant 0$ for all $0<y\leqslant\varepsilon_0$, and choose $\varepsilon=\min(\varepsilon_0, d_0m_0/7, m_0r_0/(4M_1))$. For $z\in\mathbf{R}^N\setminus K$ with $\operatorname{dist}(z,K)\leqslant\varepsilon$, we take $x\in K$ with $|z-x|=\operatorname{dist}(z,K)$. Then for $\alpha\in\mathbf{N}_0^N$, we apply (3) and 3.8(2) to get

$$|\tilde f^{(\alpha)}(z)|\leqslant|f_x^{(\alpha)}(z)|+|\tilde f^{(\alpha)}(z)-f_x^{(\alpha)}(z)|$$
$$\leqslant A\exp\Big(\frac1j\varphi^*(j|\alpha|)\Big)+(C_1+C_2)\exp\Big(\frac1P\varphi^*(P|\gamma|)\Big).$$

For $z\in K$ we use 3.8(2) alone. In both cases we see that

$$|\tilde f^{(\alpha)}(z)|\leqslant C_3\exp\Big(\frac1P\varphi^*(P|\gamma|)\Big)\quad\text{ for all } z\in\mathbf{R}^N\text{ with }\operatorname{dist}(z,K)\leqslant\varepsilon.$$

This is enough, since for all compact $Q\subset\mathbf{R}^N$ the number of indices i with

$$\operatorname{supp}\Phi_i\cap\{z\in Q\mid\operatorname{dist}(z,K)>\varepsilon\}\neq\emptyset$$

is finite, and all the products $\Phi_i f_{x_i}$ are in $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$.

3.10. Corollary. *For a weight function ω the following assertions are equivalent:*

(1) *For each closed set $A\neq\emptyset$ in $\mathbf{R}^N$ the restriction map $\varrho_A\colon\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)\to\mathscr{E}_{\{\omega\}}(A)$ is surjective.*

(2) *ω is a strong weight function.*

Proof. (1)⇒(2). Choosing $A=\{0\}$, this follows from Bonet, Meise and Taylor [3], 3.8 (see 1.7).

(2)⇒(1). For $n\in\mathbf{N}$ let $B_n:=\{x\in\mathbf{R}^N\colon|x|\leqslant n\}$ and $B_0:=\emptyset$. Then choose functions φ_n in $\mathscr{D}_{\{\omega\}}(\mathbf{R}^N)$ such that for each $n\in\mathbf{N}$ we have

$$(3)\qquad 0\leqslant\varphi_n\leqslant 1,\quad \operatorname{supp}\varphi_n\subset\mathring B_{n+1}\setminus B_{n-1},\quad\sum_{n=1}^{\infty}\varphi_n(x)=1\text{ for all }x\in\mathbf{R}^N.$$

Next fix a closed set $A \neq \emptyset$ in $\mathbf{R}^N$ and a Whitney jet $F = (f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ in $\mathscr{E}_{\{\omega\}}(A)$. For $n\in\mathbf{N}$ let F_n denote the jet $(f^\alpha | (B_{n+1}\setminus\mathring{B}_{n-1}))_{\alpha\in\mathbf{N}_0^N}$. Obviously, F_n is in $\mathscr{E}_{\{\omega\}}(B_{n+1}\setminus\mathring{B}_{n-1})$. Hence Theorem 3.9 implies the existence of $f_n \in \mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ so that $\varrho_{B_{n+1}\setminus\mathring{B}_{n-1}}(f_n) = F_n$. Now (3) implies that

$$f\colon x \mapsto \sum_{n=1}^{\infty} \varphi_n(x) f_n(x)$$

is in $\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$. To show that $\varrho_A(f) = F$, we fix $x\in A$. Then there exists a unique natural number m so that $x\in B_m\setminus B_{m-1}$. Because of (3), this implies $\varphi_n(x) = 0$ for $m\notin\{m-1, m\}$. Hence we get

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \varphi_n(x) f_n(x) = \varphi_{m-1}(x) f_{m-1}(x) + \varphi_m(x) f_m(x) \\ &= (\varphi_{m-1}(x)+\varphi_m(x)) f^0(x) = f^0(x). \end{aligned}$$

Similarly it follows that $f^{(\alpha)}(x) = f^\alpha(x)$ for each $x\in A$ and each $\alpha\in\mathbf{N}_0^N$.

In the classical theory, a simpler description of Whitney jets is possible, provided the compact set has Whitney's property (P). To show that the same holds in our setting, we first recall the definition of property (P) (see Whitney [21]).

3.11. Definition. A closed subset A of $\mathbf{R}^N$ with $\overline{\mathring{A}} = A$ has property (P) if for every compact subset K of A there is a constant $C > 0$ such that any two points x and y of K are joined by a rectifiable curve in A of length not exceeding $C|x-y|$.

3.12. Corollary. *Let ω be a strong weight function and let $A \subset \mathbf{R}^N$ be closed. Assume that $\overline{\mathring{A}} = A$ and that A has property* (P). *Let $(f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ be a family in $C(A)\cap C^\infty(\mathring{A})$ satisfying*

(1) $\qquad f^\alpha(x) = (f^0)^{(\alpha)}(x)$ *for all* $x\in\mathring{A}$, $\alpha\in\mathbf{N}_0^N$,

(2) *for each compact set $K\subset A$ there are $m\in\mathbf{N}$ and $M > 0$ such that*

$$\sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N} |f^\alpha(x)| \exp\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right) \leq M.$$

Then there exists $g\in\mathscr{E}_{\{\omega\}}(\mathbf{R}^N)$ with $g^{(\alpha)}(x) = f^\alpha(x)$ for all $x\in A$, $\alpha\in\mathbf{N}_0^N$.

Proof. In view of 3.9 it suffices to show that $F = (f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ is in $\mathscr{E}_{\{\omega\}}(A)$. Choose a compact $K\subset A$ and $k\in\mathbf{N}$ with $K\subset\{x\in A\colon |x| < k\}$. Let $K_1 = \{x\in A\colon |x|\leq k\}$ and let C be the constant belonging to K_1 in property (P). Let $K_2 = \{x\in A\colon |x|\leq (2C+1)k\}$, and apply (2) to K_2 to get $m\in\mathbf{N}$ and $M > 0$. Let now $x, y\in K$ be given. Then there are sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ in $\mathring{K}_1$ tending to x and y respectively. By property (P) there is for fixed $n\in\mathbf{N}$ a rectifiable curve ϱ in A from x_n to y_n of length not exceeding $C|x_n - y_n|$.

Since $x_n, y_n\in\mathring{A}$ we may perturb ϱ to a C^1-curve σ joining x_n and y_n of length at most $2C|x_n-y_n|$ which is parametrized by arc length. Note that σ must lie in K_2. Whitney [21], Lemma 3, states that for each $l\in\mathbf{N}$ and each $\alpha\in\mathbf{N}_0^N$ with $|\alpha|\leq l$

$$|(R^l_{x_n}F)_\alpha(y_n)| \leq \frac{N^{l-|\alpha|}}{(l-|\alpha|-1)!}(2C|x_n-y_n|)^{l-|\alpha|}\sup_{z\in\sigma}|f^\alpha(x_n)-f^\alpha(z)|.$$

Assuming $\sigma(0) = x_n$ and using that σ is parametrized by arc length, we have by the mean value theorem for all t in the domain of σ

$$|f^\alpha(x_n)-f^\alpha(\sigma(t))| \leq |t|\sup_\tau|(\operatorname{grad} f^\alpha(\tau), \sigma'(\tau))| \leq 2CN|x_n-y_n|\sup_{|\gamma|=1}\sup_{z\in K_2}|f^{\alpha+\gamma}(z)|.$$

Combining this with (2) we have

$$|(R^l_{x_n}F)_\alpha(y_n)| \leq \frac{N^{l+1-|\alpha|}}{(l-|\alpha|-1)!}(2C|x_n-y_n|)^{l+1-|\alpha|} M\exp\left(\frac{1}{m}\varphi^*(m(|\alpha|+1))\right).$$

Letting n tend to infinity we get 3.2(2) if we apply the procedure used to prove 3.8(4) to swallow the extra factor $(2CN)^{l+1-|\alpha|}(l-|\alpha|)(l+1-|\alpha|)$.

3.13. Remark. Let $(M_p)_{p\in\mathbf{N}_0}$ be a sequence of positive numbers which has the following properties (see Komatsu [13]):

(M1) $\quad M_p^2 \leq M_{p-1}M_{p+1}$ for all $p\in\mathbf{N}$,

(M2) $\quad$ there exist $A, H > 1$ with $M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}$ for all $p\in\mathbf{N}$,

(M3) $\quad$ there exists $A > 0$ with $\sum_{q=p+1}^{\infty}\frac{M_{q-1}}{M_q} \leq Ap\frac{M_p}{M_{p+1}}$ for all $p\in\mathbf{N}$,

and define $\omega_M\colon [0, \infty[\,\to [0, \infty[$ by

$$\omega_M(t) := \begin{cases} \sup_{p\in\mathbf{N}_0}\log\dfrac{t^p M_0}{M_p} & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Then Meise and Taylor [16], 3.11, shows that ω_M is a strong weight function for which we have $\mathscr{E}_{\{\omega_M\}}(\Omega) = \mathscr{E}^{\{M_p\}}(\Omega)$ for each open set Ω in $\mathbf{R}^N$, where

$$\mathscr{E}^{\{M_p\}}(\Omega) := \{f\in C^\infty(\Omega)\colon \text{for each } K\subset\Omega \text{ compact there is } h > 0 \text{ with} \sup_{\alpha\in\mathbf{N}_0^N}\sup_{x\in K}|f^{(\alpha)}(x)|(h^{|\alpha|}M_{|\alpha|})^{-1} < \infty\}.$$

From this and Theorem 3.9 it follows that Theorem 3.1 of Bruna [5] holds, whenever $(M_p)_{p\in\mathbf{N}_0}$ satisfies (M1)–(M3). Hence some of the hypotheses in [5], 3.1, are superfluous. This has been remarked independently also by Chung and Kim [7].

4. The extension theorem for the classes $\mathscr{E}_{(\omega)}$. In this section, we characterize those Whitney jets on closed subsets of $\mathbf{R}^N$ which are restrictions of functions in $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$, ω a strong weight function.

4.1. DEFINITION. Let ω be a weight function and let $A \neq \emptyset$ be a closed subset of $\mathbf{R}^N$. A Whitney jet $F = (f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ on A is called an *ω-Whitney jet of Beurling type* if the following holds: For each compact set $K \subset A$ and each $m\in\mathbf{N}$ there exists $M > 0$ such that

(1)
$$\sup_{x\in K}\sup_{\alpha\in\mathbf{N}_0^N} |f^{(\alpha)}(x)|\exp\big(-m\varphi^*(|\alpha|/m)\big) \leqslant M,$$

(2) for each $l\in\mathbf{N}_0$, each $\alpha\in\mathbf{N}_0^N$ with $|\alpha| \leqslant l$ and all $x, y\in K$
$$|(R_x^l F)_\alpha(y)| \leqslant M\frac{|x-y|^{l+1-|\alpha|}}{(l+1-|\alpha|)!}\exp\left(m\varphi^*\left(\frac{l+1}{m}\right)\right).$$

By $\mathscr{E}_{(\omega)}(A)$ we denote the linear space of all ω-Whitney jets of Beurling type on A.

4.2. Remark. For each weight function ω and each closed set $A \subset \mathbf{R}^N$ the restriction map ϱ_A: $f\mapsto (f^{(\alpha)}|A)_{\alpha\in\mathbf{N}_0^N}$ maps $\mathscr{E}_{(\omega)}(\mathbf{R}^N)$ into $\mathscr{E}_{(\omega)}(A)$.

For a singleton A, condition 4.1(2) is empty. Moreover, Meise and Taylor [16], 3.10 (see 1.7), shows that in this case ϱ_A is surjective if and only if ω is a strong weight function. We will show in the sequel that ϱ_A: $\mathscr{E}_{(\omega)}(\mathbf{R}^N) \to \mathscr{E}_{(\omega)}(A)$ is surjective for each closed set in $\mathbf{R}^N$, provided that ω is a strong weight function. To obtain this by a reduction argument from Theorem 3.9, we prove the following two lemmas.

4.3. LEMMA. *Let $(M_j)_{j\in\mathbf{N}}$ be a sequence of positive numbers and let $(\psi_j)_{j\in\mathbf{N}}$ be a sequence of differentiable functions on $[0, \infty[$ which satisfies for all $j\in\mathbf{N}$:*

(i) *ψ_j is convex and increasing and $\psi_j(0) = 0$,*
(ii) *$\psi_j'(t) > \psi_{j+1}'(t)$ for all $t > 0$,*
(iii) *$\lim_{t\to\infty}(\psi_j(t)-\psi_{j+1}(t)) = \infty$,*
(iv) *$\lim_{t\to\infty}\psi_j'(t) = \infty$ for all $j\in\mathbf{N}$.*

Then there exist a sequence $(N_j)_{j\in\mathbf{N}}$ of positive numbers and a convex function h: $[0, \infty[\to[0, \infty[$ such that

(1)
$$h(t) \geqslant \inf_{j\in\mathbf{N}}(\psi_j(t)+M_j) \quad \text{for all } t > 0,$$

(2)
$$h(t) \leqslant \psi_j(t)+N_j \quad \text{for all } t > 0 \text{ and all } j\in\mathbf{N}.$$

Proof. It is easy to check that by enlarging the numbers M_j if necessary, we can find a strictly increasing sequence $(t_j)_{j\in\mathbf{N}}$ in $[0, \infty[$ with $t_0 = 0$ so that

(3)
$$m(t) := \inf_{j\in\mathbf{N}}(\psi_j(t)+M_j) = \psi_k(t)+M_k \quad \text{for } t\in[t_{k-1}, t_k].$$

Then we distinguish two cases:

Case 1: There exist $A, B > 0$ so that $At+B \geqslant m(t)$ for all $t \geqslant 0$. Then the hypotheses imply that we can choose h: $t\mapsto At+B$.

Case 2: Not case 1. We claim that in this case we can find a strictly increasing sequence $(n_j)_{j\in\mathbf{N}}$ in $\mathbf{N}$, a sequence $(s_j)_{j\in\mathbf{N}_0}$ in $[0, \infty[$ and a continuous convex function h: $[0, \infty[\to[0, \infty[$ satisfying (1) and (2), so that the following holds:

(4)
$$s_0 = 0 \quad\text{and}\quad s_j\in\,]t_{n_{j+1}-1}, t_{n_{j+1}}] \quad\text{for all } j\in\mathbf{N},$$

(5)
$$h(t) = m(t) \qquad \text{for } s_{j-1} \leqslant t \leqslant t_{n_j},$$

(6)
$$h(t) = m(t_{n_j})+\psi_{n_j}'(t_{n_j})(s-t_{n_j}) \quad \text{for } t_{n_j} \leqslant t \leqslant s_j.$$

To prove this by induction, we define $s_0 := 0$ and $n_1 := 1$. Assume that n_j and s_{j-1} have been chosen already. Then define h_j: $[0, \infty[\to[0, \infty[$ by
$$h_j(s) := m(t_{n_j})+\psi_{n_j}'(t_{n_j})(s-t_{n_j})$$
and note that the hypothesis of case 2 implies that
$$\Sigma_j := \{s > t_{n_j}\colon h_j(s) < m(s)\}$$
is not empty. Hence
$$s_j := \inf\Sigma_j \geqslant t_{n_j},$$
and we can choose n_{j+1} so that $s_j\in\,]t_{n_{j+1}-1}, t_{n_{j+1}}]$. Because of (ii), it is clear that $n_{j+1} > n_j$. From this choice it is evident that
$$h\colon t \mapsto \begin{cases} m(t) & \text{for } s_{j-1} \leqslant t \leqslant t_{n_j},\\ h_j(t) & \text{for } t_{n_j} \leqslant t \leqslant s_j,\end{cases}$$
is continuous and satisfies (5) and (6). To prove the convexity of h, note that for $s_j\in\,]t_{n_{j+1}-1}, t_{n_{j+1}}]$ we have $m(s_j) = h_j(s_j)$, $m'_-(s_j) = h'_+(s_j)$, and for some $\delta > 0$, $m(\tau) > h_j(\tau)$ for $s_j < \tau < s_j+\delta$. Therefore an easy calculus argument shows
$$h'_-(s_j) = h_j'(s_j) \leqslant m'_+(s_j) \leqslant m'_-(s_j) = h'_+(s_j).$$
The convexity in t_{n_j} is clear.

Next note that h satisfies (1) by construction. To prove (2), fix $l\in\mathbf{N}$ and $j\in\mathbf{N}$ so that $l < n_j$. Then (5) and (3) give
$$h(t) = m(t) \leqslant \psi_l(t)+M_l \quad \text{for all } t\in[s_j, t_{n_{j+1}}].$$
For $t\in[t_{n_j}, s_j]$ we get (assuming $s_j > t_{n_j}$) from (6) and (3)
$$\psi_{n_j}(t)+M_{n_j} \geqslant h_j(t) = h(t) \quad\text{and}\quad \psi_l(t_{n_j})+M_l \geqslant \psi_{n_j}(t_{n_j})+M_{n_j}.$$
Therefore (ii) implies $\psi_l(t)+M_l \geqslant h(t)$ for all $t\in[t_{n_j}, s_j]$. Consequently, $\psi_l(t)+M_l \geqslant h(t)$ for all $t\in[t_{n_j}, t_{n_{j+1}}]$ whenever $n_j > l$. Therefore we can choose $N_l > M_l$ so that (2) holds.

4.4. Lemma. *Let ω be a strong weight function and assume that h: $[0, \infty[\to[0, \infty[$ satisfies $\omega = o(h)$. Then there exists a strong weight function σ so that $\omega = o(\sigma)$ and $\sigma = o(h)$.*

Proof. Since ω is a strong weight function, 1.7(3) implies the existence of $K > 1$ so that

$$\limsup_{t\to\infty} \omega(Kt)/\omega(t) < K.$$

Next note that 1.1(γ) and the hypothesis imply $\lim_{t\to\infty} h(t) = \infty$. Therefore we can define inductively a sequence $(x_n)_{n\in\mathbf{N}}$ in $[0, \infty[$ with $x_1 = 0$ and $\omega(x_2) > 0$ which has the following properties:

$$(1)\qquad \int_{x_{n+1}}^{\infty} \frac{\omega(t)}{1+t^2}\,dt \leqslant (n+1)^{-3},$$

$$(2)\qquad x_{n+1} \geqslant Kx_n,$$

$$(3)\qquad \omega(x_{n+1}) \geqslant 2^{n+1-i}\omega(x_i), \quad 1 \leqslant i \leqslant n,$$

$$(4)\qquad h(x) \geqslant n^2\omega(x) \quad \text{for all } x \geqslant x_n.$$

Then we define σ: $[0, \infty[\to[0, \infty[$ by

$$\sigma(x) := n\omega(x) - \sum_{i=1}^{n} \omega(x_i) \quad \text{for } x\in[x_n, x_{n+1}[.$$

Obviously, σ is continuous and satisfies 1.1(δ). Moreover, for $n \geqslant 2$ and $x\in[x_n, x_{n+1}[$ we have

$$(5)\qquad \sigma(x) = \Big(n - \sum_{i=1}^{n} \frac{\omega(x_i)}{\omega(x)}\Big)\omega(x) \geqslant \Big(n - \sum_{i=1}^{n} 2^{i-n}\Big)\omega(x) \geqslant (n-2)\omega(x)$$

and hence

$$\omega(x) \leqslant \frac{1}{n-2}\sigma(x) \quad \text{for all } x\in[x_n, x_{n+1}[\text{ and } n \geqslant 2.$$

This proves $\omega = o(\sigma)$ and consequently σ satisfies 1.1(γ). From (4) and the definition of σ we get

$$\sigma(x) \leqslant n\omega(x) \leqslant \frac{1}{n}h(x) \quad \text{for all } x \geqslant x_n,$$

and hence $\sigma = o(h)$.

Next choose $\varepsilon > 0$ and $N\in\mathbf{N}$ so that

$$(6)\qquad \omega(Kt)/\omega(t) \leqslant K-\varepsilon \quad \text{for all } t \geqslant x_N,$$

fix $x \geqslant x_N$ and distinguish the following two cases:

Case 1: $x_n \leqslant x < Kx \leqslant x_{n+1}$ for an appropriate $n \geqslant N$. Then (5) and (6) give

$$\sigma(Kx)-\sigma(x) = n(\omega(Kx)-\omega(x)) \leqslant n((K-\varepsilon)\omega(x)-\omega(x))$$
$$\leqslant n(K-\varepsilon-1)\omega(x) \leqslant \frac{n}{n-2}(K-\varepsilon-1)\sigma(x).$$

Case 2: $x_n \leqslant x < x_{n+1} < Kx$ for an appropriate $n \geqslant N$. Then (2) implies $Kx \leqslant x_{n+2}$. Therefore (5) and (6) give

$$\sigma(Kx)-\sigma(x) = (n+1)\omega(Kx)-\omega(x_{n+1})-n\omega(x)$$
$$\leqslant (n+1)(\omega(Kx)-\omega(x)) \leqslant \frac{n+1}{n-2}(K-\varepsilon-1)\sigma(x).$$

Altogether, this proves that

$$(7)\qquad \limsup_{x\to\infty} \sigma(Kx)/\sigma(x) \leqslant K-\varepsilon < K.$$

Since it is no restriction to assume $K \geqslant 2$, this implies that σ satisfies 1.1(α). Now note that (1) implies that σ satisfies 1.1(β) (see the proof of Braun, Meise and Taylor [4], 1.6). Hence (7) shows that σ is a strong weight function which has all the desired properties.

4.5. Theorem. *For each strong weight function ω and each compact set $K \neq \emptyset$ in $\mathbf{R}^N$ the restriction map ϱ_K: $\mathscr{E}_{(\omega)}(\mathbf{R}^N)\to\mathscr{E}_{(\omega)}(K)$ is surjective.*

Proof. Fix a compact set $K \neq \emptyset$ in $\mathbf{R}^N$ and fix $F = (f^\alpha)_{\alpha\in\mathbf{N}_0^N}$ in $\mathscr{E}_{(\omega)}(K)$. For $n\in\mathbf{N}_0$ let

$$a_n := \sup_{|\alpha|=n} \sup_{x\in K} |f^\alpha(x)|,$$

$$b_0 := 0, \quad b_{n+1} := \sup_{|\alpha|\leqslant n} \sup_{x,y\in K, x\neq y} |(R_x^n F)_\alpha(y)| \frac{(n+1-|\alpha|)!}{|x-y|^{n+1-|\alpha|}},$$

and define g: $[0, \infty[\to\mathbf{R}$ by

$$g(t) := \log\max(a_n, b_n, 1) \quad \text{for } n \leqslant t < n+1.$$

Since φ^* is increasing and since F is in $\mathscr{E}_{(\omega)}(K)$, we get from 4.1 the existence of a sequence $(M_j)_{j\in\mathbf{N}}$ in $[0, \infty[$ so that

$$(1)\qquad g(t) \leqslant j\varphi^*(t/j)+M_j \quad \text{for all } t \geqslant 0,\ j\in\mathbf{N}.$$

Now define ψ_j: $t\mapsto j\varphi^*(t/j)$ and note that w.l.o.g. we can assume that φ is C^1 with $\lim_{t\to\infty} \varphi'(t) = \infty$. Then $(\varphi^*)' = (\varphi')^{-1}$, and $(\psi_j)_{j\in\mathbf{N}}$ satisfies the hypotheses of Lemma 4.3. Therefore, (1) implies the existence of a convex function h: $[0, \infty[\to[0, \infty[$ and of a sequence $(N_j)_{j\in\mathbf{N}}$ so that

$$(2)\qquad g \leqslant h \leqslant \inf_{j\in\mathbf{N}}(\psi_j+N_j).$$

From this and the definition of ψ_j we get for each $j \in \mathbf{N}$

$$(3) \qquad h^*(x) \geqslant j\varphi^{**}(x) - N_j = j\varphi(x) - N_j.$$

Now define f: $t \mapsto h^*(\max(0, \log t))$ and note that by (3) we have for all $j \in \mathbf{N}$ and all $t \geqslant 1$

$$\omega(t) = \varphi(\log t) \leqslant \frac{1}{j}(h^*(\log t) + N_j) = \frac{1}{j} f(t) + \frac{1}{j} N_j.$$

This proves $\omega = o(f)$. Therefore, Lemma 4.4 implies the existence of a strong weight function σ and of $A \in]0, \infty[$ so that

$$(4) \qquad \omega = o(\sigma) \quad \text{and} \quad \sigma \leqslant f + A.$$

Hence, we have

$$\psi(x) := \sigma(e^x) \leqslant f(e^x) + A = h^*(x) + A \quad \text{for all } x \geqslant 0.$$

Therefore, (2) implies $g \leqslant h = h^{**} \leqslant \psi^* + A$. From this and from the definition of g it follows that F is in $\mathscr{E}_{\{\sigma\}}(K)$. Since σ is a strong weight function, Theorem 3.9 gives the existence of $f \in \mathscr{E}_{\{\sigma\}}(\mathbf{R}^N)$ with $\varrho_K(f) = F$. This proves the theorem, since (4) together with [4], 3.9 and 4.5, implies $\mathscr{E}_{\{\sigma\}}(\mathbf{R}^N) \subset \mathscr{E}_{(\omega)}(\mathbf{R}^N)$.

4.6. Corollary. *For a weight function ω the following assertions are equivalent*:

(1) *For each closed set $A \neq \emptyset$ in $\mathbf{R}^N$, the restriction map ϱ_A: $\mathscr{E}_{(\omega)}(\mathbf{R}^N) \to \mathscr{E}_{(\omega)}(A)$ is surjective.*

(2) *ω is a strong weight function.*

Proof. (1) ⇒ (2). Choosing $A = \{0\}$, this follows from Meise and Taylor [16], 3.10 (see 1.7).

(2) ⇒ (1). This follows from Theorem 4.5 by the same arguments which were used in the proof of Corollary 3.10.

The following corollary is the analogue to 3.12 for the Beurling case.

4.7. Corollary. *Let ω be a strong weight function and let $A \subset \mathbf{R}^N$ be closed. Assume that $\bar{\mathring{A}} = A$ and that A has property* (P) (see 3.11). *Let $(f^\alpha)_{\alpha \in \mathbf{N}_0^N}$ be a family in $C(A) \cap C^\infty(\mathring{A})$ satisfying*

$$(1) \qquad f^\alpha(x) = (f^0)^{(\alpha)}(x) \quad \textit{for all } x \in \mathring{A}, \alpha \in \mathbf{N}_0^N,$$

(2) *for each compact set $K \subset A$ and each $m \in \mathbf{N}$ there is $M > 0$ such that*

$$\sup_{x \in K} \sup_{\alpha \in \mathbf{N}_0^N} |f^\alpha(x)| \exp\left(-\frac{1}{m}\varphi^*(m|\alpha|)\right) \leqslant M.$$

Then there exists $g \in \mathscr{E}_{(\omega)}(\mathbf{R}^N)$ with $g^{(\alpha)}(x) = f^\alpha(x)$ for all $x \in A$, $\alpha \in \mathbf{N}_0^N$.

4.8. Remark. For a sequence $(M_p)_{p \in \mathbf{N}_0}$ satisfying (M1)–(M3), define ω_M as in 3.13. Then Meise and Taylor [16], 3.11, shows that $\mathscr{E}_{(\omega_M)}(\Omega) = \mathscr{E}^{(M_p)}(\Omega)$ for each open set Ω in $\mathbf{R}^N$, where

$$\mathscr{E}^{(M_p)}(\Omega) := \{f \in C^\infty(\Omega): \text{ for each } K \subset \Omega \text{ compact and each } h > 0: \sup_{\alpha \in \mathbf{N}_0^N} \sup_{x \in K} |f^{(\alpha)}(x)| (h^{|\alpha|} M_{|\alpha|})^{-1} < \infty\}.$$

Since ω_M is a strong weight function, Corollary 4.6 implies that Théorème I.3.3 of Kantor [12] holds for $\Omega = \mathbf{R}^N$, whenever $(M_p)_{p \in \mathbf{N}_0}$ satisfies (M1)–(M3). Note that Kantor [12] states Théorème I.3.3 for sequences $(M_p)_{p \in \mathbf{N}_0}$ satisfying only (M1) and (M3)′. By Petzsche [18], 3.5 and 1.1, and also by Meise and Taylor [16], 3.10, his statement is not correct.

Note that an easy modification of the proof of 4.4 together with the idea of proof of Braun, Meise and Taylor [4], 1.9, shows the following:

4.9. Lemma. *Let ω be a weight function, h_j: $[0, \infty[\to [0, \infty[$ so that $\omega = o(h_j)$ for all $j \in \mathbf{N}$. Then there exists a weight function σ with $\omega = o(\sigma)$ and $\sigma = o(h_j)$ for all $j \in \mathbf{N}$.*

From this one derives as in the proof of 4.5:

4.10. Corollary. *For each weight function ω, each open subset Ω of $\mathbf{R}^N$ and each $f \in \mathscr{E}_{(\omega)}(\Omega)$ there is a weight function σ with $\omega = o(\sigma)$ and $f \in \mathscr{E}_{\{\sigma\}}(\Omega)$.*

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