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On topologizable algebras

by

V. MÜLLER (Prague)

Abstract. We construct a commutative algebra which is not topologizable as a topological algebra (with jointly continuous multiplication) and a commutative topological algebra which cannot be topologized as a locally convex algebra.

Let A be an associative algebra. By [2] it is always possible to define a topology on A which makes A a locally convex algebra with separately continuous multiplication (i.e. $x_\alpha, x, y \in A$, $x_\alpha \to x$ implies $x_\alpha y \to xy$, $yx_\alpha \to yx$).

On the other hand (cf. [3]) in general it is not possible to introduce a topology on A which makes A a locally convex algebra with jointly continuous multiplication (i.e. $x_\alpha \to x$, $y_\beta \to y \Rightarrow x_\alpha y_\beta \to xy$). The aim of this note is to exhibit two examples which continue these investigations.

In the first example we construct a commutative algebra which admits no topology. This gives a negative answer to the question raised in [2]. In the second example we construct a topological algebra which admits no locally convex topology.

All algebras in this paper will be complex (this condition, however, is not essential).

We say that an algebra A is *topologizable* (*topologizable as a locally convex algebra*) if there exists a topology on A which makes A a topological (locally convex) algebra with jointly continuous multiplication.

It is easy to see that an algebra A is topologizable if and only if there exists a system $\mathscr{V}$ of subsets of A (zero-neighbourhoods in A) satisfying

(1) $\bigcap_{V\in\mathscr{V}} V = \{0\}$,
(2) $\lambda V \subset V$ for every $V \in \mathscr{V}$ and complex number λ, $|\lambda| \leqslant 1$,
(3) each $V \in \mathscr{V}$ is absorbent,
(4) for every $V \in \mathscr{V}$ there exists $W \in \mathscr{V}$ such that $W + W \subset V$,
(5) for every $V \in \mathscr{V}$ there exists $W \in \mathscr{V}$ such that $W \cdot W \subset V$.

For basic properties of topological algebras see e.g. [1].

THEOREM 1. *There exists a commutative algebra which is not topologizable.*

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Proof. Denote by $\mathbf{N}$ the set of all positive integers and by $\mathscr{F}$ the set of all sequences $f = \{f_j\}_{j=1}^{\infty}$ of positive integers. Consider the linear space A of all formal linear combinations of elements c, $x_i (i \in \mathbf{N})$ and $a_f (f \in \mathscr{F})$. We define a multiplication in A by

$$cz = zc = 0 \quad \text{for every } z \in A,$$

$$x_i x_j = 0 \quad (i, j \in \mathbf{N}),$$

$$a_f a_{f'} = 0 \quad (f, f' \in \mathscr{F}),$$

$$x_n a_f = a_f x_n = f_n c \quad (n \in \mathbf{N},\ f \in \mathscr{F}).$$

Clearly these relations define uniquely a multiplication on A which makes A a commutative algebra (for the associative law note that the product of any three of the basis elements is zero).

We prove that A is not topologizable. Suppose on the contrary that there exists a system $\mathscr{V}$ of zero-neighbourhoods in A satisfying (1)–(5). Let $V, W \in \mathscr{V}$ satisfy $c \notin V$ and $W \cdot W \subset V$.

For $n = 1, 2, \ldots$ choose $s_n > 0$ such that $x_n \in s_n \cdot W$. Let $f = \{f_n\}_{n=1}^{\infty}$ be a sequence of positive integers with $f_n > n s_n$. Then $a_f \in r \cdot W$ for some $r > 0$. We have

$$c = \frac{1}{f_n}(x_n a_f) = \frac{r s_n}{f_n}\left(\frac{x_n}{s_n} \cdot \frac{a_f}{r}\right) \in \frac{r s_n}{f_n} \cdot W \cdot W \subset \frac{r s_n}{f_n} V.$$

Since $c \notin V$ we have $r s_n / f_n > 1$ and $r > f_n / s_n > n$ $(n \in \mathbf{N})$, a contradiction.

Remark. Let X be a linear space of infinite dimension and let $\mathscr{L}(X)$ be the algebra of all linear mappings acting in X. By [3], $\mathscr{L}(X)$ cannot be topologized as a locally convex algebra. Using analogous method as in Theorem 1 it is possible to show that $\mathscr{L}(X)$ is not topologizable. In fact, even the algebra of all finite-dimensional operators in X is not topologizable.

THEOREM 2. *There exists a commutative topological algebra which is not topologizable as a locally convex algebra.*

Proof. Let K be an uncountable set. Denote by $\mathscr{D}$ the set of all functions $d\colon \mathbf{N} \times K \to \mathbf{N}$. For $d \in \mathscr{D}$, $n \in \mathbf{N}$ and $k \in K$ we shall write shortly d_{nk} instead of $d(n, k)$.

Clearly for every $d \in \mathscr{D}$ and $n \in \mathbf{N}$ there exists a subset $K_{dn} \subset K$ and a positive integer d_n such that $\operatorname{card} K_{dn} = d_n$ and $d_{nk} = d_n$ for every $k \in K_{dn}$. Let A be the linear space of all (finite) linear combinations of elements c, x_{nk} $(n \in \mathbf{N},\ k \in K)$, a_d $(d \in \mathscr{D})$ and y_{dnk} $(d \in \mathscr{D},\ n \in \mathbf{N},\ k \in K_{dn} \subset K)$.

We define a multiplication in A by

$$cz = zc = 0 \quad (z \in A),$$

$$y_{dnk} z = z y_{dnk} = 0 \quad (z \in A,\ d \in \mathscr{D},\ n \in \mathbf{N},\ k \in K_{dn}),$$

$$a_d a_{d'} = 0 \quad (d, d' \in \mathscr{D}),$$

$$x_{nk} x_{n'k'} = 0 \quad (n, n' \in \mathbf{N},\ k, k' \in K),$$

$$x_{nk} a_d = a_d x_{nk} = \begin{cases} d_n y_{dnk} & (d \in \mathscr{D},\ n \in \mathbf{N},\ k \in K_{dn}), \\ 0 & (k \notin K_{dn}). \end{cases}$$

Clearly A is a commutative algebra. To define a topology on A we shall need the following notations:

Let $\mathscr{L}$ be the set of all complex-valued functions $\lambda\colon k \mapsto \lambda_k$ defined on K with finite support. For $\lambda \in \mathscr{L}$ and $i \in \{0, 1, 2, \ldots\}$ define

$$m_i(\lambda) = \min_{\substack{M \subset K \\ \operatorname{card} M = i}} \max_{j \in K \setminus M} |\lambda_j|.$$

Clearly $\max_{j \in K} |\lambda_j| = m_0(\lambda) \geqslant m_1(\lambda) \geqslant \ldots$ and $\operatorname{card}\{j \in K\colon |\lambda_j| > m_i(\lambda)\} \leqslant i$.

LEMMA 3. *Let $\lambda, \mu \in \mathscr{L}$ and let $s, t \in \{0, 1, 2, \ldots\}$. Then*

$$m_{s+t}(\lambda + \mu) \leqslant m_s(\lambda) + m_t(\mu)$$

where $\lambda + \mu \in \mathscr{L}$ is defined by $(\lambda + \mu)_k = \lambda_k + \mu_k$ $(k \in K)$.

Proof. Suppose $j \in K$, $|\lambda_j + \mu_j| > m_s(\lambda) + m_t(\mu)$. Then either $|\lambda_j| > m_s(\lambda)$ or $|\mu_j| > m_t(\mu)$. Hence

$$\operatorname{card}\{j\colon |\lambda_j + \mu_j| > m_s(\lambda) + m_t(\mu)\} \leqslant \operatorname{card}\{j\colon |\lambda_j| > m_s(\lambda)\} + \operatorname{card}\{j\colon |\mu_j| > m_t(\mu)\} \leqslant s + t,$$

and we conclude that $m_{s+t}(\lambda + \mu) \leqslant m_s(\lambda) + m_t(\mu)$.

For $\lambda \in \mathscr{L}$ define $h(\lambda) = \sum_{i=0}^{\infty} (i+1) m_i(\lambda)$.

LEMMA 4. *If $\lambda, \mu \in \mathscr{L}$ then $h(\lambda + \mu) \leqslant 4[h(\lambda) + h(\mu)]$.*

Proof. We have

$$\begin{aligned} h(\lambda + \mu) &= \sum_{r=0}^{\infty} (2r+1) m_{2r}(\lambda + \mu) + \sum_{r=0}^{\infty} (2r+2) m_{2r+1}(\lambda + \mu) \\ &\leqslant \sum_{r=0}^{\infty} (2r+1)[m_r(\lambda) + m_r(\mu)] + \sum_{r=0}^{\infty} (2r+2)[m_r(\lambda) + m_{r+1}(\mu)] \\ &\leqslant \sum_{r=0}^{\infty} (4r+3)[m_r(\lambda) + m_r(\mu)] \leqslant 4[h(\lambda) + h(\mu)]. \end{aligned}$$

We now continue the proof of Theorem 2. Let $u \in A$, i.e. u can be expressed as

$$u = \alpha c + \sum_{n\in\mathbf{N}} \sum_{k\in K} \beta_{nk} x_{nk} + \sum_{d\in\mathscr{D}} \gamma_d a_d + \sum_{d\in\mathscr{D}} \sum_{n\in\mathbf{N}} \sum_{k\in K_{dn}} \delta_{dnk} y_{dnk} \tag{6}$$

where α, β_{nk}, γ_d, δ_{dnk} are complex numbers such that only a finite number of them are nonzero. For u of the form (6) define

$$f(u) = |\alpha| + \sum_{n\in\mathbf{N}} h(\{\beta_{nk}\}_{k\in K}) + \sum_{d\in\mathscr{D}} |\gamma_d| + \sum_{d\in\mathscr{D}} \sum_{n\in\mathbf{N}} \frac{2}{d_n+1} h(\{\delta_{dnk}\}_{k\in K})$$

(we put formally $\delta_{dnk} = 0$ for $k \in K \setminus K_{dn}$).

The function $f\colon A \to [0, \infty)$ has the following properties:

(a) $u \in A$, $u \neq 0 \Rightarrow f(u) \neq 0$,

(b) $f(\varepsilon u) = |\varepsilon| f(u)$ for each complex number ε and $u \in A$,

(c) $f(u+u') \leqslant 4[f(u)+f(u')]$,

(d) $f(uu') \leqslant 8 f(u) f(u')$.

The first two properties are evident, property (c) follows from Lemma 4. To prove (d) suppose that $u, u' \in A$ are of the form (6) (i.e. $u' = \alpha' c + \sum_n \sum_k \beta'_{nk} x_{nk} + \ldots$). Then

$$\begin{aligned} f(uu') &= f\Big(\sum_{d,n} \sum_{k\in K_{dn}} d_n y_{dnk} (\beta_{nk}\gamma'_d + \beta'_{nk}\gamma_d)\Big) \\ &= \sum_{d,n} \frac{2d_n}{d_n+1} h(\{\beta_{nk}\gamma'_d + \beta'_{nk}\gamma_d\}_{k\in K_{dn}}) \\ &\leqslant 8 \sum_{d,n} [|\gamma'_d| h(\{\beta_{nk}\}_{k\in K_{dn}}) + |\gamma_d| h(\{\beta'_{nk}\}_{k\in K_{dn}})] \leqslant 8 f(u) f(u'). \end{aligned}$$

Let $V = \{u \in A\colon f(u) < 1\}$ and $\mathscr{V} = \{tV\colon t \in (0, \infty)\}$. Then $\mathscr{V}$ satisfies conditions (1)–(5) so A with the topology given by $\mathscr{V}$ is a topological algebra.

Let $M \subset A$ be the subspace generated by the elements of the form

$$c - \frac{1}{d_n} \sum_{k\in K_{dn}} y_{dnk}, \quad d \in \mathscr{D},\ n \in \mathbf{N}.$$

Clearly M is an ideal in A.

Let $u \in A$ be of the form (6). If $\beta_{nk} \neq 0$ for some $n \in \mathbf{N}$, $k \in K$ or $\gamma_d \neq 0$ for some $d \in \mathscr{D}$ then $(u+\varepsilon V) \cap M = \emptyset$ for a suitable $\varepsilon > 0$, so $u \notin \bar{M}$. Similarly, $u \notin \bar{M}$ if $\delta_{dnk} \neq \delta_{dnk'}$ for some d, n, k, k'. Finally, if

$$u = \alpha c - \sum_{d,n} \sum_{k\in K_{dn}} \varepsilon_{dn} y_{dnk} \quad \text{and} \quad \alpha \neq \sum_{d,n} d_n \varepsilon_{dn}$$

we have $u \notin \bar{M}$ as $f\big((1/d_n)\sum_{k\in K_{dn}} y_{dnk}\big) = 1$ $(d \in \mathscr{D},\ n \in \mathbf{N})$.

Hence M is a closed ideal in A and $c \notin M$. Let $B = A/M$ and let $\pi\colon A \to B$ be the canonical homomorphism. Then B is a topological algebra and $\pi(c) \neq 0$.

We prove that B is not topologizable as a locally convex algebra. Suppose on the contrary that there exists a system $\mathscr{W}$ of convex zero-neighbourhoods in B satisfying (1)–(5). We shall need the following lemma:

LEMMA 5. *For every $W \in \mathscr{W}$ there exist $d \in \mathscr{D}$ and $n \in \mathbf{N}$ such that $\pi(y_{dnk}) \in W$ for every $k \in K_{dn}$.*

Proof. Let $W \in \mathscr{W}$. Suppose on the contrary that for every $d \in \mathscr{D}$ and $n \in \mathbf{N}$ there exists $k \in K_{dn}$ with $\pi(y_{dnk}) \notin W$. Let $W' \in \mathscr{W}$ satisfy $W'W' \subset W$. For $n \in \mathbf{N}$ and $k \in K$ choose $s_{nk} > 0$ such that $\pi(x_{nk}) \in s_{nk} W'$.

Choose $d = \{d_{nk}\}_{n\in\mathbf{N}} \in \mathscr{D}$ such that $d_{nk} > n s_{nk}$ $(n \in \mathbf{N},\ k \in K)$. Then $\pi(a_d) \in rW'$ for some $r > 0$.

We have supposed that for every $n \in \mathbf{N}$ there exists $k \in K_{dn}$ such that $\pi(y_{dnk}) \notin W$. On the other hand, we have

$$\pi(y_{dnk}) = \frac{1}{d_n} \pi(x_{nk}) \pi(a_d) \in \frac{1}{d_n} s_{nk} W' r W' \subset \frac{s_{nk} r}{d_n} W.$$

So $s_{nk} r/d_n > 1$, $r > d_n/s_{nk} > n$ for every $n \in \mathbf{N}$, which is a contradiction.

We now conclude the proof of Theorem 2. Let $W \in \mathscr{W}$. Let $d \in \mathscr{D}$ and $n \in \mathbf{N}$ be given by Lemma 5. Then

$$\pi(c) = \frac{1}{d_n} \sum_{k\in K_{dn}} \pi(y_{dnk})$$

and $\pi(y_{dnk}) \in W$ for every $k \in K_{dn}$. Since W is convex and card $K_{dn} = d_n$ we have $\pi(c) \in W$ for every $W \in \mathscr{W}$, a contradiction with condition (1).

PROBLEM. Is it possible to construct separable algebras with properties of Theorem 1 (Theorem 2)?

References

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INSTITUTE OF MATHEMATICS ČSAV
Žitná 25, 115 67 Praha 1, Czechoslovakia

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