On topologizable algebras

by

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Abstract. We construct a commutative algebra which is not topologizable as a topological algebra (with jointly continuous multiplication) and a commutative topological algebra which cannot be topologized as a locally convex algebra.

Let $A$ be an associative algebra. By [2] it is always possible to define a topology on $A$ which makes $A$ a locally convex algebra with separately continuous multiplication (i.e. $x_{a}, x, y \in A, x_{a} \rightarrow x$ implies $x_{a}y \rightarrow xy, yx_{a} \rightarrow xy$).

On the other hand (cf. [3]) in general it is not possible to introduce a topology on $A$ which makes $A$ a locally convex algebra with jointly continuous multiplication (i.e. $x_{a} \rightarrow x, y_{a} \rightarrow y \Rightarrow x_{a}y_{a} \rightarrow xy$). The aim of this note is to exhibit two examples which continue these investigations.

In the first example we construct a commutative algebra which admits no topology. This gives a negative answer to the question raised in [2]. In the second example we construct a topological algebra which admits no locally convex topology.

All algebras in this paper will be complex (this condition, however, is not essential).

We say that an algebra $A$ is topologizable (topologizable as a locally convex algebra) if there exists a topology on $A$ which makes $A$ a topological (locally convex) algebra with jointly continuous multiplication.

It is easy to see that an algebra $A$ is topologizable if and only if there exists a system $\mathcal{V}$ of subsets of $A$ (zero-neighbourhoods in $A$) satisfying

1. $\bigcap_{V \in \mathcal{V}} V = \{0\}$,
2. $\lambda V \subseteq V$ for every $V \in \mathcal{V}$ and complex number $\lambda, |\lambda| \leq 1$,
3. each $V \in \mathcal{V}$ is absorbent,
4. for every $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W + W \subseteq V$,
5. for every $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W \cdot W \subseteq V$.

For basic properties of topological algebras see e.g. [1].

Theorem 1. There exists a commutative algebra which is not topologizable.

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Proof. Denote by $\mathbb{N}$ the set of all positive integers and by $\mathcal{F}$ the set of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of positive integers. Consider the linear space $A$ of all formal linear combinations of elements $c, x_i (i \in \mathbb{N})$ and $a_f (f \in \mathcal{F})$. We define a multiplication in $A$ by

$$cz = zc = 0 \quad \text{for every } z \in A,$$

$$x_i x_j = 0 \quad (i, j \in \mathbb{N}),$$

$$a_f a_{f'} = 0 \quad (f, f' \in \mathcal{F}),$$

$$x_n a_f = a_f x_n = f_n c \quad (n \in \mathbb{N}, f \in \mathcal{F}).$$

Clearly these relations define uniquely a multiplication on $A$ which makes $A$ a commutative algebra (for the associative law note that the product of any three of the basis elements is zero).

We prove that $A$ is not topologizable. Suppose on the contrary that there exists a system $\mathcal{V}$ of zero-neighbourhoods in $A$ satisfying (1)–(5). Let $V, W \in \mathcal{V}$ satisfy $c \not\in V$ and $W \cdot W \subseteq V$.

For $n = 1, 2, \ldots$ choose $\varepsilon_n > 0$ such that $x_n \in \varepsilon_n W$. Let $f = \{f_n\}_{n=1}^{\infty}$ be a sequence of positive integers with $f_n > n \varepsilon_n$. Then $a_f \in r \cdot W$ for some $r > 0$. We have

$$c = \frac{1}{f_n} (x_n a_f) = \frac{x_n}{f_n} \frac{a_f}{r} \in \varepsilon_n W \cdot W \subseteq \varepsilon_n W.$$

Since $c \not\in V$ we have $r \varepsilon_n f_n > 1$ and $r > f_n/\varepsilon_n > n (n \in \mathbb{N})$, a contradiction.

Remark. Let $X$ be a linear space of infinite dimension and let $\mathcal{L}(X)$ be the algebra of all linear mappings acting in $X$. By [3], $\mathcal{L}(X)$ cannot be topologized as a locally convex algebra. Using analogous method as in Theorem 1 it is possible to show that $\mathcal{L}(X)$ is not topologizable. In fact, even the algebra of all finite-dimensional operators in $X$ is not topologizable.

Theorem 2. There exists a commutative topological algebra which is not topologizable as a locally convex algebra.

Proof. Let $K$ be an uncountable set. Denote by $\mathcal{D}$ the set of all functions $d : \mathbb{N} \times K \to \mathbb{N}$. For $d \in \mathcal{D}$, $n \in \mathbb{N}$ and $k \in K$ we shall write shortly $d_{nk}$ instead of $d(n, k)$.

Clearly for every $d \in \mathcal{D}$ and $n \in \mathbb{N}$ there exists a subset $K_{dn} \subseteq K$ and a positive integer $d_n$ such that card $K_{dn} = d_n$ and $d_{nk} = d_n$ for every $k \in K_{dn}$. Let $A$ be the linear space of all (finite) linear combinations of elements $c, x_{dn} (n \in \mathbb{N}, k \in K), a_d (d \in \mathcal{D})$ and $y_{dn} (d \in \mathcal{D}, n \in \mathbb{N}, k \in K_{dn} \subseteq K)$.

We define a multiplication in $A$ by

$$cz = zc = 0 \quad (z \in A),$$

$$y_{dn} z = z y_{dn} \quad (z \in A, d \in \mathcal{D}, n \in \mathbb{N}, k \in K_{dn}),$$

$$a_d a_{d'} = 0 \quad (d, d' \in \mathcal{D}),$$

$$x_{dn} x_{dn'} = 0 \quad (n, n' \in \mathbb{N}, k, k' \in K),$$

$$x_{dn} a_d = a_d x_{dn} = \begin{cases} d_n y_{dn} & (d \in \mathcal{D}, n \in \mathbb{N}, k \in K_{dn}), \\ 0 & (k \not\in K_{dn}) \end{cases}$$

Clearly $A$ is a commutative algebra. To define a topology on $A$ we shall need the following notations:

Let $\mathcal{L}$ be the set of all complex-valued functions $\lambda : k \to \mathbb{C}$ defined on $K$ with finite support. For $\lambda \in \mathcal{L}$ and $I \in \{0, 1, 2, \ldots\}$ define

$$m_I (\lambda) = \min_{M \in K} \max_{M \in I} |\lambda_k|,$$

Clearly $\max_{M \in K} |\lambda_k| = m_0 (\lambda) \geq m_1 (\lambda) \geq \ldots$ and card $\{j \in K : |\lambda_j| > m_1 (\lambda)\} \leq I$.

Lemma 3. Let $\lambda, \mu \in \mathcal{L}$ and let $s, t \in \{0, 1, 2, \ldots\}$. Then

$$m_{s+1} (\lambda + \mu) \leq m_s (\lambda) + m_s (\mu).$$

Proof. Suppose $j \in K, |\lambda_j + \mu_j| > m_s (\lambda) + m_s (\mu)$. Then either $|\lambda_j| > m_s (\lambda)$ or $|\mu_j| > m_s (\mu)$. Hence

$$\text{card} \{j : |\lambda_j| > m_s (\lambda)\} \leq \text{card} \{j : |\lambda_j| > m_s (\lambda) + \text{card} \{j : |\mu_j| > m_s (\mu)\} \leq s + t,$$

and we conclude that $m_{s+1} (\lambda + \mu) \leq m_s (\lambda) + m_s (\mu)$.

For $\lambda \in \mathcal{L}$ define $h (\lambda) = \sum_{M \in K} (l+1) m_l (\lambda) k_l (\lambda) k_l (\mu)$.

Lemma 4. If $\lambda, \mu \in \mathcal{L}$ then $h (\lambda + \mu) \leq 4 h (\lambda) + h (\mu)$.

Proof. We have

$$h (\lambda + \mu) = \sum_{r=0}^{\infty} (2r+1)m_{2r} (\lambda + \mu) + \sum_{r=0}^{\infty} (2r+2)m_{2r+1} (\lambda + \mu)$$

$$= \sum_{r=0}^{\infty} (2r+1)[m_r (\lambda) + m_r (\mu)] + \sum_{r=0}^{\infty} (2r+2)[m_r (\lambda) + m_{r+1} (\mu)]$$

$$\leq \sum_{r=0}^{\infty} (4r+3)[m_r (\lambda) + m_r (\mu)] \leq 4 h (\lambda) + h (\mu).$$
We now continue the proof of Theorem 2. Let \( u \in A \), i.e. \( u \) can be expressed as
\[
 u = ac + \sum_{n \in N} \sum_{k \in K} \beta_{n,k} x_{n,k} + \sum_{d \in D} \gamma_d y_d + \sum_{d \in D} \sum_{\alpha \in \Delta} \delta_{d,n} y_{d,nk}
\]
where \( a, \beta_{n,k}, \gamma_d, \delta_{d,n} \) are complex numbers such that only a finite number of them are nonzero. For \( u \) of the form (6) define
\[
 f(u) = |\alpha| + \sum_{n \in N} h((\beta_{n,k})_{k \in K}) + |\gamma| + \sum_{d \in D} \frac{2}{d+1} h((\delta_{d,n})_{n \in N})
\]
(we put formally \( \delta_{d,n} = 0 \) for \( k \in K \setminus K_{d,n} \)).

The function \( f: A \to [0, \infty) \) has the following properties:
(a) \( u \in A, \alpha \neq 0 \Rightarrow f(u) \neq 0 \),
(b) \( f(\alpha u) = |\alpha| f(u) \) for each complex number \( \alpha \) and \( u \in A \),
(c) \( f(u + u') \leq 4 \left[ f(u) + f(u') \right] \),
(d) \( f(\alpha u) \leq 8 f(u) f(u') \).

The first two properties are evident, property (c) follows from Lemma 4.

To prove (d) suppose that \( u, u' \in A \) are of the form (6) (i.e. \( u = \alpha c + \sum_{n \in N} \sum_{k \in K} \beta_{n,k} x_{n,k} + \ldots \)). Then
\[
f(uu') = f \left( \sum_{d \in D} \sum_{k \in K_{d,n}} d_{n,k} y_{d,n} (\beta_{n,k} y_d + \beta_{n,k} \gamma_d) \right)
\]
\[
= \sum_{d \in D} \frac{2d_{n,k}}{d_{n,k} + 1} h((\beta_{n,k} y_d + \beta_{n,k} \gamma_d)_{k \in K_{d,n}})
\]
\[
\leq 8 \sum_{d \in D} \left[ |\gamma_d| h((\beta_{n,k} y_d)_{k \in K_{d,n}}) + |\gamma_d| h((\delta_{d,n})_{n \in N}) \right] \leq 8 f(u) f(u')
\]

Let \( V = \{ u \in A : f(u) < 1 \} \) and \( V' = \{ tV : t \in (0, \infty) \} \). Then \( V' \) satisfies conditions (1)–(5) so \( A \) with the topology given by \( V' \) is a topological algebra.

Let \( M \lesssim A \) be the subspace generated by the elements of the form
\[
c - \frac{1}{d_{n,k}} \sum_{n \in N} \gamma_{d,n} y_{d,nk}, \quad d \in D, \quad n \in N.
\]
Clearly \( M \) is an ideal in \( A \).

Let \( u \in A \) be of the form (6). If \( \beta_{n,k} \neq 0 \) for some \( n \in N, k \in K \) or \( \gamma_d \neq 0 \) for some \( d \in D \) then \((u + \varepsilon V) \cap M = \emptyset \) for a suitable \( \varepsilon > 0 \), so \( u \notin M \). Similarly, \( u \notin M \) if \( \delta_{d,n} \neq \delta_{d,n} \) for some \( d, n, k, k' \). Finally, if
\[
u = ac - \sum_{d \in D} \sum_{n \in N} \epsilon_{d,n} y_{d,nk} \quad \text{and} \quad \alpha \neq \sum_{d \in D} \epsilon_{d,n} y_{d,nk}
\]
we have \( u \notin M \) as \( f(\sum_{d \in D} \epsilon_{d,n} y_{d,nk}) = 1 \) \((d \in D, n \in N)\).

Hence \( M \) is a closed ideal in \( A \) and \( c \notin M \). Let \( B = A/M \) and let \( \pi: A \to B \) be the canonical homomorphism. Then \( B \) is a topological algebra and \( \pi(\alpha) \neq 0 \).