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Some approximation problems in L^p-spaces of matrix-valued functions

by

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Abstract. In [7] some Banach spaces $L^p(F)$, $1 < p < \infty$, of equivalence classes of matrix-valued functions which are p-integrable with respect to a nonnegative Hermitian matrix-valued measure F were introduced. In the special case $p = 2$, we obtain the Hilbert space arising from the theory of vector-valued stationary stochastic processes. Analogously to the theory of stationary processes we introduce the notions of interpolability, minimality, $\mathcal{J}_0$-regularity, and $\mathcal{J}_c$-regularity of the spaces $L^p(F)$ and characterize them in terms of F.

Introduction. In the theory of vector-valued stationary stochastic processes certain Hilbert spaces $L^2(F)$ of equivalence classes of matrix-valued functions which are square-integrable with respect to a nonnegative Hermitian matrix-valued measure F play an important role. In fact, there exists an isometric isomorphism between the Hilbert space spanned by the values of the process and the space $L^2(F)$, which can be described explicitly (cf. [14]). This makes it possible to consider the problems of linear extrapolation and interpolation of the stationary process as approximation problems in the space $L^2(F)$.

In [7] some Banach spaces $L^p(F)$, $1 \leqslant p \leqslant \infty$, of equivalence classes of matrix-valued functions which are p-integrable w.r.t. F were introduced. As a special case one obtains the above-mentioned space $L^2(F)$. Thus it is natural to study the approximation problems arising from stationary processes in $L^p(F)$ spaces. In our paper we study problems for $L^p(F)$ spaces, $1 < p < \infty$, originating in linear interpolation of vector-valued stationary stochastic processes. We will introduce concepts for $L^p(F)$ which are well known for stationary processes, e.g. interpolability and minimality.

H. Salehi pointed out the significance of Hellinger integrals in relation to interpolation of vector-valued stationary processes (see [16]–[19]). A. Weron improved Salehi's method and applied it to processes on locally compact abelian (abbreviated to LCA) groups (see [21], compare also [9] and [10]). We generalize Salehi and Weron's method to $L^p(F)$ spaces and thus can prove some of their results for these spaces. We note that Salehi and Scheidt [20] stated some further results on linear interpolation of vector-valued stationary processes. But their method uses the existence of an orthogonal projection and thus seems to be unsuited for $L^p(F)$ if $p \neq 2$.

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The first and second sections of our paper are devoted to some preliminaries on matrix integrals and Banach modules, respectively. In order to generalize Salehi and Weron's method to $L^p(F)$ we need a description of bounded matrix-linear functionals on $L^p(F)$. This description will be obtained in the third section using the description of bounded linear functionals on $L^p(F)$ given in [7]. The fourth section deals with the definition and properties of certain Banach spaces $H^p(F)$ whose elements are matrix-valued measures. In particular, we obtain an isometric isomorphism between $L^p(F)$ and $H^p(F)$, which allows us to investigate our problem in $H^p(F)$ instead of $L^p(F)$. The $H^p(F)$ spaces are generalizations of the space $H^2(F)$ introduced in [16] with the help of the Hellinger integral. In the fifth section we apply Weron's method to $L^p(F)$. Using these results we state criteria for interpolability and minimality in the sixth section of our paper. The seventh section is devoted to $\mathscr{J}_0$-regularity, where $\mathscr{J}_0$ is the family of singletons. Finally, in the eighth section we generalize Avetisyan and Dobrushin's result [1] on $\mathscr{J}_c$-regularity, where $\mathscr{J}_c$ is the family of nonempty and proper compact subsets.

Throughout the paper, we use the following notations. By $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ we denote the sets of positive integers, integers, real numbers, and complex numbers, respectively. The symbol $\mathbf{C}^n$, $n\in\mathbf{N}$, stands for the linear space of column vectors of length n. The entries of all matrices considered in our paper are complex numbers. For a matrix X, we denote by X^*, $X^\#$, $r(X)$, and $\operatorname{tr} X$ the adjoint matrix, the Moore–Penrose inverse, the rank, and the trace of X, respectively. The unit matrix and the zero matrix are denoted by I and 0, respectively. We will not specify their order in the notations, since no confusion may occur. By $|\cdot|$ we denote an arbitrary norm on a linear space of matrices and by $|\cdot|_E$ the euclidean norm: $|X|_E := (\operatorname{tr}(X^*X))^{1/2}$ for a matrix X. The symbols $\operatorname{Ker} U$ and $\mathscr{R}(U)$ stand for the kernel and the range of a linear operator U, respectively. Finally, $\operatorname{supp}\varphi$ denotes the support of a function φ on a topological space.

1. Integration of matrix-valued functions. Let $(\Omega, \mathscr{B}, \mu)$ be a positive measure space. As usual, all relations between measurable functions on Ω are to be understood as holding almost everywhere w.r.t. the measure μ. Furthermore, for an integrable function φ, we will often write $\int_\Omega \varphi\, d\mu$ instead of $\int_\Omega \varphi(\omega)\,\mu(d\omega)$.

For $n\in\mathbf{N}$, we denote by $\mathscr{M}_n$ the linear space of all $n\times n$-matrices. A function $\Phi\colon \Omega\to\mathscr{M}_n$, $\Phi := (\varphi_{ij})_{i,j=1}^n$, is called *measurable* (*integrable*, respectively) if all entries φ_{ij}, $i, j = 1, \ldots, n$, are measurable (integrable, respectively). If Φ is integrable, we set $\int_\Omega \Phi\, d\mu := (\int_\Omega \varphi_{ij}\, d\mu)_{i,j=1}^n$. For an integrable function Φ and X, $Y\in\mathscr{M}_n$, the functions $X\Phi$ and ΦY are integrable and we have the equalities $\int_\Omega X\Phi\, d\mu = X\int_\Omega \Phi\, d\mu$ and $\int_\Omega \Phi Y\, d\mu = \int_\Omega \Phi\, d\mu\, Y$.

For a function $\Phi\colon \Omega\to\mathscr{M}_n$, we define

$$\Phi^*\colon\quad \Phi^*(\omega) := \Phi(\omega)^*,\qquad \omega\in\Omega,$$
$$\Phi^\#\colon\quad \Phi^\#(\omega) := \Phi(\omega)^\#,\qquad \omega\in\Omega,$$
$$|\Phi|\colon\quad |\Phi|(\omega) := |\Phi(\omega)|,\qquad \omega\in\Omega.$$

LEMMA 1.1 (cf. [13, Lemma 3.1]). *If Φ is measurable, then so is $\Phi^\#$.*

LEMMA 1.2. *The function Φ is integrable if and only if $|\Phi|$ is integrable.*

Proof. Since all norms on $\mathscr{M}_n$ are equivalent, and since the lemma is obvious for the norm $|\cdot|_E$, the result follows immediately.

By $\mathscr{M}_n^+$ we denote the set of all nonnegative Hermitian $n\times n$-matrices. For a function $\Phi\colon \Omega\to\mathscr{M}_n^+$ and a positive real number α, we set

$$\Phi^\alpha(\omega) := \Phi(\omega)^\alpha,\qquad \omega\in\Omega.$$

The function Φ is measurable if and only if Φ^α is measurable (cf. [4, p. 391]).

LEMMA 1.3. *The function Φ^α is integrable if and only if $|\Phi|^\alpha$ is integrable.*

Proof. For the spectral norm $|\cdot|_s$ the result is true because of $|\Phi(\omega)^\alpha|_s = |\Phi(\omega)|_s^\alpha$, $\omega\in\Omega$, and Lemma 1.2. Since all norms on $\mathscr{M}_n$ are equivalent, the lemma remains true for an arbitrary norm $|\cdot|$.

2. Banach $\mathscr{M}_n$-modules. In this section we recall some definitions and basic facts from the theory of $\mathscr{M}_n$-modules, which will be used later on.

DEFINITION 2.1. A linear space $\mathscr{F}$ is called a *unitary left $\mathscr{M}_n$-module* if there is defined a map $\mathscr{M}_n\times\mathscr{F}\ni(X, \Phi)\to X\Phi\in\mathscr{F}$ having the following properties:

1) $X(\Phi+\Psi) = X\Phi + X\Psi$, $X\in\mathscr{M}_n$, $\Phi, \Psi\in\mathscr{F}$,
2) $(X+Y)\Phi = X\Phi + Y\Phi$, $X, Y\in\mathscr{M}_n$, $\Phi\in\mathscr{F}$,
3) $X(Y\Phi) = (XY)\Phi$, $X, Y\in\mathscr{M}_n$, $\Phi\in\mathscr{F}$,
4) $I\Phi = \Phi$, $\Phi\in\mathscr{F}$,
5) $\alpha\Phi = \alpha I\Phi$, $\alpha\in\mathbf{C}$, $\Phi\in\mathscr{F}$.

Furthermore, let $\mathscr{F}$ be a Banach space under the norm $\|\cdot\|$. Then $\mathscr{F}$ is called a *unitary left Banach $\mathscr{M}_n$-module* if additionally

6) $\|X\Phi\| \leqslant c|X|\|\Phi\|$, $X\in\mathscr{M}_n$, $\Phi\in\mathscr{F}$, for some positive constant c.

Let $\mathscr{F}$ be a unitary left Banach $\mathscr{M}_n$-module. A (not necessarily closed) linear subspace $\mathscr{F}_1$ of $\mathscr{F}$ is called a *left $\mathscr{M}_n$-submodule* of $\mathscr{F}$ if $X\in\mathscr{M}_n$ and $\Phi\in\mathscr{F}_1$ imply $X\Phi\in\mathscr{F}_1$. If the left $\mathscr{M}_n$-submodule is closed, it is called a *left Banach $\mathscr{M}_n$-submodule* of $\mathscr{F}$.

A map $L\colon \mathscr{F}\to\mathscr{M}_n$ is called a *bounded left $\mathscr{M}_n$-linear functional* on $\mathscr{F}$ if L is a continuous map from $\mathscr{F}$ to the normed linear space $\mathscr{M}_n$ and if

$L(X\Phi+Y\Psi) = XL(\Phi)+YL(\Psi)$, $X, Y\in\mathcal{M}_n$, $\Phi, \Psi\in\mathcal{F}$. If $\mathcal{L}$ is a subset of $\mathcal{F}$, then $\bar{\mathcal{L}}$ denotes the closure of $\mathcal{L}$ and $\bigvee\mathcal{L}$ denotes the left $\mathcal{M}_n$-submodule of $\mathcal{F}$ generated by $\mathcal{L}$, i.e.

$$\bigvee\mathcal{L} := \{X\Phi+Y\Psi:\ X, Y\in\mathcal{M}_n,\ \Phi, \Psi\in\mathcal{L}\}.$$

Clearly, if $\mathcal{L}$ is a left $\mathcal{M}_n$-submodule of $\mathcal{F}$, then $\bar{\mathcal{L}}$ is a left Banach $\mathcal{M}_n$-submodule of $\mathcal{F}$.

In the sequel we need the following two facts on $\mathcal{M}_n$-modules, which were proved in [2] even for a more general situation.

LEMMA 2.1 (cf. [2, Theorem 4]). *Let $\mathcal{F}$ be a unitary left Banach $\mathcal{M}_n$-module. The map $L\to \operatorname{tr} L$ is a one-to-one correspondence between the set of bounded left $\mathcal{M}_n$-linear functionals on $\mathcal{F}$ and the set of bounded* **C***-linear functionals on $\mathcal{F}$.*

The fact that the correspondence $L\to\operatorname{tr} L$ is one-to-one can be formulated in the following form.

LEMMA 2.2 (cf. [2, Lemma 1]). *Let L be a bounded left $\mathcal{M}_n$-linear functional on $\mathcal{F}$. If $\operatorname{tr}L(\Phi) = 0$ for all $\Phi\in\mathcal{F}$, then $L(\Phi) = 0$ for all $\Phi\in\mathcal{F}$.*

3. The spaces $L^p(F)$

3.1. By an *$\mathcal{M}_n$-valued measure* on the σ-algebra $\mathcal{B}$ we will mean a σ-additive function M from $\mathcal{B}$ into the normed space $\mathcal{M}_n$. Obviously, $M := (m_{ij})_{i,j=1}^n$ is an $\mathcal{M}_n$-valued measure if and only if each m_{ij} is a finite complex measure on $\mathcal{B}$. If φ: $\Omega\to\mathbf{C}$ is a function integrable w.r.t. all m_{ij}, we set $\int_\Omega \varphi\, dM := (\int_\Omega \varphi\, dm_{ij})_{i,j=1}^n$.

By (DS) we will denote the set of all nonnegative measures on $\mathcal{B}$ having the direct sum property (for the definition see [3, p. 179]). We recall that every σ-finite measure has the direct sum property (cf. [3, p. 179]), and that for an arbitrary complex measure ν on $\mathcal{B}$ absolutely continuous w.r.t. μ the Radon–Nikodym derivative $d\nu/d\mu$ exists if $\mu\in$(DS) (cf. [3, p. 182]). Moreover, it can be easily proved that μ, $\nu\in$(DS) implies $\mu+\nu\in$(DS). We will say that the $\mathcal{M}_n$-valued measure M is *absolutely continuous* w.r.t. $\mu\in$(DS) and write $M\ll\mu$ if all entries m_{ij} of M are absolutely continuous w.r.t. μ. In this case we set $dM/d\mu := (dm_{ij}/d\mu)_{i,j=1}^n$.

Let $F := (f_{ij})_{i,j=1}^n$ be a nonnegative Hermitian $\mathcal{M}_n$-valued measure on $\mathcal{B}$. In the sequel we will call such measures *$\mathcal{M}_n^+$-valued measures* on $\mathcal{B}$. Consider a measure $\mu\in$(DS) such that $F\ll\mu$.

For $1\leqslant p<\infty$, let Φ: $\Omega\to\mathcal{M}_n$ be a function with the following properties:

(i) $\Phi(dF/d\mu)^{1/p}$ is measurable,

(ii) $\|\Phi\|_p := (\int_\Omega|\Phi(dF/d\mu)^{1/p}|^p\, d\mu)^{1/p}<\infty$.

Two $\mathcal{M}_n$-valued functions Φ and Ψ with the properties (i) and (ii) are called *equivalent* if $\Phi(dF/d\mu) = \Psi(dF/d\mu)$ μ-a.e.

DEFINITION 3.1 (cf. [7, Section 3]). Let $1\leqslant p<\infty$. The set of equivalence classes of functions Φ with the properties (i) and (ii) is denoted by $L^p(F)$.

As usual, we will work with representatives, i.e. with functions instead of equivalence classes. Of course, the definition of $\|\cdot\|_p$ depends on the norm $|\cdot|$. But since all results of our paper are independent of the choice of $|\cdot|$, we will omit the dependence on $|\cdot|$ in the notations. Similarly, the dimension n will not be shown in the notations, since no confusion may occur. On the other hand, it is not hard to see that $\|\cdot\|_p$ and $L^p(F)$ do not depend on the choice of μ, i.e., μ can be replaced by any measure ν such that $\nu\in$(DS) and $F\ll\nu$ (cf. [21, Lemma 2.1] for the case $p = 2$). In the sequel, we will often use the finite measure

$$\tau := \operatorname{tr} F,$$

which has the property $F\ll\tau$.

The results of [7, Section 3] yield the following theorem.

THEOREM 3.2. *Let $1\leqslant p<\infty$ and let F be an $\mathcal{M}_n^+$-valued measure. Then $L^p(F)$ is a unitary left Banach $\mathcal{M}_n$-module under the norm $\|\cdot\|_p$.*

3.2. Using Lemma 2.1 and the description of all bounded **C**-linear functionals on $L^p(F)$ (see [7, Theorem 9]) we will obtain the form of the bounded left $\mathcal{M}_n$-linear functionals on $L^p(F)$.

From now on, let $1<p<\infty$ and let q be defined by $1/p+1/q = 1$.

LEMMA 3.3. *Let $1<p<\infty$. Let $\Phi\in L^p(F)$ and $\Psi\in L^q(F)$. Let $\mu\in$(DS) be such that $F\ll\mu$. Then the function $\Phi(dF/d\mu)\Psi^*$ is integrable and the integral $\int_\Omega\Phi(dF/d\mu)\Psi^*\, d\mu$ does not depend on the choice of μ.*

Proof. The inequality

$$\int_\Omega\left|\Phi\frac{dF}{d\mu}\Psi^*\right|_E d\mu \leqslant \int_\Omega\left|\Phi\left(\frac{dF}{d\mu}\right)^{1/p}\right|_E\left|\Psi\left(\frac{dF}{d\mu}\right)^{1/q}\right|_E d\mu \tag{3.1}$$
$$\leqslant\left(\int_\Omega\left|\Phi\left(\frac{dF}{d\mu}\right)^{1/p}\right|_E^p d\mu\right)^{1/p}\left(\int_\Omega\left|\Psi\left(\frac{dF}{d\mu}\right)^{1/q}\right|_E^q d\mu\right)^{1/q}<\infty$$

and Lemma 1.2 imply the integrability of $\Phi(dF/d\mu)\Psi^*$. The independence of the choice of μ can be proved as in [21, Lemma 2.1].

For $\Phi\in L^p(F)$, $\Psi\in L^q(F)$, we define the $n\times n$-matrix

$$\langle\Phi, \Psi\rangle := \int_\Omega\Phi\frac{dF}{d\mu}\Psi^*\, d\mu.$$

LEMMA 3.4. *Let $\Phi\in L^p(F)$, $\Psi\in L^q(F)$, Then*

(a) $\langle\Phi, \Psi\rangle = \langle\Psi, \Phi\rangle^*$,

(b) $|\langle\Phi, \Psi\rangle|\leqslant c\|\Phi\|_p\|\Psi\|_q$ *for some positive constant c not depending on Φ and Ψ.*

If $X, Y \in \mathcal{M}_n$, $\Phi, \Theta \in L^p(F)$, $\Psi \in L^q(F)$, *then*

(c) $\langle X\Phi + Y\Theta, \Psi\rangle = X\langle\Phi, \Psi\rangle + Y\langle\Theta, \Psi\rangle$.

Proof. (b) follows from the inequality

$$|\langle\Phi, \Psi\rangle| \leqslant \int_\Omega \left|\Phi \frac{dF}{d\mu} \Psi^*\right| d\mu \leqslant c_1 \int_\Omega \left|\Phi \frac{dF}{d\mu} \Psi^*\right|_E d\mu$$

with some positive constant c_1 and from (3.1). The other statements of the lemma are trivial.

Using [7, Theorem 9], Lemma 3.3, and the equivalence of all norms on the finite-dimensional space $\mathcal{M}_n$ we obtain the following result.

LEMMA 3.5. *Let* $1 < p < \infty$. *Then for each* $\Psi \in L^q(F)$

$$l(\Phi) := \operatorname{tr}\langle\Phi, \Psi\rangle, \quad \Phi \in L^p(F), \tag{3.2}$$

defines a bounded **C**-*linear functional on* $L^p(F)$. *Conversely, for each bounded* **C**-*linear functional* l *on* $L^p(F)$, *there exists a unique* $\Psi \in L^q(F)$ *such that* (3.2) *holds.*

Combining Lemma 3.5 and Lemma 2.1 we can give a description of the bounded left $\mathcal{M}_n$-linear functionals on $L^p(F)$.

THEOREM 3.6. *Let* $1 < p < \infty$. *Then for each* $\Psi \in L^q(F)$

$$L(\Phi) := \langle\Phi, \Psi\rangle, \quad \Phi \in L^p(F), \tag{3.3}$$

defines a bounded left $\mathcal{M}_n$-*linear functional on* $L^p(F)$. *Conversely, for each bounded left* $\mathcal{M}_n$-*linear functional* L *on* $L^p(F)$, *there exists a unique* $\Psi \in L^q(F)$ *such that* (3.3) *holds.*

3.3. We will say that $\Phi \in L^p(F)$ and $\Psi \in L^q(F)$ are *orthogonal* if $\langle\Phi, \Psi\rangle = 0$. If $\mathcal{L}$ is a subset of $L^p(F)$, then $\mathcal{L}^\perp$ will denote the orthogonal complement of $\mathcal{L}$, i.e., $\mathcal{L}^\perp := \{\Psi \in L^q(F): \langle\Phi, \Psi\rangle = 0 \text{ for each } \Phi \in \mathcal{L}\}$.

LEMMA 3.7. *Let* $1 < p < \infty$ *and let* $\mathcal{L}$ *be a left Banach* $\mathcal{M}_n$-*submodule of* $L^p(F)$. *Then* $\mathcal{L}^\perp$ *is a left Banach* $\mathcal{M}_n$-*submodule of* $L^q(F)$ *and*

$$(\mathcal{L}^\perp)^\perp = \mathcal{L}. \tag{3.4}$$

Proof. Using Lemma 3.4 it is not hard to see that $\mathcal{L}^\perp$ is a left Banach $\mathcal{M}_n$-submodule of $L^q(F)$. Obviously $(\mathcal{L}^\perp)^\perp \supseteq \mathcal{L}$. Consider $\Phi_0 \notin \mathcal{L}$. By the Hahn–Banach theorem and Lemma 3.5, there exists $\Psi \in L^q(F)$ such that $\operatorname{tr}\langle\Phi_0, \Psi\rangle \neq 0$ and $\operatorname{tr}\langle\Phi, \Psi\rangle = 0$ if $\Phi \in \mathcal{L}$. But Lemma 2.2 implies $\langle\Phi, \Psi\rangle = 0$ for all $\Phi \in \mathcal{L}$, hence, $\Psi \in \mathcal{L}^\perp$. From $\langle\Phi_0, \Psi\rangle \neq 0$ it follows that $\Phi_0 \notin (\mathcal{L}^\perp)^\perp$.

LEMMA 3.8. *Let* $\{\mathcal{L}_\lambda: \lambda \in \Lambda\}$ *be a family of left Banach* $\mathcal{M}_n$-*submodules of* $L^p(F)$. *Then*

$$\overline{\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda^\perp} = \Big(\bigcap_{\lambda\in\Lambda} \mathcal{L}_\lambda\Big)^\perp. \tag{3.5}$$

Proof. It is easy to prove the equality $(\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda)^\perp = \bigcap_{\lambda\in\Lambda} \mathcal{L}_\lambda^\perp$. Since $(\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda)^\perp = (\overline{\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda})^\perp$, we obtain

$$(\overline{\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda})^\perp = \bigcap_{\lambda\in\Lambda} \mathcal{L}_\lambda^\perp. \tag{3.6}$$

Applying (3.6) to the set $\{\mathcal{L}_\lambda^\perp: \lambda \in \Lambda\}$ of left Banach $\mathcal{M}_n$-submodules of $L^q(F)$ and using (3.4) we obtain $(\overline{\bigvee_{\lambda\in\Lambda} \mathcal{L}_\lambda^\perp})^\perp = \bigcap_{\lambda\in\Lambda} \mathcal{L}_\lambda$. Now take the orthogonal complements of both sides and use (3.4) again.

4. The space $H^p(F)$

4.1. Following [13, Section 5], we say that an $\mathcal{M}_n$-valued measure M on $\mathcal{B}$ is *strongly absolutely continuous* w.r.t. the $\mathcal{M}_n^+$-valued measure F on $\mathcal{B}$ ($M \lll F$) if and only if there exists a measure μ such that $\mu \in$ (DS), $M \ll \mu$, $F \ll \mu$, and $\operatorname{Ker} dM/d\mu \supseteq \operatorname{Ker} dF/d\mu$ μ-a.e. Note that the definition of absolute continuity does not depend on the choice of μ, i.e., if ν is another measure such that $\nu \in$ (DS), $M \ll \nu$, and $F \ll \nu$, then $\operatorname{Ker} dM/d\nu \supseteq \operatorname{Ker} dF/d\nu$ ν-a.e. if and only if $\operatorname{Ker} dM/d\mu \supseteq \operatorname{Ker} dF/d\mu$ μ-a.e. (cf. [13, Section 5]).

Because of Lemma 1.1 we can define

$$|||M|||_p := \left(\int_\Omega \left|\frac{dM}{d\mu}\left(\left(\frac{dF}{d\mu}\right)^{\#}\right)^{1/q}\right|^p d\mu\right)^{1/p}, \quad 1 < p < \infty. \tag{4.1}$$

LEMMA 4.1 (cf. [16, Lemma 1]). $|||M|||_p$ *does not depend on the choice of* μ.

Proof. Let ν be another measure such that $\nu \in$ (DS), $M \ll \nu$, $F \ll \nu$, and $\operatorname{Ker} dM/d\nu \supseteq \operatorname{Ker} dF/d\nu$ ν-a.e. Consider $\sigma := \mu + \nu$. We have $\sigma \in$ (DS) and

$$\int_\Omega \left|\frac{dM}{d\mu}\left(\left(\frac{dF}{d\mu}\right)^{\#}\right)^{1/q}\right|^p d\mu = \int_\Omega \left|\frac{dM}{d\sigma}\left(\left(\frac{dF}{d\sigma}\right)^{\#}\right)^{1/q}\right|^p \left(\left(\frac{d\mu}{d\sigma}\right)^{\#}\left(\frac{d\mu}{d\sigma}\right)^{1/q}\right)^p d\mu$$

$$= \int_\Omega \left|\frac{dM}{d\sigma}\left(\left(\frac{dF}{d\sigma}\right)^{\#}\right)^{1/q}\right|^p \left(\frac{d\mu}{d\sigma}\right)^{\#} d\mu = \int_\Omega \left|\frac{dM}{d\sigma}\left(\left(\frac{dF}{d\sigma}\right)^{\#}\right)^{1/q}\right|^p d\sigma.$$

In the same way we obtain

$$\int_\Omega \left|\frac{dM}{d\nu}\left(\left(\frac{dF}{d\nu}\right)^{\#}\right)^{1/q}\right|^p d\nu = \int_\Omega \left|\frac{dM}{d\sigma}\left(\left(\frac{dF}{d\sigma}\right)^{\#}\right)^{1/q}\right|^p d\sigma,$$

hence, the lemma is proved.

DEFINITION 4.2 (cf. [9, Definition (2.10)]). Let $1 < p < \infty$ and let F be an $\mathcal{M}_n^+$-valued measure on $\mathcal{B}$. By $H^p(F)$ we denote the class of all $\mathcal{M}_n$-valued measures M on $\mathcal{B}$ such that $M \lll F$ and the integral (4.1) is finite.

Clearly, $H^p(F)$ is a unitary left $\mathcal{M}_n$-module. Furthermore, it is not hard to see that $|||\cdot|||_p$ defines a norm on $H^p(F)$.

Let $\Phi \in L^p(F)$. Because of Lemma 3.3 we can define an $\mathcal{M}_n$-valued measure M_Φ by

$$M_\Phi(B) := \int_B \Phi \frac{dF}{d\tau} d\tau, \quad B \in \mathcal{B}. \tag{4.2}$$

We set

$$U_p \Phi := M_\Phi, \quad \Phi \in L^p(F). \tag{4.3}$$

The following result plays a central role in our investigations.

THEOREM 4.3. *Let $1 < p < \infty$. The map U_p is an isometric isomorphism between $L^p(F)$ and $H^p(F)$.*

Proof. The definition of M_Φ implies $M_\Phi \ll \tau$ and $dM_\Phi/d\tau = \Phi dF/d\tau$, hence, $\operatorname{Ker} dM_\Phi/d\tau \supseteq \operatorname{Ker} dF/d\tau$ τ-a.e., i.e. $M_\Phi \lll F$. Moreover, since

$$\int_\Omega \left|\frac{dM_\Phi}{d\tau}\left(\left(\frac{dF}{d\tau}\right)^{\#}\right)^{1/q}\right|^p d\tau = \int_\Omega \left|\Phi \frac{dF}{d\tau}\left(\left(\frac{dF}{d\tau}\right)^{\#}\right)^{1/q}\right|^p d\tau = \int_\Omega \left|\Phi \left(\frac{dF}{d\tau}\right)^{1/p}\right|^p d\tau < \infty,$$

we see that U_p is an isometry from $L^p(F)$ into $H^p(F)$. Obviously, U_p is $\mathcal{M}_n$-linear. Now consider an arbitrary element M of $H^p(F)$. Set $\Phi := (dM/d\mu)(dF/d\mu)^{\#}$, where μ is a measure such that $\mu \in (\mathrm{DS})$, $M \ll \mu$, and $F \ll \mu$. The equality

$$\int_\Omega \left|\Phi \left(\frac{dF}{d\mu}\right)^{1/p}\right|^p d\mu = \int_\Omega \left|\frac{dM}{d\mu}\left(\frac{dF}{d\mu}\right)^{\#}\left(\frac{dF}{d\mu}\right)^{1/p}\right|^p d\mu = \int_\Omega \left|\frac{dM}{d\mu}\left(\left(\frac{dF}{d\mu}\right)^{\#}\right)^{1/q}\right|^p d\mu$$

implies that Φ is an element of $L^p(F)$. Finally, since $\operatorname{Ker} dM/d\mu \supseteq \operatorname{Ker} dF/d\mu$ μ-a.e., we have

$$M_\Phi(B) = \int_B \Phi \frac{dF}{d\tau} d\tau = \int_B \frac{dM}{d\mu}\left(\frac{dF}{d\mu}\right)^{\#} \frac{dF}{d\mu} d\mu$$
$$= \int_B \frac{dM}{d\mu} d\mu = M(B) \quad \text{for } B \in \mathcal{B}.$$

Hence, $U_p \Phi = M$ and $\mathcal{R}(U_p) = H^p(F)$.

COROLLARY 4.4. *The space $H^p(F)$ is a unitary left Banach $\mathcal{M}_n$-module.*

COROLLARY 4.5. *The elements of $H^p(F)$ are absolutely continuous w.r.t. τ.*

4.2. With the aid of Theorem 4.3 we can transfer the results on $L^p(F)$ of Section 3 to the space $H^p(F)$.

Let $M \in H^p(F)$, $N \in H^q(F)$. We set

$$\langle\langle M, N\rangle\rangle := \int_\Omega \frac{dM}{d\tau}\left(\frac{dF}{d\tau}\right)^{\#}\left(\frac{dN}{d\tau}\right)^{*} d\tau.$$

THEOREM 4.6. *Let $1 < p < \infty$. Then for each $N \in H^q(F)$ the map*

$$L(M) := \langle\langle M, N\rangle\rangle, \quad M \in H^p(F), \tag{4.4}$$

defines a bounded $\mathcal{M}_n$-linear functional on $H^p(F)$. Conversely, for each bounded $\mathcal{M}_n$-linear functional L on $H^p(F)$, there exists a unique $N \in H^q(F)$ such that (4.4) holds.

Proof. Consider $M \in H^p(F)$ and $N \in H^q(F)$. Set $\Phi := U_p^{-1} M$ and $\Psi := U_q^{-1} N$. We obtain

$$\langle\langle M, N\rangle\rangle = \int_\Omega \frac{dM}{d\tau}\left(\frac{dF}{d\tau}\right)^{\#}\left(\frac{dN}{d\tau}\right)^{*} d\tau$$
$$= \int_\Omega \Phi \frac{dF}{d\tau}\left(\frac{dF}{d\tau}\right)^{\#} \frac{dF}{d\tau} \Psi^* d\tau = \int_\Omega \Phi \frac{dF}{d\tau} \Psi^* d\tau = \langle \Phi, \Psi\rangle.$$

Now use Theorem 3.6.

We say that $M \in H^p(F)$ and $N \in H^q(F)$ are *orthogonal* if $\langle\langle M, N\rangle\rangle = 0$. If $\mathcal{L}$ is a subset of $H^p(F)$, then $\mathcal{L}^\perp$ will denote the set $\mathcal{L}^\perp := \{N \in H^q(F): \langle\langle M, N\rangle\rangle = 0 \text{ for all } M \in \mathcal{L}\}$.

5. The space $\mathfrak{R}_{C,q}$

5.1. Let G be an LCA group. In order to avoid trivialities we will assume throughout this paper that G contains more than one element. Let Γ be the dual group of G. The value of a character $\gamma \in \Gamma$ on an element $g \in C$ will be denoted by (g, γ). By λ and $\tilde{\lambda}$ we denote Haar measures of G and Γ, respectively. We assume that λ and $\tilde{\lambda}$ are normalized in such a way that the inversion formula for the Fourier transform holds (cf. [15, p. 22]). By $L^1(\lambda)$ and $L^1(\tilde{\lambda})$ we denote the linear space of $\mathcal{M}_n$-valued functions on G and Γ, respectively, which are integrable w.r.t. λ and $\tilde{\lambda}$, respectively. For $S \in L^1(\lambda)$ and $T \in L^1(\tilde{\lambda})$, we set

$$\hat{S}(\gamma) := \int_G (g, \gamma)^* S(g)\, \lambda(dg), \quad \gamma \in \Gamma,$$
$$\check{T}(g) := \int_\Gamma (g, \gamma)\, T(\gamma)\, \tilde{\lambda}(d\gamma), \quad g \in G,$$

i.e., $\hat{S}$ is the Fourier transform of S and $\check{T}$ is the inverse Fourier transform of T.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets of Γ. A nonnegative measure μ on $\mathcal{B}$ is called *regular* if $\mu(B) = \inf \mu(V) = \sup \mu(K)$, $B \in \mathcal{B}$, where the infimum is taken over all open sets $V \supseteq B$ and the supremum is taken over all compact sets $K \subseteq B$. A complex measure on $\mathcal{B}$ is called *regular* if its total variation is regular. An $\mathcal{M}_n$-valued measure on $\mathcal{B}$ is called *regular* if all its entries are regular measures. Note that the Haar measure $\tilde{\lambda}$ of Γ is regular if and only if Γ is discrete or σ-compact (cf. [6, (16.14)]).

Let $\mathcal{D}(G)$ denote the set of inverse Fourier–Stieltjes transforms of all $\mathcal{M}_n$-valued measures on $\mathcal{B}$, i.e., $\Phi \in \mathcal{D}(G)$ if and only if $\Phi(g) = \int_\Gamma (g, \gamma)\, M(d\gamma)$, $g \in G$, for some $\mathcal{M}_n$-valued measure M.

Let $\mathscr{K}$ denote the set of all proper and nonempty compact subsets of G. Let F be an $\mathscr{M}_n^+$-valued measure on $\mathscr{B}$, and $L^p(F)$ and $H^p(F)$, $1 < p < \infty$, the unitary left Banach $\mathscr{M}_n$-modules introduced in Sections 3 and 4, respectively.

Let $C \in \mathscr{K}$. We define

$$\mathfrak{M}_{G\setminus C,p} := \bigvee_{g \in G\setminus C} \{(g, \cdot)^* I\},$$

where the closure is taken in $L^p(F)$. Obviously. $\mathfrak{M}_{G\setminus C,p}$ is a left Banach $\mathscr{M}_n$-submodule of $L^p(F)$ and

$$\mathfrak{M}_{G\setminus C,p} = \Big\{\sum_{j=1}^{k} X_j (g_j, \cdot)^*\colon X_j \in \mathscr{M}_n,\ g_j \in G\setminus C,\ j = 1, \ldots, k,\ k \in \mathbf{N}\Big\}.$$

Finally, we set

$$\mathfrak{N}_{C,q} := \mathfrak{M}^{\perp}_{G\setminus C,p}.$$

By Lemma 3.7, $\mathfrak{N}_{C,q}$ is a left Banach $\mathscr{M}_n$-submodule of $L^q(F)$.

5.2. Now we give a characterization of the set $U_p \mathfrak{N}_{C,p} := \{U_p \Phi\colon \Phi \in \mathfrak{N}_{C,p}\}$, where U_p is the map defined in (4.3), $1 < p < \infty$. We follow the method developed by A. Weron [21] in the case $p = 2$ (see also [9] and [10]). First we introduce some notations.

DEFINITION 5.1. For $C \in \mathscr{K}$, we denote by $\mathscr{S}_C$ the set of all $\mathscr{M}_n$-valued functions S on G such that $S \in L^1(\lambda) \cap \mathscr{D}(G)$ and $\operatorname{supp} S \subseteq C$, and by $\mathscr{T}_C$ the set of the Fourier transforms of functions from $\mathscr{S}_C$.

The functions in $\mathscr{T}_C$ belong to $L^1(\tilde{\lambda})$. Hence, for $T \in \mathscr{T}_C$, we can define an $\mathscr{M}_n$-valued measure M^T by

$$M^T(B) := \int_B T\, d\tilde{\lambda}, \quad B \in \mathscr{B}. \tag{5.1}$$

Since the entries of M^T, $T \in \mathscr{T}_C$, are finite measures, we conclude that M^T is a regular $\mathscr{M}_n$-valued measure on $\mathscr{B}$ (cf. [6, (11.12) and (11.32)]). The set of all measures M^T, $T \in \mathscr{T}_C$, is denoted by $\mathscr{N}_C$.

Let $1 < p < \infty$. Let $C \in \mathscr{K}$ and $\Phi \in \mathfrak{N}_{C,p}$. Since $\Phi \in L^1(F)$, we obtain $(g, \cdot)\Phi(\cdot) \in L^1(F)$ for all $g \in G$. We set

$$S_\Phi(g) := \int_\Gamma (g, \gamma)\Phi(\gamma) \frac{dF}{d\tau}(\gamma)\tau(d\gamma), \quad g \in G.$$

LEMMA 5.2 (cf. [21, Lemma 4.2]). *The function S_Φ belongs to $\mathscr{S}_C$.*

Proof. From the definition of S_Φ it follows that $S_\Phi(g) = \int_\Gamma (g, \gamma) M_\Phi(d\gamma)$, where $M_\Phi = U_p \Phi$ is the measure defined in (4.2). Hence, $S_\Phi \in \mathscr{D}(G)$. Thus, S_Φ is continuous (cf. [15, p. 15]). Since from the definition of $\mathfrak{N}_{C,p}$ it follows easily that $\operatorname{supp} S_\Phi \subseteq C$, we conclude that $S_\Phi \in L^1(\lambda)$.

Let $T_\Phi := \hat{S}_\Phi$ and let M^{T_Φ} be the measure defined from T_Φ according to (5.1).

LEMMA 5.3 (cf. [21, Lemma 4.5(b)]). *For $\Phi \in \mathfrak{N}_{C,p}$, we have $M^{T_\Phi} = M_\Phi = U_p \Phi$.*

Proof. Since $dM^{T_\Phi}/d\tilde{\lambda} = T_\Phi$, we obtain

$$S_\Phi(g) = \check{T}_\Phi(g) = \int_\Gamma (g, \gamma) T_\Phi(\gamma) \tilde{\lambda}(d\gamma) = \int_\Gamma (g, \gamma) M^{T_\Phi}(d\gamma), \quad g \in G.$$

On the other hand, $S_\Phi(g) = \int_\Gamma (g, \gamma) M_\Phi(d\gamma)$, $g \in G$, and the uniqueness of the inverse Fourier transform (cf. [15, p. 17]) yields $M^{T_\Phi} = M_\Phi$.

Now we can characterize the set $U_p \mathfrak{N}_{C,p}$.

THEOREM 5.4 (cf. [21, Theorem 4.9] for $p = 2$). *Let $1 < p < \infty$ and let $C \in \mathscr{K}$. Then $U_p \mathfrak{N}_{C,p} = \mathscr{N}_C \cap H^p(F)$.*

Proof. Lemma 5.3 and Theorem 4.3 imply the inclusion $U_p \mathfrak{N}_{C,p} \subseteq \mathscr{N}_C \cap H^p(F)$. On the other hand, let $M^T \in \mathscr{N}_C \cap H^p(F)$, where M^T is defined by (5.1) and $T \in \mathscr{T}_C$. According to Theorem 4.3 there exists a $\Phi \in L^p(F)$ such that $U_p \Phi = M_\Phi = M^T$, where M_Φ is defined by (4.2). It only remains to prove that Φ is an element of $\mathfrak{N}_{C,p}$. But for $g \in G\setminus C$,

$$\begin{aligned}\langle \Phi, (g, \cdot)^* I\rangle &= \int_\Gamma (g, \gamma)\Phi(\gamma)\frac{dF}{d\tau}(\gamma)\tau(d\gamma) = \int_\Gamma (g, \gamma) M_\Phi(d\gamma) = \int_\Gamma (g, \gamma) M^T(d\gamma) \\ &= \int_\Gamma (g, \gamma) T(\gamma)\tilde{\lambda}(d\gamma) = \check{T}(g) = 0\end{aligned}$$

because $\check{T} \in \mathscr{S}_C$. Hence, $\Phi \in \mathfrak{N}_{C,p}$.

5.3. The Haar measure $\tilde{\lambda}$ is regular if and only if Γ is discrete or σ-compact (cf. [6, (16.14)]); note that the σ-compactness of Γ is equivalent to the σ-finiteness of $\tilde{\lambda}$. But if $\tilde{\lambda}$ is regular, then $\tilde{\lambda} \in (\mathrm{DS})$ (cf. [3, p. 337]). In this case we can give another useful characterization of $U_p \mathfrak{N}_{C,p}$.

Let F be an $\mathscr{M}_n^+$-valued measure on $\mathscr{B}$ and let F' be the Radon–Nikodym derivative of the absolutely continuous part of F w.r.t. $\tilde{\lambda}$. It is well known that F' is an $\mathscr{M}_n^+$-valued measurable function.

Now we can state the following result. Since its proof is analogous to the proof of Corollary 3.16 in [10], we omit it.

THEOREM 5.5. *Let Γ be discrete or σ-compact. Let $1 < p < \infty$ and $C \in \mathscr{K}$. A measure M^T defined by* (5.1) *belongs to $U_p \mathfrak{N}_{C,p}$ if and only if the following three conditions hold:*

(i) $T \in \mathscr{T}_C$,
(ii) $\operatorname{Ker} T \supseteq \operatorname{Ker} F'$ $\tilde{\lambda}$-*a.e.*,
(iii) $\int_\Gamma |T((F')^{\#})^{1/q}|^p\, d\tilde{\lambda} < \infty$.

Remark 5.6. In the already mentioned Corollary 3.16 of [10] the authors assumed that Γ is σ-compact. Using the concept of the direct sum property of a measure we obtain the result for discrete groups Γ, too. However, this generalization does not seem to be very useful (cf. Remark 8.3 and Theorem 8.5).

6. Interpolability and minimality

6.1. Let G be an LCA group containing more than one element. Let F be an $\mathcal{M}_n^+$-valued measure on the σ-algebra $\mathcal{B}$ of Borel sets of the dual group Γ.

Definition 6.1. Let $1 < p < \infty$ and $C \in \mathcal{K}$. The set C is called *interpolable in* $L^p(F)$ if $\mathfrak{M}_{G\setminus C,p} = L^p(F)$. The space $L^p(F)$ is called *interpolable* if each set $C \in \mathcal{K}$ is interpolable in $L^p(F)$. The space $L^p(F)$ is called *minimal* if for each $g \in G$ the set $\{g\}$ is not interpolable in $L^p(F)$.

By Theorem 3.6, Lemmas 2.1 and 2.2, and the Hahn–Banach theorem, the set $C \in \mathcal{K}$ is not interpolable in $L^p(F)$ if and only if $\mathfrak{N}_{C,q} = \{0\}$. Thus, using Theorem 5.4 we immediately obtain criteria for interpolability.

Theorem 6.2 (cf. [21, Theorem 5.2] for $p = 2$). *Let $1 < p < \infty$ and $C \in \mathcal{K}$. Then C is interpolable in $L^p(F)$ if and only if $\mathcal{N}_C \cap H^q(F) = \{0\}$.*

Corollary 6.3 (cf. [21, Corollary 5.4] for $p = 2$ and [22, Corollary 3.2] for $n = 1$). *The space $L^p(F)$ is interpolable if and only if $\left(\bigcup_{C\in\mathcal{K}} \mathcal{N}_C\right) \cap H^q(F) = \{0\}$.*

If G is discrete, then a subset $C \subseteq G$ is compact if and only if it is finite, say $C = \{g_1, \ldots, g_k\}$, $k \in \mathbf{N}$.

Lemma 6.4 [21, Lemma 4.7(b)]. *The set $\mathcal{T}_C$ consists of all $\mathcal{M}_n$-valued trigonometric polynomials of the form $\sum_{j=1}^k X_j(g_j, \cdot)^*$, $X_j \in \mathcal{M}_n$, $j = 1, \ldots, k$.*

From this observation and Corollary 6.3 we can easily deduce the following result, whose proof will be omitted.

Corollary 6.5 (cf. [21, Theorem 5.5] for $p = 2$). *Let G be discrete and let $F \ll \tilde{\lambda}$. The space $L^p(F)$ is interpolable if and only if for any $\mathcal{M}_n$-valued trigonometric polynomial W the integral $\int_\Gamma |W((F')^{\#})^{1/p}|^q \, d\tilde{\lambda}$ is equal to 0 or to ∞.*

6.2. If G is not discrete, then using the uniform continuity of the Fourier–Stieltjes transform (cf. [15, p. 15]), it can easily be shown that $L^p(F)$ cannot be minimal. Thus, we will assume that G is discrete. Since for $g \in G$ the operator of multiplication by $(g, \cdot)$ is an isometry in $L^p(F)$, we conclude that $L^p(F)$ is minimal if and only if the set $\{e\}$ consisting of the unit e of G is not interpolable in $L^p(F)$. By Theorem 3.6, Lemmas 2.1 and 2.2, and the Hahn–Banach theorem, it follows that the space $L^p(F)$ is minimal if and only if $\mathfrak{N}_{\{e\},q} = \{0\}$. Combining this fact with Theorems 5.4, 5.5, and Lemma 6.4 we obtain the following criterion for minimality.

Theorem 6.6. *Let $1 < p < \infty$ and let G be discrete. Let F be an $\mathcal{M}_n^+$-valued measure on $\mathcal{B}$ and F' the Radon–Nikodym derivative of its absolutely continuous part w.r.t. $\tilde{\lambda}$. The space $L^p(F)$ is minimal if and only if there exists an $X \in \mathcal{M}_n$ such that* $\operatorname{Ker} X \supseteq \operatorname{Ker} F'$ *$\tilde{\lambda}$-a.e. and* $0 < \int_\Gamma |X((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty$.

If $p = 2$ and $|\cdot|$ is the euclidean norm, $L^2(F)$ becomes a Hilbert space. In this case one can derive several conditions equivalent to the minimality of $L^2(F)$ (see [9, Theorem 4.6], compare also [5, Ch. III.6]). In the scalar case, i.e. $n = 1$, M. Pourahmadi [12, Proof of Theorem 3.3] and A. Weron [22, Corollary 3.1] proved explicit formulas for the distance of the function $(e, \cdot)$ from the space $\mathfrak{M}_{G\setminus\{e\},p}$ in $L^p(F)$, $1 < p \leqslant 2$ (see also [11, Corollary 3.2]). From these formulas one immediately deduces the minimality conditions of Theorem 6.6, $1 < p \leqslant 2$ (cf. [12, Theorem 3.3] and [22, Theorem 3.1]). In the general situation it seems difficult to obtain deeper results on the distance of $(e, \cdot)$ from $\mathfrak{M}_{G\setminus\{e\},p}$. However, we still have the following fact.

Corollary 6.7 (cf. [9, Theorem 4.6] for $p = 2$). *The space $L^p(F)$ is minimal if and only if there exists an orthoprojector $P \in \mathcal{M}_n$ such that* $\operatorname{Ker} P \supseteq \operatorname{Ker} F'$ *$\tilde{\lambda}$-a.e. and* $0 < \int_\Gamma |P((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty$.

Proof. The sufficiency is clear from Theorem 6.6. Assume, conversely, that $L^p(F)$ is minimal. Then by Theorem 6.6 there exists an $X \in \mathcal{M}_n$ such that $\operatorname{Ker} X \supseteq \operatorname{Ker} F'$ $\tilde{\lambda}$-a.e. and $0 < \int_\Gamma |X((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty$. Let $P := X^{\#} X$. Then P is the orthoprojector onto $\mathcal{R}(X^*)$ and we have $\int_\Gamma |P((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty$. The last integral is not zero, since otherwise $\int_\Gamma |X((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} = 0$ because $XP = X$.

7. $\mathcal{J}_0$-regularity. Let G be an LCA group containing more than one element.

Definition 7.1. Let $1 < p < \infty$ and let $\mathcal{J}$ be a family of nonempty subsets of G. Let F be an $\mathcal{M}_n^+$-valued measure on the σ-algebra $\mathcal{B}$ of Borel sets of the dual group Γ. The space $L^p(F)$ is called *$\mathcal{J}$-regular* if $\bigcap_{A\in\mathcal{J}} \mathfrak{M}_{G\setminus A,p} = \{0\}$, and *$\mathcal{J}$-singular* if $\mathfrak{M}_{G\setminus A,p} = L^p(F)$ for each $A \in \mathcal{J}$.

Lemma 7.2. *Let $\mathcal{J} \subseteq \mathcal{K}$. The space $L^p(F)$ is $\mathcal{J}$-regular if and only if*

$$\overline{\bigvee_{C\in\mathcal{J}} U_q \mathfrak{N}_{C,q}} = H^q(F).$$

Proof. By (3.5), Theorem 3.6, Lemmas 2.1 and 2.2, and the Hahn–Banach theorem, $L^p(F)$ is $\mathcal{J}$-regular if and only if

$$\overline{\bigvee_{C\in\mathcal{J}} \mathfrak{N}_{C,q}} = \overline{\bigvee_{C\in\mathcal{J}} \mathfrak{M}_{G\setminus C,p}^{\perp}} = \Big(\bigcap_{C\in\mathcal{J}} \mathfrak{M}_{G\setminus C,p}\Big)^{\perp} = \{0\}^{\perp} = L^q(F).$$

Now use Theorem 4.3.

In our paper we consider two families of subsets of G:

$$\mathcal{J}_0 := \{\{g\} : g \in G\}, \quad \mathcal{J}_c := \mathcal{K}.$$

Comparing Definitions 6.1 and 7.1 we see that $L^p(F)$ is $\mathcal{J}_0$-singular or $\mathcal{J}_c$-singular if and only if it is not minimal or interpolable, respectively. Thus, we can use the results of Section 6 to obtain conditions for the $\mathcal{J}_0$-singularity and the $\mathcal{J}_c$-singularity of $L^p(F)$. The details are left to the reader. In this section we study $\mathcal{J}_0$-regularity and Section 8 is devoted to $\mathcal{J}_c$-regularity.

If G is not discrete, then $L^p(F)$ cannot be $\mathcal{J}_0$-regular (cf. the beginning of Section 6.2). Hence, we will assume in the remaining part of this section that G is discrete.

THEOREM 7.3 (cf. [10, Theorem 5.3] for $p = 2$). *Let $1 < p < \infty$ and let G be discrete. Let F be an $\mathcal{M}_n^+$-valued measure on $\mathcal{B}$ and F' the Radon–Nikodym derivative of its absolutely continuous part w.r.t. $\tilde{\lambda}$. The space $L^p(F)$ is $\mathcal{J}_0$-regular if and only if the following three conditions hold:*

(i) $F \ll \tilde{\lambda}$.

(ii) *There exists a subspace $\mathcal{A}$ of $\mathbf{C}^n$ such that $\mathcal{R}(F') = \mathcal{A}$ $\tilde{\lambda}$-a.e.*

(iii) $((F')^{\#})^{q/p}$ *is integrable w.r.t.* $\tilde{\lambda}$.

In the proof of Theorem 7.3 we need the following fact.

LEMMA 7.4. *Let $\{M_j\}_{j=1}^{\infty}$ be a sequence of measures in $H^p(F)$ which tends to a measure M in $H^p(F)$ as $j \to \infty$. Then $\lim_{j\to\infty} M_j(B) = M(B)$ for each $B \in \mathcal{B}$.*

Proof. Set $\Phi_j := U_p^{-1} M_j$, $j \in \mathbf{N}$, and $\Phi := U_p^{-1} M$. By Theorem 4.3 we have $\lim_{j\to\infty} \Phi_j = \Phi$ in $L^p(F)$. Since F is finite, $\lim_{j\to\infty} \Phi_j = \Phi$ in $L^1(F)$ (cf. [8, Lemma 5]). Hence, the inequality

$$|M_j(B) - M(B)| = \left|\int_B \Phi_j \frac{dF}{d\tau} d\tau - \int_B \Phi \frac{dF}{d\tau} d\tau\right|$$

$$\leqslant \int_B \left|(\Phi_j - \Phi)\frac{dF}{d\tau}\right| d\tau \leqslant \|\Phi_j - \Phi\|_1, \qquad B \in \mathcal{B},$$

yields the result of the lemma.

Proof of Theorem 7.3. *Necessity.* Assume that $L^p(F)$ is $\mathcal{J}_0$-regular.

Proof of (i). Assume that F is not absolutely continuous w.r.t. $\tilde{\lambda}$. Then there exists a nonzero measure $M \in H^q(F)$ whose support is contained in the support of the singular part of F. But (5.1) and Theorem 5.4 show that the elements of $U_q \mathfrak{N}_{\{g\},q}$, $g \in G$, are absolutely continuous w.r.t. $\tilde{\lambda}$. By Lemma 7.2, this contradicts the $\mathcal{J}_0$-regularity of $L^p(F)$. Thus, $F \ll \tilde{\lambda}$.

The proof of (ii) and (iii) is adapted from the proof of Theorem 5.2 in [10].

Proof of (ii). Since F is a finite measure, we have $\int_\Gamma |(F')^{1/q}|^q \, d\tilde{\lambda} < \infty$ by Lemmas 1.2 and 1.3. Hence,

$$\int_\Gamma \left|\frac{dF}{d\tau}\left(\left(\frac{dF}{d\tau}\right)^{\#}\right)^{1/p}\right|^q d\tau = \int_\Gamma |F'((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} = \int_\Gamma |(F')^{1/q}|^q \, d\tilde{\lambda} < \infty,$$

i.e., $F \in H^q(F)$. According to Lemma 7.2, there exist a sequence $\{C_j\}_{j=1}^{\infty}$ of finite subsets of G and a sequence $\{\sum_{g\in C_j} M_{g,j}\}_{j=1}^{\infty}$ such that $M_{g,j} \in U_q \mathfrak{N}_{\{g\},q}$, $g \in C_j$, $j \in \mathbf{N}$, and

$$\lim_{j\to\infty} \sum_{g\in C_j} M_{g,j} = F \tag{7.1}$$

in $H^q(F)$. Without loss of generality we may assume that $e \in C_j$, $j \in \mathbf{N}$. By Theorems 5.4 and 5.5, and Lemma 6.4, $M_{g,j}$ is of the form $M_{g,j}(B) = X_{g,j} \int_B (g, \gamma)^* \tilde{\lambda}(d\gamma)$, $B \in \mathcal{B}$, where $X_{g,j}$ is an $n \times n$-matrix such that

$$\operatorname{Ker} X_{g,j} \supseteq \operatorname{Ker} F' \quad \tilde{\lambda}\text{-a.e.}, \tag{7.2}$$

$$\int_\Gamma |X_{g,j}((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty, \quad g \in C_j,\ j \in \mathbf{N}. \tag{7.3}$$

Using (7.1), Lemma 7.4, and the equality $\int_\Gamma (g, \gamma)^* \tilde{\lambda}(d\gamma) = 0$, for $g \in G$, $g \neq e$, we get

$$\lim_{j\to\infty} X_{e,j} = F(\Gamma). \tag{7.4}$$

The relations (7.2) and (7.4) yield

$$\operatorname{Ker} F(\Gamma) \supseteq \operatorname{Ker} F' \quad \tilde{\lambda}\text{-a.e.}$$

On the other hand, $\operatorname{Ker} F(\Gamma) = \operatorname{Ker}(\int_\Gamma F' \, d\tilde{\lambda}) \subseteq \operatorname{Ker} F'$ $\tilde{\lambda}$-a.e. (cf. [13, Lemma 3.2 (a)]). Thus $\operatorname{Ker} F(\Gamma) = \operatorname{Ker} F'$ $\tilde{\lambda}$-a.e. Since F' is $\mathcal{M}_n^+$-valued, there is a subspace $\mathcal{A}$ of $\mathbf{C}^n$ such that $\mathcal{R}(F') = \mathcal{A}$ $\tilde{\lambda}$-a.e.

Proof of (iii). (7.3) and (7.4) yield $\int_\Gamma |F(\Gamma)((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty$. This gives

$$\int_\Gamma |F(\Gamma)^{\#} F(\Gamma)((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty.$$

Since $F(\Gamma)^{\#} F(\Gamma)$ is the orthoprojector onto $\mathcal{R}(F(\Gamma)^*) = \mathcal{R}(F(\Gamma))$ and since $\mathcal{R}(F') = \mathcal{R}(F(\Gamma))$ $\tilde{\lambda}$-a.e., we finally conclude that $\int_\Gamma |((F')^{\#})^{1/p}|^q \, d\tilde{\lambda} < \infty$. Now Lemmas 1.2 and 1.3 imply the integrability of $((F')^{\#})^{q/p}$.

Sufficiency. Assume (i)–(iii). Let Q be the orthoprojector in $\mathbf{C}^n$ onto $\mathcal{A}$. For each $g \in G$, the measure M_g defined by $M_g(B) := Q \int_B (g, \gamma)^* \tilde{\lambda}(d\gamma)$, $B \in \mathcal{B}$, belongs to $U_q \mathfrak{N}_{\{g\},q}$. If $L^p(F)$ were not $\mathcal{J}_0$-regular, then, by Lemma 7.2, Theorem 4.6, Lemmas 2.1 and 2.2, and the Hahn–Banach theorem, there would exist a measure $N \in H^p(F)$ such that $N \neq 0$ in $H^p(F)$ and $\langle\langle N, M_g \rangle\rangle = 0$ for each $g \in G$. But

$$\langle\langle N, M_g \rangle\rangle = \int_\Gamma (g, \gamma) \frac{dN}{d\tilde{\lambda}}(\gamma) F'(\gamma)^{\#} \tilde{\lambda}(d\gamma).$$

Thus, the uniqueness theorem for the inverse Fourier–Stieltjes transform would imply $(dN/d\tilde{\lambda})(F')^{\#} = 0$ $\tilde{\lambda}$-a.e., and hence, $\operatorname{Ker} dN/d\tilde{\lambda} \supseteq \mathcal{R}(F')$ $\tilde{\lambda}$-a.e. On the other hand, $\operatorname{Ker} dN/d\tilde{\lambda} \supseteq \operatorname{Ker} F'$ $\tilde{\lambda}$-a.e., since $N \in H^p(F)$. Thus, $dN/d\tilde{\lambda} = 0$ $\tilde{\lambda}$-a.e. This means $N = 0$ in $H^p(F)$, a contradiction.

8. $\mathcal{J}_c$-regularity. M. G. Avetisyan and R. L. Dobrushin (cf. [1, Theorem 1]) proved a result which in the situation of our paper can be stated in the following form.

THEOREM 8.1. *Let $G := \mathbf{Z}^l$, $l \in \mathbf{N}$. Let F be an $\mathcal{M}_n^+$-valued measure on the σ-algebra $\mathcal{B}$ of Borel sets of the dual group Γ, and let F' be the Radon–Nikodym derivative of the absolutely continuous part of F w.r.t. $\tilde{\lambda}$. The space $L^2(F)$ is $\mathcal{J}_c$-regular if and only if the following three conditions hold:*

(i) $F \ll \tilde{\lambda}$.

(ii) *The rank $r(F')$ is constant $\tilde{\lambda}$-a.e.*

(iii) *There exists an $\mathcal{M}_n$-valued trigonometric polynomial W such that* $\operatorname{Ker} W = \operatorname{Ker} F'$ *$\tilde{\lambda}$-a.e. and* $\int_\Gamma |W((F')^{\#})^{1/2}|^2\, d\tilde{\lambda} < \infty$.

In [1] the authors also sketch the proof of the corresponding result for $G := \mathbf{R}^l$, $l \in \mathbf{N}$. The present section deals with some generalizations of the result of Theorem 8.1, yet I was not able to generalize that result to an arbitrary LCA group G whose dual group is discrete or σ-compact. The main obstacle is the fact that the rank of a function in $\mathcal{T}_C$, $C \in \mathcal{K}$, is in general not necessarily constant $\tilde{\lambda}$-a.e. Thus we give the following definition.

DEFINITION 8.2. We will say that an LCA group G containing more than one element has the *property* ($\mathcal{C}$) if for each $C \in \mathcal{K}$ an arbitrary function from $\mathcal{T}_C$ has constant rank $\tilde{\lambda}$-a.e.

Remark 8.3. Note that the groups $\mathbf{Z}^l$ and $\mathbf{R}^l$, $l \in \mathbf{N}$, have the property ($\mathcal{C}$). If G is compact and hence Γ is discrete, then G does not have the property ($\mathcal{C}$). If G is discrete and can be ordered, then G has the property ($\mathcal{C}$) (cf. [20, Lemma 4.6]). Note further that a discrete group can be ordered if and only if it does not contain a finite subgroup (cf. [15, p. 194]).

Now we will prove a generalization of Theorem 8.1. We start with the following lemma.

LEMMA 8.4. *For $1 < p < \infty$ and $C \in \mathcal{K}$,*

$$\overline{\bigvee_{C \in \mathcal{K}} U_p \mathfrak{N}_{C,p}} = \overline{\bigcup_{C \in \mathcal{K}} U_p \mathfrak{N}_{C,p}}. \tag{8.1}$$

Proof. The inclusion $\overline{\bigcup_{C \in \mathcal{K}} \mathfrak{N}_{C,p}} \subseteq \overline{\bigvee_{C \in \mathcal{K}} \mathfrak{N}_{C,p}}$ is obvious. On the other hand, for $C, D \in \mathcal{K}$, we have

$$\overline{\mathfrak{N}_{C,p} \vee \mathfrak{N}_{D,p}} = \overline{\mathfrak{M}^\perp_{G\setminus C,q} \vee \mathfrak{M}^\perp_{G\setminus D,q}} = (\mathfrak{M}_{G\setminus C,p} \cap \mathfrak{M}_{G\setminus D,q})^\perp$$
$$\subseteq \mathfrak{M}^\perp_{G\setminus (C\cup D),q} = \mathfrak{N}_{C\cup D,p}$$

by (3.5). This implies $\overline{\bigvee_{C \in \mathcal{K}} \mathfrak{N}_{C,p}} \subseteq \overline{\bigcup_{C \in \mathcal{K}} \mathfrak{N}_{C,p}}$. Now use Theorem 4.3.

THEOREM 8.5. *Let G be an LCA group with property ($\mathcal{C}$) and whose dual group Γ is σ-compact. Let $1 < p < \infty$. Let F be an $\mathcal{M}_n^+$-valued measure on $\mathcal{B}$ and F' the Radon–Nikodym derivative of its absolutely continuous part w.r.t. $\tilde{\lambda}$. The space $L^p(F)$ is $\mathcal{J}_c$-regular if and only if the following three conditions hold:*

(i) $F \ll \tilde{\lambda}$.

(ii) *The rank $r(F')$ is constant $\tilde{\lambda}$-a.e.*

(iii) *There exists a $C \in \mathcal{K}$ and a function $W \in \mathcal{T}_C$ such that* $\operatorname{Ker} W = \operatorname{Ker} F'$ *$\tilde{\lambda}$-a.e. and* $\int_\Gamma |W((F')^{\#})^{1/p}|^q\, d\tilde{\lambda} < \infty$.

Proof. *Necessity.* Assume that $L^p(F)$ is $\mathcal{J}_c$-regular. According to Lemma 7.2 and (8.1), we have

$$\overline{\bigcup_{C \in \mathcal{K}} U_q \mathfrak{N}_{C,q}} = H^q(F). \tag{8.2}$$

Now (i) can be proved in the same way as (i) of Theorem 7.3.

Proof of (ii). In the proof of Theorem 7.3 it was shown that $F \in H^q(F)$. From (8.2) follows the existence of sequences $\{C_j\}_{j=1}^\infty \subseteq \mathcal{K}$ and $\{M_j\}_{j=1}^\infty \subseteq H^q(F)$ such that $M_j \in U_q \mathfrak{N}_{C_j,q}$, $j \in \mathbf{N}$, and $\lim_{j\to\infty} M_j = F$ in $H^q(F)$. But the relation

$$\lim_{j\to\infty} \int_\Gamma |(dM_j/d\tilde{\lambda} - F')((F')^{\#})^{1/p}|^q\, d\tilde{\lambda} = 0$$

implies the existence of a subsequence $\{M_{j_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} (dM_{j_k}/d\tilde{\lambda} - F')((F')^{\#})^{1/p} = 0$$

and hence

$$\lim_{k\to\infty} \frac{dM_{j_k}}{d\tilde{\lambda}}((F')^{\#})^{1/p} = (F')^{1/q} \quad \tilde{\lambda}\text{-a.e.} \tag{8.3}$$

Let r be the maximal rank of F' and let $B \in \mathcal{B}$ be a Borel set such that $\tilde{\lambda}(B) > 0$ and $r(F'(\gamma)) = r$ for $\tilde{\lambda}$-a.e. $\gamma \in B$. Since rank is a lower continuous function, we conclude from (8.3) that there exist an $i \in \mathbf{N}$ and a Borel subset B_1 of B such that $\tilde{\lambda}(B_1) > 0$ and $r((dM_i/d\tilde{\lambda})(\gamma)) \geq r$ for $\tilde{\lambda}$-a.e. $\gamma \in B_1$. Since G has the property ($\mathcal{C}$), we find that $r(dM_i/d\tilde{\lambda}) \geq r$ $\tilde{\lambda}$-a.e. But from (4.2) it follows that $dM_i/d\tilde{\lambda} = (U_q^{-1} M_i) F'$ $\tilde{\lambda}$-a.e. Hence, $r(F') \geq r$ $\tilde{\lambda}$-a.e. Since r is the maximal rank of F', we obtain $r(F') = r$ $\tilde{\lambda}$-a.e.

Proof of (iii). It is not hard to see that for the function $W := dM_i/d\tilde{\lambda}$ all conditions hold.

Sufficiency. Assume (i)–(iii). If $L^p(F)$ were not $\mathcal{J}_c$-regular, there would exist an $N \in H^p(F)$ such that $N \neq 0$ in $H^p(F)$ and $\langle\langle N, M^T \rangle\rangle = 0$ for arbitrary $C \in \mathcal{K}$ and each $T \in \mathcal{T}_C$ (cf. the proof of Theorem 7.3). Here M^T denotes the measure defined in (5.1). In particular, we would have

$$\int_\Gamma \frac{dN}{d\tilde{\lambda}}(\gamma) F'(\gamma)^{\#} (g, \gamma) W(\gamma)^* \tilde{\lambda}(d\gamma) = 0 \quad \text{for each } g \in G.$$

The uniqueness theorem for the inverse Fourier–Stieltjes transform implies $(dN/d\tilde{\lambda})(F')^{\#} W^* = 0$ $\tilde{\lambda}$-a.e. If P denotes the orthoprojector onto $\mathscr{R}(W^*)$, we get

$$0 = \frac{dN}{d\tilde{\lambda}}(F')^{\#} W^*(W^{\#})^* = \frac{dN}{d\tilde{\lambda}}(F')^{\#} P = \frac{dN}{d\tilde{\lambda}}(F')^{\#} \quad \tilde{\lambda}\text{-a.e.},$$

hence, $\operatorname{Ker} dN/d\tilde{\lambda} \supseteq \mathscr{R}(F')$ $\tilde{\lambda}$-a.e. But since $\operatorname{Ker} dN/d\tilde{\lambda} \supseteq \operatorname{Ker} F'$ $\tilde{\lambda}$-a.e. because $N \in H^p(F)$, we deduce $dN/d\tilde{\lambda} = 0$ $\tilde{\lambda}$-a.e. Thus, $N = 0$ in $H^p(F)$, a contradiction.

Remark 8.6. Analyzing the proof of Theorem 8.5 we see that the conditions (i)–(iii) are sufficient and (i) is necessary for the $\mathscr{J}_c$-regularity of $L^p(F)$ even in the case that G does not have the property ($\mathscr{C}$). However, condition (ii) is in general not necessary for the $\mathscr{J}_c$-regularity of $L^p(F)$ if we do not require that G has the property ($\mathscr{C}$). In fact, consider a group consisting of three elements. The dual group Γ also has three elements. Consider a nonnegative scalar measure μ on $\mathscr{B}$ such that μ has positive masses on two elements of Γ and is 0 on the third element. Clearly, (ii) does not hold for μ. But it is not hard to see that $L^p(\mu)$ is $\mathscr{J}_c$-regular.

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