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Sums of square-zero operators

by

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Dedicated to Professor Paul Halmos on his 75th birthday

Abstract. This paper is concerned with characterizations of bounded linear operators on a complex Hilbert space which are expressible as a sum of two or more square-zero operators. We characterize sums of two square-zero operators among invertible operators, normal operators and operators on a finite-dimensional space. In particular, we show that if T is such a sum, then T and $-T$ have the same spectra modulo the maximal ideal in the algebra of all bounded linear operators. This, together with a result of Pearcy and Topping's, yields a characterization of sums of four square-zero operators: T is such a sum if and only if it is a commutator. We also obtain various necessary or sufficient conditions for sums of three square-zero operators on a finite-dimensional space.

1. Introduction. In this paper, we are concerned with the problems of characterizing bounded linear operators on a complex Hilbert space which are expressible as a sum of two or more square-zero operators (an operator T is *square-zero* if $T^2 = 0$). Such problems were first considered by Pearcy and Topping [9]: they showed, using the structure of commutators, that on an infinite-dimensional space H any operator of class (F) is the sum of four square-zero operators and that any operator on H is the sum of five such operators. Recall that the class (F) consists of operators not of the form $\lambda I + K$, where λ is a scalar and K belongs to the unique maximal ideal J of the algebra $\mathscr{B}(H)$ of bounded linear operators on H. One of our main results (Theorem 3.8) complements these: an operator T is the sum of four square-zero operators if and only if T is a commutator. It is interesting to contrast this with a result of Fong and Sourour [5] that T is the sum of two quasinilpotent operators if and only if T is a commutator.

We start in Section 2 by studying sums of two square-zero operators. We are able to completely characterize such operators among invertible operators, normal operators and operators on finite-dimensional spaces (Theorems 2.4, 2.9, and 2.11). For noninvertible operators on an infinite-dimensional space, a complete characterization seems difficult. We obtain various necessary

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and/or sufficient conditions. Among other things, we show that if T is the sum of two square-zero operators then T and $-T$ have the same spectra (modulo J) (Theorem 2.12). This, together with Pearcy and Topping's result, is the main ingredient in proving the above-mentioned result on sums of four square-zero operators.

In Section 3, we consider the problem of expressing operators as sums of three or more square-zero ones. Here we mainly confine ourselves to the finite-dimensional case. We show that every finite matrix with trace zero is the sum of four square-zero matrices (Theorem 3.6) and proceed to determine whether fewer will do. Depending on the dimension of the underlying space, the minimal such number can be completely determined. This is achieved through an examination of matrices which are expressible as sums of three square-zero ones. Although we have not been able to give a complete characterization of such matrices, we do obtain some necessary and/or sufficient conditions on matrices in order that they be expressible as such. In particular, we show that if the $n\times n$ matrix T is the sum of three square-zero matrices then the geometric multiplicity of any eigenvalue of T is at most three fourths of n (Theorem 3.1). The paper is concluded in Section 4 with some open problems.

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2. Two squaro-zero operators. Recall that an operator T is an *involution* if $T^2=I$. We start with the following

LEMMA 2.1. *Let T be an operator. If there exists an involution V such that $TV=-VT$, then T is the sum of two square-zero operators.*

Proof. Let $T_1=\frac{1}{2}T(I-V)$ and $T_2=\frac{1}{2}T(I+V)$. Then it is easily seen that $T_1^2=T_2^2=0$ and $T=T_1+T_2$. ∎

For an operator T, $\sigma(T)$ (resp. $\sigma_J(T)$) denotes its spectrum (resp. spectrum modulo J, that is, the spectrum of the coset $T+J$ in $\mathscr{B}(H)/J$). We say that $\sigma(T)$ *does not surround* 0 if 0 belongs to the unbounded component of $\mathbf{C}\backslash\sigma(T)$. It is well known that if $\sigma(T)$ does not surround 0, then T has a square root which is an analytic function of T (cf. [11, p. 246]). We next extend Lemma 2.1 slightly by relaxing the restriction on V.

PROPOSITION 2.2. *Let T be an operator. If there exists an operator X such that $TX=-XT$ and $\sigma(X^2)$ does not surround 0, then T is the sum of two square-zero operators.*

Proof. Let Y be an analytic function of X^2 satisfying $Y^2=X^2$ and let $V=XY^{-1}$. Since Y^{-1} commutes with X, it is easily seen that V is an involution. On the other hand, $TX=-XT$ implies that $TX^2=X^2T$, whence $TY=YT$. We infer that $TV=-VT$. Our assertion then follows from Lemma 2.1. ∎

COROLLARY 2.3. *Let T be an operator. If either $TX=-XT$ for some operator X with no real spectrum or T is unitarily equivalent to $-T$, then T is the sum of two square-zero operators.*

Proof. The first assumption implies that $\sigma(X^2)$ does not surround 0, whence the conclusion follows by Proposition 2.2. As for the second one, note that, by the spectral theorem, every unitary operator U has a square root which commutes with every operator that commutes with U. Hence we can proceed as in the proof of Proposition 2.2 to reach the conclusion. ∎

For invertible operators, the converses of Lemma 2.1 and Proposition 2.2 are also true.

THEOREM 2.4. *Let T be an inverible operator. Then the following statements are equivalent:*

(i) *T is the sum of two square-zero operators;*
(ii) *there exists an involution V such that $TV=-VT$;*
(iii) *there exists an invertible X such $TX=-XT$ and $\sigma(X^2)$ does not surround* 0.

Proof. We need only prove (i) ⇒ (ii). Assume that $T=T_1+T_2$, where $T_1^2=T_2^2=0$. Let $V=(T_1-T_2)T^{-1}$. Since

$$\begin{aligned}(T_1-T_2)T&=(T_1-T_2)(T_1+T_2)=T_1T_2-T_2T_1\\&=-(T_1+T_2)(T_1-T_2)=-T(T_1-T_2)\end{aligned}$$

and $(T_1-T_2)^2=-T_1T_2-T_2T_1=-(T_1+T_2)^2=-T^2$, we have

$$V^2=(T_1-T_2)T^{-1}(T_1-T_2)T^{-1}=-(T_1-T_2)^2T^{-2}=I.$$

Moreover, $TV=T(T_1-T_2)T^{-1}=-(T_1-T_2)TT^{-1}=-VT$ as desired. ∎

The next is an immediate corollary of Theorem 2.4.

COROLLARY 2.5. *An invertible operator is the sum of two square-zero operators if and only if its inverse is.*

If more restrictions are imposed on the spectrum of T in Theorem 2.4, then we can have a better description of its structure.

COROLLARY 2.6. *If $\sigma(T^2)$ does not surround 0, then the conditions* (i)–(iii) *in Theorem 2.4 are equivalent to*

(iv) *T is similar to $S\oplus(-S)$ for some invertible S.*

Proof. (ii) ⇒ (iv). Let V be an involution such that $TV=-VT$. Since $\sigma(V)=\{\pm1\}$, V is similar to an operator of the form $I_1\oplus(-I_2)$, where I_1 and I_2 are the identity operators on some spaces H_1 and H_2, respectively. Let X be an invertible operator implementing this similarity: $XV=(I_1\oplus(-I_2))X$.

We have

$$XTX^{-1}(I_1\oplus(-I_2)) = -(I_1\oplus(-I_2))XTX^{-1}.$$

If $XTX^{-1} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ on the decomposition $H_1\oplus H_2$, then carrying out the above matrix multiplication yields that $\left[\begin{smallmatrix} A & -B \\ C & -D \end{smallmatrix}\right] = \left[\begin{smallmatrix} -A & -B \\ C & D \end{smallmatrix}\right]$. Therefore, $A = 0$ and $D = 0$. In other words, T is similar to $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]$ on $H_1\oplus H_2$. Since T^2 is similar to $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} BC & 0 \\ 0 & CB \end{smallmatrix}\right]$, BC and CB are both invertible. Thus so are B and C. Hence we may assume, for simplicity, that $H_1 = H_2$. We have $\sigma(BC) = \sigma(CB) = \sigma(T^2)$ (cf. [6, Problem 76]). By our assumption, $S \equiv (CB)^{1/2}$ exists. If

$$X = \begin{bmatrix} C^{-1}S & -C^{-1}S \\ I & I \end{bmatrix},$$

then X is invertible (by [6, Problem 71]) and $\left[\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right]X = X\left[\begin{smallmatrix} S & 0 \\ 0 & -S \end{smallmatrix}\right]$. This shows that T is similar to $S\oplus(-S)$ as required.

(iv) $\Rightarrow$ (i). Since

$$\begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix} = \frac{1}{2}\begin{bmatrix} S & -S \\ S & -S \end{bmatrix} + \frac{1}{2}\begin{bmatrix} S & S \\ -S & -S \end{bmatrix}$$

is the sum of two square-zero operators, the same holds for T. ∎

For noninvertible operators, it seems difficult to give a complete characterization of sums of two square-zero operators. This we achieve only for two classes: normal operators and operators on finite-dimensional spaces. The proofs depend largely on the fact that we can handle separately the "invertible part" and the "zero part" of the operators under consideration. The next two lemmas are the results needed for normal operators.

LEMMA 2.7. *Let $T = T_1\oplus 0$, where T_1 is one-to-one or has dense range. Then T is the sum of two square-zero operators if and only if T_1 is.*

Proof. We need only prove the necessity part. Assume that T_1 is one-to-one and $T = S+R$, where

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \quad\text{and}\quad R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}$$

are square-zero operators. We have $T_1 = S_1+R_1$, $0 = S_2+R_2$ and $0 = S_4+R_4$. Since $S^2 = R^2 = 0$, a little computation yields that $S_1S_2+S_2S_4 = 0$ and $R_1R_2+R_2R_4 = 0$. Hence

$$\begin{aligned} T_1S_2 &= (S_1+R_1)S_2 = -S_2S_4+R_1(-R_2) \\ &= -S_2S_4+R_2R_4 = -S_2S_4+(-S_2)(-S_4) = 0. \end{aligned}$$

Since T_1 is one-to-one, we infer that $S_2 = 0$. Similarly, $R_2 = 0$. Therefore, $T_1 = S_1+R_1$ is the sum of two square-zero operators. If T_1 has dense range, then proceed as above with T replaced by T^*. ∎

LEMMA 2.8. *T is the sum of two square-zero operators if and only if there exists an operator X such that $TX = -XT$ and $X^2 = -T^2$.*

Proof. If $T = T_1+T_2$, where $T_1^2 = T_2^2 = 0$, then, letting $X = T_1-T_2$, we have $TX = T_2T_1-T_1T_2 = -XT$ and $X^2 = -T_1T_2-T_2T_1 = -T^2$. Conversely, if X is an operator satisfying $TX = -XT$ and $X^2 = -T^2$, then $T = \frac{1}{2}(T+X)+\frac{1}{2}(T-X)$ is the sum of two square-zero operators. ∎

Now we are ready for the characterization of sums of two square-zero operators among normal operators.

THEOREM 2.9. *Let T be a normal operator on H. Then the following statements are equivalent:*

(i) *T is the sum of two square-zero operators;*
(ii) *T is unitarily equivalent to $S\oplus(-S)\oplus 0$ for some normal S;*
(iii) *T is unitarily equivalent to $-T$.*

Proof. Since (ii) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (i) was proved in Corollary 2.3, we need only show that (i) $\Rightarrow$ (ii).

Let $E(\cdot)$ be the spectral measure of T, and let $\sigma_1 = \{z\in\mathbb{C}\colon \operatorname{Re} z > 0$ or $\operatorname{Re} z = 0$ and $\operatorname{Im} z > 0\}$ and $\sigma_2 = \{z\in\mathbb{C}\colon -z\in\sigma_1\}$. If $T_j = T|E(\sigma_j)H$, $j = 1, 2$, then T is unitarily equivalent to $T_1\oplus T_2\oplus 0$. Since $T_1\oplus T_2$ is one-to-one, Lemma 2.7 implies that $T_1\oplus T_2$ is the sum of two square-zero operators. By Lemma 2.8, there exists an operator

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

such that $(T_1\oplus T_2)X = -X(T_1\oplus T_2)$ and $X^2 = -(T_1\oplus T_2)^2$. The former yields that $T_1X_{11} = -X_{11}T_1$, $T_2X_{22} = -X_{22}T_2$, $T_1X_{12} = -X_{12}T_2$ and $T_2X_{21} = -X_{21}T_1$. Since the spectral measures of T_1 and $-T_1$ are mutually singular, we obtain $X_{11} = 0$ (cf. [3, Proposition 2.4]). Similarly, $X_{22} = 0$. Thus $X^2 = -(T_1\oplus T_2)^2$ yields that $X_{12}X_{21} = -T_1^2$ and $X_{21}X_{12} = -T_2^2$. By our construction, both T_1 and T_2 are one-to-one with dense range. From the above, the same is true for X_{12} and X_{21}. Applying [2, Lemma 4.1] to $T_1X_{12} = -X_{12}T_2$ gives the unitary equivalence of T_1 and $-T_2$. Thus T is unitarily equivalent to $T_1\oplus(-T_1)\oplus 0$ as asserted. ∎

Next we consider sums of two square-zero operators on a finite-dimensional space. To separate the "invertible part", we need the following lemma.

LEMMA 2.10. *Let $T = T_1\oplus T_2$, where $\sigma(T_1)\cap\sigma(-T_2) = \emptyset$. Then T is the sum of two square-zero operators if and only if T_1 and T_2 are.*

Proof. To prove the necessity part, we proceed as in the proof of Lemma 2.7. Now, instead of $T_1S_2 = 0$, we would obtain $T_1S_2 = -S_2T_2$. Thus $\sigma(T_1)\cap\sigma(-T_2) = \emptyset$ implies that $S_2 = 0$ (cf. [10]). Similarly, $R_2 = 0$. Hence

$T_1 = S_1 + R_1$ is the sum of two square-zero operators. The same holds for T_2. ∎

THEOREM 2.11. *Let T be a finite matrix. Then the following statements are equivalent:*

(i) *T is the sum of two square-zero matrices;*
(ii) *there exists an involution V such that $TV = -VT$;*
(iii) *T is similar to $-T$;*
(iv) *T is similar to $S \oplus (-S) \oplus N$, where S is invertible and N is nilpotent.*

Proof. (i) ⇒ (ii). By the Jordan canonical form, T is similar to $T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. Lemma 2.10 then implies that both T_1 and T_2 are sums of square-zero matrices. Theorem 2.4 applied to T_1 yields an involution intertwining T_1 and $-T_1$. As for T_2, we need only consider a nilpotent Jordan block

$$J = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$$

of size, say, k. It is easily seen that

$$V = \begin{bmatrix} 1 & & & & & 0 \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ 0 & & & & & (-1)^{k+1} \end{bmatrix}$$

is an involution and $JV = -VJ$. (ii) follows immediately.

(ii) ⇒ (iii). This is trivial.

(iii) ⇒ (iv). As above, T is similar to $T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. The similarity of T and $-T$ implies the existence of an invertible matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

such that

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \begin{bmatrix} -T_1 & 0 \\ 0 & -T_2 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

Hence we have $X_{12} T_2 = -T_1 X_{12}$. Since $\sigma(T_2) \cap \sigma(-T_1) = \emptyset$, using [10] we deduce that $X_{12} = 0$. Similarly, $X_{21} = 0$. Therefore, X_{11} is invertible and

$X_{11} T_1 = -T_1 X_{11}$. Since both $\sigma(X_{11}^2)$ and $\sigma(T_1^2)$ do not surround 0, (iv) is a consequence of Corollary 2.6.

(iv) ⇒ (i). For any operator S,

$$\begin{bmatrix} S & 0 \\ 0 & -S \end{bmatrix} = \frac{1}{2} \begin{bmatrix} S & -S \\ S & -S \end{bmatrix} + \frac{1}{2} \begin{bmatrix} S & S \\ -S & -S \end{bmatrix}$$

is the sum of two square-zero operators. On the other hand, any $k \times k$ nilpotent Jordan block J can be written as

$$J = \begin{bmatrix} 0 & 1 & & & & 0 \\ & 0 & & & & \\ & & 0 & 1 & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & \end{bmatrix} + \begin{bmatrix} 0 & & & & & 0 \\ & 0 & 1 & & & \\ & & 0 & & & \\ & & & 0 & 1 & \\ & & & & 0 & \\ 0 & & & & & \ddots \end{bmatrix},$$

the latter two matrices being square-zero. This proves (i). ∎

We end this section with a necessary condition on the spectrum (modulo J) for sums of two square-zero operators. This condition is useful in Section 3 in characterizing sums of four square-zero operators.

THEOREM 2.12. *If T is the sum of two square-zero operators on an infinite-dimensional space, then $\sigma(T) = \sigma(-T)$ and $\sigma_J(T) = \sigma_J(-T)$.*

Proof. Let $T = T_1 + T_2$, where $T_1^2 = T_2^2 = 0$. If $S = T_1 - T_2$ and λ is any complex number, then

$$\begin{aligned}(T-\lambda I)(S-T-\lambda I) &= TS - \lambda S - T^2 + \lambda T - \lambda T + \lambda^2 I \\ &= (T_1+T_2)(T_1-T_2) - \lambda S - T^2 + \lambda^2 I \\ &= -ST - \lambda S - T^2 + \lambda^2 I \\ &= (S+T-\lambda I)(-T-\lambda I).\end{aligned}$$

It is easily seen that $(S-T)^2 = (S+T)^2 = 0$. If $\lambda \neq 0$, then both $S-T-\lambda I$ and $S+T-\lambda I$ are invertible. We deduce from the above that $T-\lambda I$ is invertible (resp. invertible modulo J) if and only if $-T-\lambda I$ is. Thus $\sigma(T)\backslash\{0\} = \sigma(-T)\backslash\{0\}$ and $\sigma_J(T)\backslash\{0\} = \sigma_J(-T)\backslash\{0\}$. Therefore $\sigma(T) = \sigma(-T)$ and $\sigma_J(T) = \sigma_J(-T)$. ∎

3. Three or more square-zero operators. For sums of three square-zero operators, we confine ourselves to the finite-dimensional case and start with the following necessary condition.

THEOREM 3.1. *If the $n \times n$ matrix T is a sum of three square-zero matrices, then* $\dim\ker(T-\lambda I) \leqslant \frac{3}{4}n$ *for any* $\lambda \in \mathbf{C}$, $\lambda \neq 0$.

Proof. Let $T = T_1 + T_2 + T_3$, where $T_j^2 = 0$ for $j = 1, 2, 3$. Note that $\dim\ker T_j \geqslant n/2$ for all j. Indeed, if $\dim\ker T_j < n/2$, then $\operatorname{ran} T_j \subseteq \ker T_j$ implies that $\operatorname{rank} T_j < n/2$, whence $n = \dim\ker T_j + \operatorname{rank} T_j < n/2 + n/2 = n$, which is impossible.

Let $K = \ker(T - \lambda I) \cap \ker T_3$ and $m = \dim\ker(T - \lambda I)$. Then

$$\begin{aligned}\dim K &= \dim\ker(T-\lambda I) + \dim\ker T_3 - \dim\big(\ker(T-\lambda I) + \ker T_3\big)\\ &\geqslant m + n/2 - n = m - n/2.\end{aligned}$$

Since K is invariant for both T and T_3, it is invariant for $T_1 + T_2$ and $(T_1+T_2)|K = T|K - T_3|K = \lambda I_K$, I_K being the identity matrix on K. Hence $K \subseteq \ker(T_1 + T_2 - \lambda I)$, which implies that $\dim\ker(T_1 + T_2 - \lambda I) \geqslant \dim K \geqslant m - n/2$. By Theorem 2.11, $T_1 + T_2$ is similar to $-(T_1 + T_2)$ whence we also have $\dim\ker(T_1 + T_2 + \lambda I) \geqslant m - n/2$.

Let $L = \ker(T - \lambda I) \cap \ker(T_1 + T_2 + \lambda I)$. We repeat the above arguments:

$$\begin{aligned}\dim L &= \dim\ker(T-\lambda I) + \dim\ker(T_1 + T_2 + \lambda I)\\ &\quad - \dim\big(\ker(T-\lambda I) + \ker(T_1 + T_2 + \lambda I)\big)\\ &\geqslant m + (m - n/2) - n = 2m - \tfrac{3}{2}n.\end{aligned}$$

Since L is invariant for both T and $T_1 + T_2$, it is invariant for T_3 and $T_3|L = T|L - (T_1+T_2)|L = \lambda I_L - (-\lambda I_L) = 2\lambda I_L$, where I_L is the identity matrix on L. Thus $L \subseteq \ker(T_3 - 2\lambda I)$, which implies that $\dim\ker(T_3 - 2\lambda I) \geqslant \dim L \geqslant 2m - \frac{3}{2}n$. Since $T_3^2 = 0$, $T_3 - 2\lambda I$ is invertible for any $\lambda \neq 0$. Therefore $\dim\ker(T_3 - 2\lambda I) = 0$. From the above, we infer that $m \leqslant \frac{3}{4}n$ as asserted. ∎

Next we consider sufficient conditions for sums of three square-zero matrices. Our main tool is the following lemma. Recall that a matrix T on H is cyclic if there exists a vector $x \in H$ such that H is the span of the vectors $T^k x$, $k = 0, 1, 2, \ldots$; $\operatorname{tr} T$ denotes the trace of T.

LEMMA 3.2. *If T is an $n \times n$ cyclic matrix and $\lambda_1, \ldots, \lambda_n$ are complex numbers satisfying $\sum_{j=1}^n \lambda_j = \operatorname{tr} T$, then there exist matrices A and B such that $T = A + B$, $A^2 = 0$ and B is cyclic with $\sigma(B) = \{\lambda_1, \ldots, \lambda_n\}$.*

Proof. Since T is similar to a companion matrix of the form

$$C = \begin{bmatrix} 0 & & & a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & a_{n-2} \\ 0 & & 1 & a_{n-1}\end{bmatrix},$$

we need only prove the lemma for C. For $j = 0, 1, \ldots, n-2$, let b_j be the coefficient of z^j in the expansion of $(z-\lambda_1)\ldots(z-\lambda_n)$. Let

$$A = \begin{bmatrix} 0 & & & a_0 + b_0 \\ & \ddots & & \vdots \\ & & \ddots & a_{n-2} + b_{n-2} \\ 0 & & & 0\end{bmatrix}, \quad B = \begin{bmatrix} 0 & & & -b_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -b_{n-2} \\ 0 & & 1 & a_{n-1}\end{bmatrix}.$$

It is easily seen that $C = A + B$, $A^2 = 0$ and B is cyclic with characteristic polynomial $(z-\lambda_1)\ldots(z-\lambda_n)$ (noting that $a_{n-1} = \operatorname{tr} T = \sum_{j=1}^n \lambda_j$). Hence $\sigma(B) = \{\lambda_1, \ldots, \lambda_n\}$. ∎

PROPOSITION 3.3. *Let T be an $n \times n$ matrix with $\operatorname{tr} T = 0$. If $T = T_1 \oplus \ldots \oplus T_m$, where each T_j is cyclic with size at least 2, then T is the sum of three square-zero matrices.*

Proof. Let $t_j = \operatorname{tr} T_j$, $j = 1, \ldots, m$, and $c > \sum_{j=1}^m |t_j|$. By Lemma 3.2, there exist, for each j, matrices A_j and B_j such that $T_j = A_j + B_j$, $A_j^2 = 0$ and

$$\sigma(B_j) = \Big\{c - \sum_{i=1}^{j-1} t_i,\ \sum_{i=1}^{j} t_i - c,\ 0, \ldots, 0\Big\}.$$

If $A = A_1 \oplus \ldots \oplus A_m$ and $B = B_1 \oplus \ldots \oplus B_m$, then $T = A + B$, $A^2 = 0$ and $\sigma(B) = \{b_1, \ldots, b_{2m}, 0, \ldots, 0\}$, where b_j's are distinct nonzero numbers which are pairwise negative to each other. In particular, B is similar to $B_1' \oplus B_2'$, where

$$B_1' = \begin{bmatrix} b_1' & 0 \\ 0 & -b_1'\end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} b_m' & 0 \\ 0 & -b_m'\end{bmatrix}$$

and B_2' is nilpotent. By Theorem 2.11, B is the sum of two square-zero matrices, whence T is the sum of three such matrices. ∎

Here are two corollaries of the preceding proposition, the second of which characterizes sums of three $n \times n$ square-zero matrices for n up to 5.

COROLLARY 3.4. *If the $n \times n$ matrix T is such that $\operatorname{tr} T = 0$ and $\dim\ker(T - \lambda I) \leqslant 3$ for any $\lambda \neq 0$, then T is the sum of three square-zero matrices.*

Proof. Using Proposition 3.3 and the rational form for matrices, we are reduced to considering T in one of the following forms:

$$(1)\ \begin{bmatrix} T_1 & 0 \\ 0 & a\end{bmatrix}, \quad (2)\ \begin{bmatrix} T_1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a\end{bmatrix}, \quad (3)\ \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & a\end{bmatrix},$$

where $a \neq 0$ and T_1 and T_2 are cyclic with size at least 2 and respective characteristic polynomials p_1 and p_2 satisfying $p_2 | p_1$ and $p_1(a) = p_2(a) = 0$. All these cases can be handled by judiciously choosing the matrices A and B in Lemma 3.2:

(1) Let A and B be such that $T_1 = A+B$, $A^2 = 0$ and B is cyclic with $\sigma(B) = \{-a, 0, \ldots, 0\}$. Then B is similar to $\left[\begin{smallmatrix} -a & 0 \\ 0 & N \end{smallmatrix}\right]$, where N is nilpotent. Hence T is the sum of $\left[\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and a matrix similar to

$$\begin{bmatrix} -a & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & a \end{bmatrix},$$

the latter being a sum of two square-zero matrices by Theorem 2.11. This proves our assertion for T.

(2) Let the matrices A and B be as above except that this time $\sigma(B) = \{-a, -a, 0, \ldots, 0\}$. Then B is similar to $\left[\begin{smallmatrix} -a & 1 \\ 0 & -a \end{smallmatrix}\right] \oplus N$, where N is nilpotent. Hence T is the sum of $A \oplus \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and a matrix similar to $\left[\begin{smallmatrix} -a & 1 \\ 0 & -a \end{smallmatrix}\right] \oplus N \oplus \left[\begin{smallmatrix} a & -1 \\ 0 & a \end{smallmatrix}\right]$. Again, the latter matrix is a sum of two square-zero ones by Theorem 2.11.

(3) Let $t = \operatorname{tr} T_1$. We consider three subcases:

(i) $t = -2a$. Apply Lemma 3.2 to obtain A_j, B_j, $j = 1, 2$, such that $T_j = A_j + B_j$, $A_j^2 = 0$ and B_j is cyclic with $\sigma(B_1) = \{-2a, 0, \ldots, 0\}$ and $\sigma(B_2) = \{2a, -a, 0, \ldots, 0\}$. Then B_1 and B_2 are similar to

$$\begin{bmatrix} -2a & 0 \\ 0 & N_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & N_2 \end{bmatrix},$$

respectively, where N_1 and N_2 are nilpotent. Hence T is the sum of

$$\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and a matrix similar to $\left[\begin{smallmatrix} 2a & 0 \\ 0 & -2a \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right] \oplus N_1 \oplus N_2$. The latter is a sum of two square-zero matrices by Theorem 2.11.

(ii) $t = -a$. Applying Lemma 3.2 yields A_j, B_j, $j = 1, 2$, as above except that $\sigma(B_1) = \{-a, 0, \ldots, 0\}$ and $\sigma(B_2) = \{0, \ldots, 0\}$. Then B_1 is similar to $\left[\begin{smallmatrix} -a & 0 \\ 0 & N_1 \end{smallmatrix}\right]$ and B_2 is itself nilpotent. Hence T is the sum of three square-zero matrices as above.

(iii) $t \neq -2a, -a$. Obtain A_j, B_j, $j = 1, 2$, as above except that $\sigma(B_1) = \{t+a, -a, 0, \ldots, 0\}$ and $\sigma(B_2) = \{-t-a, 0, \ldots, 0\}$. In this case, B_1 and B_2 are similar to

$$\begin{bmatrix} t+a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & N_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -t-a & 0 \\ 0 & N_2 \end{bmatrix},$$

respectively, with nilpotent N_1 and N_2. Our assertion on T then follows as above. ∎

Corollary 3.5. (1) *A 2×2 matrix T is the sum of two square-zero matrices if and only if* $\operatorname{tr} T = 0$.

(2) *A 3×3 or 4×4 matrix T is the sum of three square-zero matrices if and only if* $\operatorname{tr} T = 0$.

(3) *A 5×5 matrix T is the sum of three square-zero matrices if and only if* $\operatorname{tr} T = 0$ *and* $\dim \ker(T - \lambda I) \leqslant 3$ *for any* $\lambda \neq 0$.

Proof. (1) If $\operatorname{tr} T = 0$, then T is similar to $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right]$. In either case, T is the sum of two square-zero matrices by Theorem 2.11.

(2) and (3) follow immediately from Theorem 3.1 and Corollary 3.4. ∎

The next theorem says that matrices with trace zero can always be written as a sum of four square-zero ones.

Theorem 3.6. *An $n \times n$ matrix T is the sum of finitely many square-zero matrices if and only if* $\operatorname{tr} T = 0$. *In this case, the minimal number of square-zero matrices required is n if $n = 1$ or 2; 3 if $n = 3$ or 4; and 4 if $n \geqslant 5$.*

Proof. If $\operatorname{tr} T = 0$, then T is unitarily equivalent to a matrix $[t_{ij}]$ with zero diagonals $t_{ii} = 0$ (cf. [4]). The latter matrix is the sum of two nilpotent ones:

$$\begin{bmatrix} 0 & & t_{1n} \\ & \ddots & \\ t_{n1} & & 0 \end{bmatrix} = \begin{bmatrix} 0 & t_{12} & \cdots & t_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t_{n-1,n} \\ 0 & & & 0 \end{bmatrix} + \begin{bmatrix} 0 & & & 0 \\ t_{21} & \ddots & & \\ \vdots & \ddots & \ddots & \\ t_{n1} & \cdots & t_{n,n-1} & 0 \end{bmatrix}.$$

That T is the sum of four square-zero matrices follows from Theorem 2.11. If $n \geqslant 5$, then

$$T = \begin{bmatrix} -(n-1) & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

cannot be written as a sum of three square-zero matrices by Theorem 3.1. Other assertions on the minimal number follow from Theorem 2.11 and Corollary 3.5. ∎

We conclude this section with a characterization of sums of four square-zero operators on an infinite-dimensional space. We start with

Lemma 3.7. *Let $T = \lambda I + K$ on an infinite-dimensional space H, where $\lambda \in \mathbf{C}$ and K belongs to J, the maximal ideal of $\mathscr{B}(H)$. Then T is the sum of four square-zero operators if and only if $\lambda = 0$.*

Proof. If $T = T_1 + T_2$, where each T_j is the sum of two square-zero operators, and $\alpha \in \sigma_J(T_1)$, then Theorem 2.12 implies that $\alpha \in \sigma_J(-T_1) = \sigma_J(T_2 - \lambda I - K) = \sigma_J(T_2 - \lambda I)$, whence $\alpha + \lambda \in \sigma_J(T_2)$. Using Theorem 2.12 again, we have $\alpha + \lambda \in \sigma_J(-T_2) = \sigma_J(T_1 - \lambda I - K) = \sigma_J(T_1 - \lambda I)$. Thus $\alpha + 2\lambda \in \sigma_J(T_1)$.

Repeating this process, we see that $\alpha+n\lambda\in\sigma_J(T_1)$ for any even n. The boundedness of $\sigma_J(T_1)$ then implies that $\lambda=0$.

Conversely, if T is in J, we may follow the arguments in the proof of [9, Theorem 2], using the fact that operators in J are commutators [1, Theorem 4], to conclude that T is the sum of four square-zero operators. ∎

Finally, our promised characterization of sums of four square-zero operators on an infinite-dimensional space. Such operators coincide with sums of two quasinilpotent operators [5].

THEOREM 3.8. *On an infinite-dimensional space, an operator is the sum of four square-zero operators if and only if it is a commutator.*

Proof. This is an easy consequence of Lemma 3.7, [9, Theorem 2] and the characterization of commutators [1]. ∎

4. Open problems. On an infinite-dimensional space, which operator is the sum of two square-zero operators? This problem seems difficult to answer. For invertible operators, it may become manageable. We conjecture that *if T is invertible then T is the sum of two square-zero operators if and only if T is similar to $-T$.* As demonstrated in Section 2, the necessity always holds and the sufficiency is true under various extra conditions: the similarity of T and $-T$ is implemented by an invertible operator X with $\sigma(X^2)$ not surrounding 0; T is unitarily equivalent to $-T$; T is normal; or T acts on a finite-dimensional space.

As for sums of three square-zero operators, a complete characterization is beyond reach at present. For finite matrices, the following might be true: *an $n\times n$ matrix T is the sum of three square-zero matrices if and only if* $\operatorname{tr} T=0$ *and* $\dim\ker(T-\lambda I)\leqslant\frac{3}{4}n$ *for any* $\lambda\neq 0$. As proved in Corollary 3.5, this is the case for $1\leqslant n\leqslant 5$; note that the necessity is always true by Theorem 3.1. For infinite-dimensional spaces, it might be worthwhile to find an operator which is expressible as a sum of four square-zero operators, but not of three. Such an operator must be searched for among those not of the form $\lambda I+K$, where λ is a nonzero scalar and K is in J, or, in other words, among commutators (by Theorem 3.8).

We conclude this paper by bringing attention to the close resemblance between the theory of sums of square-zero operators developed here and that of sums of idempotents. This is already evident in Pearcy and Topping's paper [9] (cf. also [7], [8] and [13] for the finite-dimensional case). These additive theories are also parallel to the multiplicative ones of products of Hermitian operators, involutions and unipotents of index 2 (cf. [12] for references in these latter theories). We will explore these and other related topics in subsequent papers.

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