References

[1] C. Borell, *Convex set functions in d-space*, Period. Math. Hungar. 6 (1975), 111–136.
[2] —, *Convex measures on locally convex spaces*, Ark. Mat. 12 (1974), 239–252.
[3] T. Byczkowski and M. Ryznar, *Series of independent vector valued random variables and absolute continuity of seminorms*, Math. Scand. 62 (1988), 59–74.
[4] —, —, *Smoothness of the distribution of the norm in uniformly convex Banach spaces*, J. Theoret. Probab. 80 (1990), 433–448.
[5] J. Hoffmann-Jörgensen, L. A. Shepp and R. M. Dudley, *On the lower tail of Gaussian seminorms*, Ann. Probab. 7 (1979), 319–342.
[6] A. Prekopa, *Logarithmic concave measures with application to stochastic programming*, Acta Sci. Math. (Szeged) 32 (1971), 301–316.
[7] M. Talagrand, *Sur l'intégrabilité des vecteurs gaussiens*, Z. Wahrsch. Verw. Gebiete 68 (1984), 1–8.
[8] V. S. Tsirel'son, *The density of the distribution of the maximum of a Gaussian process*, Theory Probab. Appl. 20 (1975), 847–855.

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Improved ratio inequalities for martingales

by

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Abstract. We show that the martingale inequality

$$E[\langle M\rangle_\infty^p \exp(\alpha\langle M\rangle_\infty/M_\infty^{*2})] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^p] \qquad (0<p<\infty)$$

is valid only for $0\leqslant\alpha<\pi^2/8$. In a previous paper [3] we proved it for sufficiently small $\alpha>0$. Our new result is a sharp estimate on α.

1. Introduction. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, P)$ be a filtered probability space satisfying the usual conditions. Throughout this short note, we deal only with continuous (local) martingales adapted to the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$, and such a (local) martingale is called "(local) martingale" simply. Moreover, unless otherwise precisely stated, we assume that (local) martingales vanish at $t=0$.

As usual, for every martingale $M=(M_t)_{t\geqslant 0}$, we set $M_t^*=\sup_{s\leqslant t}|M_s|$ and denote by $\langle M\rangle$ its quadratic variation process. The following result, which is an improvement of results in Gundy [2] and Yor [5], have been established in [3]: for *sufficiently small* $\alpha>0$ and every $p>0$, we have

$$E[M_\infty^{*p}\exp(\alpha M_\infty^*/\langle M\rangle_\infty^{1/2})] \leqslant C_{\alpha,p}E[M_\infty^{*p}], \tag{1}$$

$$E[\langle M\rangle_\infty^{p/2}\exp(\alpha\langle M\rangle_\infty/M_\infty^{*2})] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^{p/2}], \tag{2}$$

where $C_{\alpha,p}$ denotes an absolute constant depending only on α and p. Note that it is not necessarily the same from line to line, and we shall use this notation also in what follows. We should be careful with the difference between the powers of ratios appearing in (1) and (2).

2. Statement of results. Our new estimates for the inequalities (1) and (2), which are themselves our main object, are the following.

THEOREM. (i) *If* $0\leqslant\alpha<1/2$, *the ratio inequalities*

$$E[M_\infty^{*p}\exp(\alpha M_\infty^{*2}/\langle M\rangle_\infty)] \leqslant C_{\alpha,p}E[M_\infty^{*p}] \qquad (p>0), \tag{3}$$

$$E[\langle M\rangle_\infty^{p/2}\exp(\alpha M_\infty^{*2}/\langle M\rangle_\infty)] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^{p/2}] \qquad (p>0) \tag{3'}$$

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hold for every $p>0$ and every continuous martingale M with $M_0=0$. However, these inequalities are no longer valid for any $p>0$ when $\alpha \geqslant 1/2$.

(ii) *On the other hand, if $0 \leqslant \alpha < \pi^2/8$, the ratio inequalities*

$$(4)\qquad E[\langle M\rangle_\infty^{p/2}\exp(\alpha\langle M\rangle_\infty/M_\infty^{*2})] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^{p/2}] \quad (p>0),$$

$$(4')\qquad E[M_\infty^{*p}\exp(\alpha\langle M\rangle_\infty/M_\infty^{*2})] \leqslant C_{\alpha,p}E[M_\infty^{*p}] \quad (p>0)$$

hold for every $p>0$ and every continuous martingale M with $M_0=0$. However, these inequalities are no longer valid for any $p>0$ when $\alpha \geqslant \pi^2/8$.

Furthermore, $E[M_\infty^{*p}]$ and $E[\langle M\rangle_\infty^{p/2}]$ are interchangeable in the above four inequalites.

Remark. Using the Hölder inequality, we can deduce some inequalities such as

$$(5)\qquad E[M_\infty^{*p}] \leqslant C_{\alpha,p}E[M_\infty^{*p}\exp(-\alpha M_\infty^{*2}/\langle M\rangle_\infty)]$$

for every $\alpha>0$ and $p>0$, provided $M_\infty^* \in L^p$. Furthermore, we have

$$E[M_\infty^{*p}] \leqslant C_{\alpha,p}\sup_T E[M_T^{*p}\exp(-\alpha M_T^{*2}/\langle M\rangle_T)]$$

even if the left-hand side is infinite, where the supremum is taken over all finite stopping times T. However, we are unable to say anything about the problem whether the restriction $M_\infty^* \in L^p$ can be removed in the inequality (5).

It was already shown in [3] that neither (3) nor (3′) is valid if $\alpha \geqslant 1/2$, and that neither (4) nor (4′) is valid if $\alpha \geqslant \pi^2/8$. In fact, let $\alpha = 1/2$ and let $M_t = B_{t\wedge 1}$, where (B_t) is a one-dimensional Brownian motion. Then the left-hand sides of both (3) and (3′) are infinite although both right-hand sides are finite. On the other hand, if we set $\alpha = \pi^2/8$ and $\tau = \inf\{t\colon |B_t| = 1\}$, then the stopped martingale $M = (B_{t\wedge\tau})_{t\geqslant 0}$ satisfies neither (4) nor (4′).

The rest of this note is devoted to the proofs of the inequalities (3), (3′), (4), and (4′) for suitable α.

3. Lemmas and proof of Theorem. We need some preliminary lemmas. The first one is purely analytic and shows that, to prove the theorem, it suffices to establish a distribution function inequality for M_∞^* and $\langle M\rangle_\infty$.

LEMMA 1. *Let X and Y be two nonnegative random variables such that*

$$(6)\qquad P(X>\gamma\lambda,\ Y\leqslant\lambda) \leqslant c\exp[-a(\sqrt{\gamma}-b)^2]\cdot P(X>\lambda)$$

for every $\lambda>0$ and $\gamma>1$, where a and c are two positive constants and b is an arbitrary constant. If $0<\alpha<a$ and $p>0$, there exists a constant $C = C(a, b, c, \alpha, p)$ depending only on the parenthesized numbers such that

$$(7)\qquad E[X^p\exp(\alpha X/Y)] \leqslant CE[X^p],$$

$$(8)\qquad E[Y^p\exp(\alpha X/Y)] \leqslant CE[Y^p].$$

Furthermore, $E[X^p]$ can be replaced by $E[Y^p]$ in (7).

This lemma was also used in [3] without proof. We give here a simple proof which probably has not been published yet.

Proof. First we note that the ratio X/Y is meaningful, which follows from the fact that X vanishes almost surely on the set of ω such that $Y(\omega)=0$. To see this, it suffices to let $\lambda\to 0$ and then $\gamma\to\infty$ in (6). Moreover, it is well known (cf. [1]) that (6) implies $E[X^p] \leqslant K_pE[Y^p]$ for some constant K_p depending only on p.

For the sake of brevity, let $\varphi(\gamma) = c\exp[-a(\sqrt{\gamma}-b)^2]$ and let $\alpha/a<\delta<1$. Integrating both sides of (6) with respect to the measure $d(\lambda^p)$ and using Fubini's theorem, we have

$$E[(X/\gamma)^p - Y^p : \delta X/Y \geqslant \gamma] \leqslant \varphi(\gamma)E[X^p]$$

for every $\gamma\geqslant 1$, since $\delta<1$. Now we integrate again both sides of this inequality with respect to the measure $\gamma^p\exp((\alpha/\delta)\gamma)d\gamma$ over the interval $[1,\infty)$, and apply Fubini's theorem. It then follows that

$$(9)\qquad E\Big[\int_1^{\delta X/Y}(X^p-\gamma^pY^p)\exp((\alpha/\delta)\gamma)\,d\gamma : X/Y\geqslant 1/\delta\Big] \leqslant C_0E[X^p],$$

where $C_0 = \int_1^\infty \varphi(\gamma)\gamma^p\exp((\alpha/\delta)\gamma)\,d\gamma$. The constant C_0, which depends on a, b, c, α, and p, is finite since $\alpha/\delta - a < 0$.

For every γ such that $1\leqslant\gamma<\delta X/Y$ we have $X^p-\gamma^pY^p \geqslant (1-\delta^p)X^p$. Then it follows from (9) that

$$E[X^p\{\exp(\alpha X/Y)-\exp(\alpha/\delta)\} : X/Y\geqslant 1/\delta] \leqslant \frac{\alpha C_0}{(1-\delta^p)\delta}E[X^p]$$

and hence that

$$(10)\qquad E[X^p\exp(\alpha X/Y) : X/Y\geqslant 1/\delta] \leqslant \frac{\alpha C_0}{(1-\delta^p)\delta}E[X^p] + e^{\alpha/\delta}E[X^p : X/Y\geqslant 1/\delta].$$

Thus (7) follows immediately from the inequality

$$E[X^p\exp(\alpha X/Y)] \leqslant E[X^p\exp(\alpha X/Y) : X/Y\geqslant 1/\delta] + e^{\alpha/\delta}E[X^p : X/Y<1/\delta].$$

The inequality (8) also follows from (10). In fact, we have

$$E[Y^p\exp(\alpha X/Y)] \leqslant \delta^pE[X^p\exp(\alpha X/Y) : X/Y\geqslant 1/\delta] + e^{\alpha/\delta}E[Y^p].$$

Hence we obtain (8) by (10) and the inequality $E[X^p] \leqslant K_pE[Y^p]$. ∎

The following lemma plays an essential role in deducing the distribution function inequalities for M_∞^{*2} and $\langle M\rangle_\infty$ of the form of (5).

LEMMA 2. (i) *If $\|\langle M\rangle_\infty\|_\infty<\infty$, then for every $\lambda>0$ we have*

$$(11)\qquad P(M_\infty^{*2}>\lambda) \leqslant 2\exp\Big(-\frac{\lambda}{2\|\langle M\rangle_\infty\|_\infty}\Big).$$

(ii) *On the other hand, if* $\|M^*_\infty\|_\infty < \infty$, *then for every* $a < \pi^2/8$ *and* $\lambda > 0$ *we have*

$$P(\langle M\rangle_\infty > \lambda) \leqslant \frac{e^a}{\cos(\sqrt{2a})}\exp\left(-\frac{a\lambda}{\|M^*_\infty\|^2_\infty}\right) \tag{12}$$

without assuming that $M_0 = 0$.

Proof. (11) is a classical result due to Stroock–Varadhan [4]. Their proof is as follows: for each real number θ, we denote by $(Z_t^{(\theta)})$ the "exponential martingale" $(\exp(\theta M_t - \frac{1}{2}\theta^2\langle M\rangle_t))_{t\geqslant 0}$. Since $(Z_t^{(\theta)})$ is a positive martingale such that $Z_0^{(\theta)} = 1$ a.s., Doob's inequality implies that

$$P(\sup_t M_t > \sqrt{\lambda}) \leqslant P(\sup_t Z_t^{(\theta)} > \exp(\theta\sqrt{\lambda} - \tfrac{1}{2}\theta^2\|\langle M\rangle_\infty\|_\infty))$$
$$\leqslant \exp(-\theta\sqrt{\lambda} + \tfrac{1}{2}\theta^2\|\langle M\rangle_\infty\|_\infty).$$

Noticing that we may replace M by $-M$ in the above, we obtain (11) by setting $\theta = \sqrt{\lambda}\|\langle M\rangle_\infty\|_\infty^{-1}$.

A similar argument is available also for a proof of (12). Now, for each real number θ, let $\tilde{Z}_t^{(\theta)} = \cos(\theta M_t)\exp(\frac{1}{2}\theta^2\langle M\rangle_t)$. Then the process $(\tilde{Z}_t^{(\theta)})$ is a martingale since it is the real part of a martingale $(\exp\{i\theta M_t - \frac{1}{2}\theta^2\langle M\rangle_t\})$. Although we do not assume that $M_0 = 0$ now, we have $E[\tilde{Z}_0^{(\theta)}] \leqslant \exp(\frac{1}{2}\theta^2\|M^*_\infty\|^2_\infty)$ as $\langle M\rangle_0 = M_0^2$. For given $a < \pi^2/8$, we set $\theta = \sqrt{2a}\|M^*_\infty\|_\infty^{-1}$; then it is clear that $\cos(\sqrt{2a}) \leqslant \cos(\theta M_\infty)$ as $\theta\|M^*_\infty\|_\infty < \pi/2$. Hence

$$\cos(\sqrt{2a})E[\exp(a\langle M\rangle_\infty/\|M^*_\infty\|^2_\infty)] \leqslant E[\tilde{Z}_\infty^{(\theta)}] \leqslant E[\tilde{Z}_0^{(\theta)}] \leqslant e^a.$$

Then (12) follows imediately from Chebyshev's inequality and the above one. ■

To utilize the inequalities (11) and (12), we need to put them in conditional form. Let S and T be two stopping times such that $P(S < T) > 0$. We set $\tilde{\Omega} = \{S < T\}$, $\tilde{\mathfrak{F}}_t = \mathfrak{F}_{S+t}|_{\tilde{\Omega}}$, and $\tilde{P}(d\omega) = P(d\omega\,|\,S < T)$. For every martingale M on Ω, the process $\tilde{M}_t = M_{(S+t)\wedge T} - M_S$ is an $(\tilde{\mathfrak{F}}_t, \tilde{P})$-martingale on $\tilde{\Omega}$ satisfying $\langle\tilde{M}\rangle_\infty \leqslant \langle M\rangle_T$ and $\tilde{M}^*_\infty \geqslant M^*_T - M^*_S$ $\tilde{P}$-a.s. Then we can apply (11) to $\tilde{M}$ in order to get

$$P((M^*_T - M^*_S)^2 > \lambda,\ S < T) \leqslant 2\exp\left(-\frac{\lambda}{2\|\langle M\rangle_T\|_\infty}\right)P(S < T). \tag{11'}$$

In a similar way, we have the conditional form of (12):

$$P(\langle M\rangle_T - \langle M\rangle_S > \lambda,\ S < T) \leqslant C_a\exp(-a\lambda/\|M^*_T\|^2_\infty)P(S < T), \tag{12'}$$

where $a < \pi^2/8$ and $C_a = e^a/\cos(\sqrt{2a})$. Now we must set $\tilde{M}_t = M_{(S+t)\wedge T}$ so that the relation $\tilde{M}^*_\infty \leqslant M^*_T$ holds. For this reason, we have proved (12) without the asumption $\tilde{M}_0 = 0$. We should note that, in this case, $\langle\tilde{M}\rangle_t$ is not equal to $\langle M\rangle_{(S+t)\wedge T}$ but $\langle M\rangle_{(S+t)\wedge T} - \langle M\rangle_S + M_S^2$. We now prove the theorem.

3. Proof of Theorem. The last statement of the theorem is obvious by the Burkholder–Davis–Gundy inequality.

For each fixed $\lambda > 0$ we define two stopping times S and T by $\inf\{t\colon M_t^{*2} > \lambda\}$ and $\inf\{t\colon \langle M\rangle_t > \lambda\}$ respectively. It is then obvious that $M_S^{*2} \leqslant \lambda$ and $\langle M\rangle_T \leqslant \lambda$ a.s., and hence for fixed $\gamma > 1$ we have

$$P(M_\infty^{*2} > \gamma\lambda,\ \langle M\rangle_\infty \leqslant \lambda) \leqslant P((M^*_\infty - M^*_S)^2 > (\sqrt{\gamma}-1)^2\lambda,\ S < \infty,\ T = \infty)$$
$$\leqslant P((M^*_\infty - M^*_S)^2 > (\sqrt{\gamma}-1)^2\lambda,\ S < T)$$
$$\leqslant 2\exp\left(-\frac{(\sqrt{\gamma}-1)^2\lambda}{2\|\langle M\rangle_T\|_\infty}\right)P(S < T)$$
$$\leqslant 2\exp(-\tfrac{1}{2}(\sqrt{\gamma}-1)^2)P(M_\infty^{*2} > \lambda).$$

We have used (11′) to get the third inequality in the above. Similarly, using (12′), we get

$$P(\langle M\rangle_\infty > \gamma\lambda,\ M_\infty^{*2} \leqslant \lambda) \leqslant \frac{e^{2a}}{\cos(\sqrt{2a})}\exp(-a\gamma)P(\langle M\rangle_\infty > \lambda)$$

for every $\lambda > 0$ and $\gamma > 1$ if $a < \pi^2/8$. Then the theorem immediately follows from Lemma 1. ■

References

[1] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probab. 1 (1973), 19–42.

[2] R. Gundy, *The density of the area integral*, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. 1, Wadsworth, Belmont, Calif., 1982, 138–149.

[3] N. Kazamaki and M. Kikuchi, *Some remarks on ratio inequalities for continuous martingales*, Studia Math. 94 (1989), 97–102.

[4] D. W. Stroock and S. R. S. Varadhan, *Diffusion processes with continuous coefficients, I*, Comm. Pure Appl. Math. 22 (1969), 345–400.

[5] M. Yor, *Application de la relation de domination à certains renforcements des inégalités de martingales*, in: Séminaire de Probabilités XVI, Lecture Notes in Math. 920, Springer, 1982, 221–233.

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