STUDIA MATHEMATICA 99 (2) (1991)

On the density of log concave seminorms on vector spaces

by

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Abstract. Let E be a complete separable metric vector space and let q be a measurable seminorm on E. Suppose further that $S = \sum X_i$ is a series a.s. convergent with respect to q with independent components which are log concave and finite-dimensional. We prove that $F(t) = P\{q(S) \leqslant t\}$ is positive and continuously differentiable for all $t > 0$.

Let μ be a symmetric Gaussian measure on a separable Banach space $(E, \|\ \|)$ and let $F(t) = \mu\{x \in E;\ \|x\| \leqslant t\}$. It is well known [5], [8] that $F(t) > 0$ for all $t > 0$ and that the function F is absolutely continuous on $(0, \infty)$. M. Talagrand [7] claimed that $F'(t)$ is continuous on $(0, \infty)$. Unfortunately, the statement on p. 7, l. 15 of [7] is erroneous and it does not seem possible to rectify that argument.

The aim of this note is to prove that this claim is true, even in a greater generality, namely for log concave measures.

We begin with introducing some notations and terminology. By E we denote a complete separable metric vector space endowed with its Borel σ-algebra $\mathscr{B}_E$. A probability measure μ on $(E, \mathscr{B}_E)$ is called *log concave* [2], [6] if for any Borel subsets A, B and all λ, $0 < \lambda < 1$,

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^{\lambda} \mu(B)^{1-\lambda}.$$

The log concave measures are closed with respect to convolution and weak convergence [2]. It is also well known that when E is locally convex then all Gaussian measures are log concave [2]. When $E = \mathbf{R}^n$ then a full probability measure μ (i.e. one with $\operatorname{supp}\mu = \mathbf{R}^n$) is log concave if and only if it is absolutely continuous with respect to the Lebesgue measure and its density g is of the form

$$g(\mathbf{x}) = C\exp(-Q(\mathbf{x}))$$

where $Q \geqslant 0$ is a finite convex function such that $\lim_{|\mathbf{x}| \to \infty} Q(\mathbf{x}) = \infty$ and $C > 0$ is a suitable constant [1], [6].

1980 *Mathematics Subject Classification*: Primary 60G15; Secondary 60B11.

Now, let q be a measurable seminorm on E, that is, a Borel measurable function $q\colon E\to \mathbf{R}^+\cup\{\infty\}$ which is subadditive and positively homogeneous.

Next, suppose that (X_i) is a sequence of independent E-valued finite-dimensional random vectors such that $S=\sum X_i$ converges in E a.s. We will assume throughout the paper that S converges also with respect to q. When the X_i are additionally symmetric, this is equivalent to a.s. q-separability of E with respect to the distribution of S [3].

Now, set $S_n=\sum_{i=1}^n X_i$, $R_n=\sum_{i=n+1}^\infty X_i$ and let μ, μ_n, ν_n be the distributions of S, S_n, R_n, respectively. For arbitrary u, $t>0$ we set $F(t)=\mu\{q\leqslant t\}$ and

$$\begin{aligned}F^n_{q\leqslant u}(t)&=\mu\{(x,y)\in E_n\times E^n;\ q(x+y)\leqslant t,\ q(y)\leqslant u\}\\&=\int_{\{q\leqslant u\}}\mu_n\{x\in E_n;\ q(x+y)\leqslant t\}\,\nu_n(dy)=\int_{\{q\leqslant u\}}F^n_y(t)\,\nu_n(dy)\end{aligned}$$

where E_n, E^n denote the linear span of $\operatorname{supp}\mu_n$, $\operatorname{supp}\nu_n$, respectively. Analogously,

$$\begin{aligned}F^n_{q>u}(t)&=\mu\{(x,y)\in E_n\times E^n;\ q(x+y)\leqslant t,\ q(y)>u\}\\&=\int_{\{q>u\}}F^n_y(t)\,\nu_n(dy).\end{aligned}$$

Before formulating and proving our theorem we make some comments explaining the idea of the proof. As usual, we try to reduce our problem to a finite-dimensional situation, which can be handled by applying polar coordinates. To do this we write $F(t)$ in the form

$$F(t)=F^n_{q\leqslant u}(t)+F^n_{q>u}(t),$$

where $t>0$ is fixed and $0<u<t$. Then the jumps of $F'(t)$ can be estimated by the sum of the corresponding jumps of $dF^n_{q\leqslant u}(t)/dt$ and the function $dF^n_{q>u}(t)/dt$. Since $F^n_{q\leqslant u}(t)$ is expressed as an integral of the distribution function $F^n_y(t)$ of a finite-dimensional random vector S_n, shifted by $y\in E^n$, and integrated over $\{q\leqslant u\}$, the jumps of its derivative can be estimated using polar coordinates. More precisely, we have the following result, proved in [4]:

LEMMA. *Let (X_i) be a sequence of independent finite-dimensional random vectors with values in E such that $\sum_{i=1}^\infty X_i$ converges with respect to q. Assume that the μ_n are full and that they have bounded and continuous densities. Then for every $t>0$ and every $0<u<t$ the function $F^n_{q\leqslant u}(t)$ is positive and absolutely continuous for large n and has left and right derivatives with respect to t satisfying*

$$0\leqslant\frac{dF^n_{q\leqslant u}(t)^-}{dt}-\frac{dF^n_{q\leqslant u}(t)^+}{dt}\leqslant\frac{2u}{t-u}\,\frac{dF^n_{q\leqslant u}(t)^+}{dt}.\tag{1}$$

Remark. The above result was proved in [4] under slightly different assumptions. What remains to be shown here is that $F^n_{q\leqslant u}$ is positive for large n. This, however, follows at once by the obvious inequality

$$F^n_{q\leqslant s}(t)\geqslant\mu_n\{q\leqslant t-u\}\,\nu_n\{q\leqslant u\},$$

if we recall that $q(R_n)\to 0$ a.s.

Now, the next and most essential problem is to estimate the derivative of $F^n_{q>u}(t)$. Recall that $q(R_n)\to 0$ a.s., which implies that $F^n_{q>u}(t)\to 0$ as $n\to\infty$. The difficult part of the proof is to show that this is also true for the derivative of $F^n_{q>u}(t)$. More precisely, we only have to consider the left derivative. Oddly enough, the corresponding statement for the right derivative is much simpler.

We are now able to state and prove our theorem.

THEOREM. *Let (X_i) be a sequence of independent finite-dimensional random vectors with values in E such that $S=\sum_{i=1}^\infty X_i$ converges a.s. with respect to a measurable seminorm q. Assume that the distributions of X_i are log concave and full. Then the distribution function of $q(S)$ is differentiable at every point $t>0$ and its derivative is continuous on $(0,\infty)$.*

Proof. We divide the proof into several steps.

1. For all n and for every $t>0$ and $u>0$ we have

$$F(t)=F^n_{q\leqslant u}(t)+F^n_{q>u}(t).\tag{2}$$

Assume that $0<u<t$. Then, by the remark made after our lemma and the log concavity of $F(t)$ and $F^n_{q\leqslant u}(t)$ (as a function of t) we infer that for n large enough these functions have left and right derivatives at t. Because of (2) the function $F^n_{q>u}(t)$ has the same property. Using the log concavity of F once again we also have

$$\frac{dF(t)^+}{dt}\leqslant\frac{dF(t)^-}{dt}.$$

Therefore, using (1) and (2) we obtain for $0<u<t$

$$0\leqslant\frac{dF(t)^-}{dt}-\frac{dF(t)^+}{dt}\leqslant\frac{dF^n_{q>u}(t)^-}{dt}+\frac{2u}{t-u}\,\frac{dF^n_{q\leqslant u}(t)^+}{dt}.$$

Since obviously

$$\frac{dF^n_{q\leqslant u}(t)^+}{dt}\leqslant\frac{dF(t)^+}{dt},$$

and u is arbitrary such that $0<u<t$, to prove the theorem it is enough to show that for every such u

$$\lim_{n\to\infty}\frac{dF^n_{q>u}(t)^-}{dt}=0.\tag{3}$$

Note that the difficulty of analyzing $F^n_{q>u}(t)$ and its derivatives stems from the fact that it is no longer log concave.

2. Let u and t be fixed positive numbers such that $u < t$. First, observe that if for all n large enough $F^n_{q>u}(t) = 0$ then (3) is obviously fulfilled. Therefore, passing to subsequences if necessary, we may assume that $F^n_{q>u}(t) > 0$ for all n. Under this assumption we show that

$$\limsup_{n\to\infty} \frac{\partial F^n_{q>u}(t)^-}{\partial t} \leqslant \frac{u}{t} \limsup_{n\to\infty} \frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u}. \tag{4}$$

Note that the right-hand side of the above inequality involves $F^n_{q\leqslant u}(t)$, which is easier to deal with, being a log concave function of (u, t).

To show (4), let s be a fixed positive number such that $s < t$. Then for every positive h, $h < 1$, and arbitrary Borel subsets $A, B \subseteq E^n$ we have

$$(1-h)\{(x, y);\ q(x+y) \leqslant t,\ y\in A\} + h\{(x, y);\ q(x+y) \leqslant t-s,\ y\in B\} \subseteq \{(x, y);\ q(x+y) \leqslant t-hs,\ y\in (1-h)A+hB\}. \tag{5}$$

By log concavity of μ we get

$$\int\limits_{(1-h)A+hb} F^n_y(t-hs)\, v_n(dy) \geqslant \Big(\int\limits_A F^n_y(t)\, v_n(dy)\Big)^{1-h} \Big(\int\limits_B F^n_y(t-s)\, v_n(dy)\Big)^h. \tag{6}$$

Now, take positive u, w such that $u < w$ and take $A = \{y\in E^n;\ q(y) > u\}$ and $B = \{y\in E^n;\ q(y) \leqslant w-u\}$. Then we clearly have $\{y\in E^n;\ q(y) > u-hw\} \supseteq (1-h)A+hB$. The inequality (6) then yields

$$F^n_{q>u-hw}(t-hs) \geqslant (F^n_{q>u}(t))^{1-h} (F^n_{q\leqslant w-u}(t-s))^h.$$

Since we have assumed that $F^n_{q>u}(t) > 0$, the right-hand side of the above inequality becomes positive for large n and we can take logarithms of both sides to get

$$\frac{1}{h}[\ln F^n_{q>u-hw}(t-hs) - \ln F^n_{q>u}(t)] \geqslant \ln F^n_{q\leqslant w-u}(t-s) - \ln F^n_{q>u}(t). \tag{7}$$

Setting $\Psi(t, u) = \ln F^n_{q>u}(t)$, we can rewrite the left-hand side of (7) in the form

$$\frac{1}{h}[\Psi(t-hs, u-hw) - \Psi(t, u)].$$

It is plausible that the above expression converges, under appropriate assumptions on Ψ, to what we would call the left-hand directional derivative of Ψ in the direction of $(-s, -w)$. If this indeed were the case, the left-hand side of (7) would become

$$(-s)\frac{\partial \Psi(t, u)^-}{\partial t} + (-w)\frac{\partial \Psi(t, u)^-}{\partial u}. \tag{8}$$

Taking into account the form of Ψ and the fact that

$$\frac{\partial F^n_{q>u}(t, u)^-}{\partial u} = -\frac{\partial F^n_{q\leqslant u}(t, u)^-}{\partial u}$$

we would then obtain

$$w\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u} - s\frac{\partial F^n_{q>u}(t)^-}{\partial t} \geqslant F^n_{q>u}(t)[\ln F^n_{q\leqslant w-u}(t-s) - \ln F^n_{q>u}(t)]. \tag{9}$$

Observe now that the expression on the right-hand side of (9) is of the form $z[\ln x - \ln z]$ with $z = F^n_{q>u}(t) \to 0$, so it converges to 0, too, as $n\to\infty$. Thus, we would finally obtain

$$\limsup_{n\to\infty} \frac{\partial F^n_{q>u}(t)^-}{\partial t} \leqslant \frac{w}{s} \limsup_{n\to\infty} \frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u},$$

which clearly yields (4). It thus remains to show why the left-hand side of (7) can be replaced by (8) and this is done in the next step.

3. In this step we complete the proof of (4) by supplying technical details lacking in the last part of the previous step.

We first write the left side of (7) in the form

$$\frac{1}{h}[\Psi(t-hs, u-hw) - \Psi(t-hs, u)] + \frac{1}{h}[\Psi(t-hs, u) - \Psi(t, u)].$$

Observe that, as $h\to 0$, the second term converges to

$$(-s)\frac{\partial \Psi(t, u)^-}{\partial t} = (-s)\frac{\partial F^n_{q>u}(t)^-}{\partial t} (F^n_{q>u}(t))^{-1}.$$

Next, using the fact that $(\partial F^n_{q\leqslant v}(t)/\partial v)(F^n_{q\leqslant v}(t))^{-1}$ is nonincreasing as a function of v, and that $-\partial F^n_{q>v}(t)/\partial v = \partial F^n_{q\leqslant v}(t)/\partial v$ is nondecreasing as a function of t, we estimate the first term:

$$\begin{aligned}
\frac{1}{h}[\Psi(t-hs, u-hw) - \Psi(t-hs, u)] &= \frac{-1}{h}\int\limits_{u-hw}^{u} \frac{\partial \Psi(t-hs, v)}{\partial v}\, dv \\
&= \frac{1}{h}\int\limits_{u-hw}^{u} \frac{\partial F^n_{q\leqslant v}(t-hs)}{\partial v} (F^n_{q>v}(t-hs))^{-1}\, dv \\
&\leqslant w\frac{\partial F^n_{q\leqslant u-hw}(t-hs)^-}{\partial u} (F^n_{q\leqslant u-hw}(t-hs))^{-1} \frac{1}{h}\int\limits_{u-hw}^{u} \frac{F^n_{q\leqslant v}(t-hs)}{F^n_{q>v}(t-hs)}\, dv \\
&\leqslant w\frac{\partial F^n_{q\leqslant u-hw}(t)^-}{\partial u} (F^n_{q\leqslant u-hw}(t-hs))^{-1} \frac{1}{h}\int\limits_{u-hw}^{u} \frac{F^n_{q\leqslant v}(t)}{F^n_{q>v}(t-hs)}\, dv.
\end{aligned}$$

Now, as $h\to 0$, the last expression converges to

$$w\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u} (F^n_{q>u}(t))^{-1}.$$

This clearly justifies the inequality (9) and ends the proof of (4).

4. Next, we show that $F^n_{q\leqslant u}(t)$ is asymptotically flat in u for every t, that is,

$$\lim_{n\to\infty}\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u}=0. \tag{10}$$

To show this we repeat Step 2 with different sets A and B in (6):

$$A=\{y\in E^n;\ q(y)\leqslant u\},\qquad B=\{y\in E^n;\ q(y)\leqslant u-w\},$$

where $0<w<u$. Then for every positive h, $h<1$, we have

$$\{y\in E^n;\ q(y)\leqslant u-hw\}\supseteq(1-h)A+hB.$$

Hence, replacing in (6) $(t-s)$ by $(t+s)$ we now obtain

$$F^n_{q\leqslant u-hw}(t+hs)\geqslant\big(F^n_{q\leqslant u}(t)\big)^{1-h}\big(F^n_{q\leqslant u-w}(t+s)\big)^h.$$

This again yields

$$\frac{1}{h}[\ln F^n_{q\leqslant u-hw}(t+hs)-\ln F^n_{q\leqslant u}(t)]\geqslant\ln F^n_{q\leqslant u-w}(t+s)-\ln F^n_{q\leqslant u}(t). \tag{11}$$

Recall that for every positive t and u the function $F^n_{q\leqslant u}(t)$ is positive for large n, so the above inequality makes sense then. Putting $\Phi(t,u)=\ln F^n_{q\leqslant u}(t)$ we repeat the arguments of Step 2. As before we anticipate that the left-hand side of (11), after letting $h\to 0$, can be replaced by a directional derivative of Φ in the direction of $(s,-w)$. The proof that this is indeed the case is postponed until the last step. Thus, the inequality (11) will yield

$$s\frac{\partial F^n_{q\leqslant u}(t)^+}{\partial t}-w\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u}\geqslant F^n_{q\leqslant u}(t)[\ln F^n_{q\leqslant u-w}(t+s)-\ln F^n_{q\leqslant u}(t)].$$

Since $F^n_{q\leqslant u-w}(t+s)\to F(t+s)$ and $F^n_{q\leqslant u}(t)\to F(t)$ as $n\to\infty$, the expression on the right-hand side of the above inequality becomes positive for large n. Thus, we will obtain

$$\limsup_{n\to\infty}\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u}\leqslant\frac{s}{w}\limsup_{n\to\infty}\frac{\partial F^n_{q\leqslant u}(t)^+}{\partial u}\leqslant\frac{s}{w}\frac{dF(t)^+}{dt}.$$

Since $s>0$ is arbitrary, this will give (10) and complete the proof of the theorem.

5. In this step we show that the left-hand side of (11), after letting $h\to 0$, can be replaced by

$$s\frac{\partial F^n_{q\leqslant u}(t)^+}{\partial t}-w\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u}\big(F^n_{q\leqslant u}(t)\big)^{-1}.$$

This will complete the proof of (10) and end the proof of the whole theorem. Note that we deal here with a concave function Φ, which makes things a little easier than in Step 3.

First, we write the left-hand side of (11) in the form

$$\frac{1}{h}[\Phi(t+hs,\ u-hw)-\Phi(t,\ u-hw)]+\frac{1}{h}[\Phi(t,\ u-hw)-\Phi(t,\ u)]. \tag{12}$$

Now, it is clear that, as $h\to 0$, the second term above tends to

$$(-w)\frac{\partial\Phi(t,u)^-}{\partial u}=(-w)\frac{\partial F^n_{q\leqslant u}(t)^-}{\partial u}\big(F^n_{q\leqslant u}(t)\big)^{-1}.$$

Next, using concavity of Φ we estimate the first term of (12):

$$\frac{1}{h}[\Phi(t+hs,\ u-hw)-\Phi(t,\ u-hw)]=\frac{1}{h}\int_t^{t+hs}\frac{\partial\Phi(r,\ u-hw)}{\partial r}dr$$
$$\leqslant s\frac{\partial\Phi(t,\ u-hw)^+}{\partial t}=s\frac{\partial F^n_{q\leqslant u-hw}(t)^+}{\partial t}\big(F^n_{q\leqslant u-hw}(t)\big)^{-1}$$
$$\leqslant s\frac{\partial F^n_{q\leqslant u}(t)^+}{\partial t}\big(F^n_{q\leqslant u-hw}(t)\big)^{-1}.$$

This clearly completes the last step.

Remark. (i) If we only assume that the series $S=\sum_{i=1}^{\infty}X_i$ converges a.s. in E and $q(S)$ is a.s. bounded, then $q(R_n)\to c$ a.s., under the assumption that the X_i are symmetric [3]. If $c>0$ then $F(t)=0$ for $t<c$ and F' may have jumps. In this situation, the above proof yields the following estimate of the relative size of the jumps of F':

$$\frac{dF(t)^-}{dt}\leqslant\frac{t+c}{t-c}\frac{dF(t)^+}{dt}$$

for all $t>c$.

(ii) Note that the whole proof remains valid also for α-concave measures, for $-\infty<\alpha<0$ [1], [2]. These measures, however, are not closed under convolution. In this case instead of distributions of series $S=\sum_{i=1}^{\infty}X_i$ with independent components we may consider random vectors S having the following property: for all n, $S=S_n+R_n$, with S_n finite-dimensional and full, S, S_n, R_n α-concave, and $q(R_n)\to 0$ in probability as $n\to\infty$. This property, however, still seems to be too restrictive.

At the end we mention one of the central problems in this area (posed by M. Talagrand).

Problem. Suppose that μ is a Gaussian measure on a separable Banach space $(E,\|\ \|)$. Is then the function $(0,\infty)\ni t\to F(t)$ real-analytic?

The author would like to express his gratitude to Dr. Michał Ryznar for stimulating conversations on the subject.

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Received March 28, 1990 (2667)
Revised version August 20, 1990

STUDIA MATHEMATICA 99 (2) (1991)

Improved ratio inequalities for martingales

by

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Abstract. We show that the martingale inequality

$$E[\langle M\rangle_\infty^p \exp(\alpha\langle M\rangle_\infty/M_\infty^{*2})] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^p] \quad (0<p<\infty)$$

is valid only for $0 \leqslant \alpha < \pi^2/8$. In a previous paper [3] we proved it for sufficiently small $\alpha > 0$. Our new result is a sharp estimate on α.

1. Introduction. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geqslant 0}, P)$ be a filtered probability space satisfying the usual conditions. Throughout this short note, we deal only with continuous (local) martingales adapted to the filtration $(\mathfrak{F}_t)_{t\geqslant 0}$, and such a (local) martingale is called "(local) martingale" simply. Moreover, unless otherwise precisely stated, we assume that (local) martingales vanish at $t = 0$.

As usual, for every martingale $M = (M_t)_{t\geqslant 0}$, we set $M_t^* = \sup_{s\leqslant t}|M_s|$ and denote by $\langle M\rangle$ its quadratic variation process. The following result, which is an improvement of results in Gundy [2] and Yor [5], have been established in [3]: for *sufficiently small* $\alpha > 0$ and every $p > 0$, we have

$$E[M_\infty^{*p}\exp(\alpha M_\infty^*/\langle M\rangle_\infty^{1/2})] \leqslant C_{\alpha,p}E[M_\infty^{*p}], \tag{1}$$

$$E[\langle M\rangle_\infty^{p/2}\exp(\alpha\langle M\rangle_\infty/M_\infty^{*2})] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^{p/2}], \tag{2}$$

where $C_{\alpha,p}$ denotes an absolute constant depending only on α and p. Note that it is not necessarily the same from line to line, and we shall use this notation also in what follows. We should be careful with the difference between the powers of ratios appearing in (1) and (2).

2. Statement of results. Our new estimates for the inequalities (1) and (2), which are themselves our main object, are the following.

THEOREM. (i) *If* $0 \leqslant \alpha < 1/2$, *the ratio inequalities*

$$E[M_\infty^{*p}\exp(\alpha M_\infty^{*2}/\langle M\rangle_\infty)] \leqslant C_{\alpha,p}E[M_\infty^{*p}] \quad (p>0), \tag{3}$$

$$E[\langle M\rangle_\infty^{p/2}\exp(\alpha M_\infty^{*2}/\langle M\rangle_\infty)] \leqslant C_{\alpha,p}E[\langle M\rangle_\infty^{p/2}] \quad (p>0) \tag{3'}$$

1985 *Mathematics Subject Classification*: Primary 60G44.