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Unbounded Toeplitz operators in the Bargmann–Segal space

by

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Abstract. Toeplitz operators T_φ in the Bargmann–Segal space with various symbols φ are investigated. In particular, the form of the adjoint of an analytic Toeplitz operator is computed, and sufficient conditions for the compactness of the resolvent $R(\lambda, T_\varphi)$ are found.

0. Introduction. In recent years a few works appeared devoted to bounded Toeplitz operators in the Bargmann–Segal space B of Gaussian square integrable entire functions in \mathbb{C}^n [4], [5], [7], [8]. Unbounded Toeplitz operators were studied mostly by Berezin [3]. But even earlier Bargmann in his well known paper [1] considered special unbounded Toeplitz operators: the creation and the annihilation operators. There is a natural equivalence between Toeplitz operators in B and pseudodifferential operators in $L^2(\mathbb{R}^n)$ found explicitly in [7].

Let $L^p(\mu)$ ($p \geq 1$) be the Banach space of all $f: \mathbb{C}^n \rightarrow \mathbb{C}$ such that $\int |f|^p d\mu < \infty$, where $d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z)$ is the Gaussian measure, and $dV(z)$ is the Lebesgue measure in \mathbb{C}^n . Denote by Q the orthogonal projection from $L^2(\mu)$ onto B . Given a measurable function φ on \mathbb{C}^n one defines the Toeplitz operator T_φ in B by

$$(0.1) \quad T_\varphi f := Q(\varphi f).$$

What is the domain of T_φ ? The above equality suggests that a “natural” domain $D(T_\varphi)$ of T_φ should be

$$D(T_\varphi) = \{f \in B: \varphi f \in L^2(\mu)\}.$$

However, for certain φ it may happen (as we shall see later) that φf does not belong to $L^2(\mu)$ but one can still define another “Toeplitz” operator \tilde{T}_φ which has this f in its domain. Namely, we have the following definition.

DEFINITION 0.1. Let φ be a measurable function on \mathbb{C}^n . The operator \tilde{T}_φ associated to φ has its domain given by

$$D(\tilde{T}_\varphi) = \{f \in B: \varphi f = h + r, h \in B \text{ and } \int r \bar{p} d\mu = 0, \forall p \in P\},$$

where P is the set of all polynomials, and we put

$$(0.2) \quad \tilde{T}_\varphi f := h$$

Note that T_φ is well defined. Indeed, suppose that $h_1 + r_1 = \varphi h = h_2 + r_2$, where $h_k \in B$ and $\int r_k \bar{p} d\mu = 0$ for all $p \in P$. Then $r_1 - r_2 = h_2 - h_1 \in B$ and is orthogonal to P . Since P is dense in B we have $r_1 - r_2 = 0 = h_2 - h_1$.

It turns out that we still need one more definition of "Toeplitz" operator corresponding to a given symbol φ .

DEFINITION 0.2. Let φ be a measurable function in \mathbb{C}^n . We define the operator Π_φ in B by

$$(0.3) \quad \Pi_\varphi f(z) := \varphi(a) f(a) e^{z\bar{a}} d\mu(a),$$

provided the integral exists and belongs to B as a function of z , where $z\bar{a} = z_1 \bar{a}_1 + \dots + z_n \bar{a}_n$. Then we put

$$D(\Pi_\varphi) = \{f \in B: \int \varphi(a) f(a) e^{z\bar{a}} d\mu(a) \in B\}.$$

As we shall see later the above definition appears naturally during our study of T_φ .

Let us describe briefly the content of this paper. The first section contains some general results on various relations between the above definitions of Toeplitz operators. In particular, it is shown that all the definitions coincide for entire symbols φ . The second section deals with subnormal Toeplitz operators given by entire φ . It is proved that $\Pi_\varphi^* = \Pi_\varphi$ under a certain assumption on φ . The third section is devoted to selfadjoint Toeplitz operators. Sufficient conditions are given in terms of φ implying the selfadjointness of T_φ . The last, fourth section concerns Toeplitz operators with compact resolvent. Two different conditions on φ are found which guarantee the compactness of the resolvent of T_φ .

We end this introduction by recalling that the canonical orthonormal basis in B we shall use in this work is formed by

$$f_k(z) := z^k / \sqrt{k!},$$

where $z^k = z_1^{k_1} \dots z_n^{k_n}$, $k! = k_1! \dots k_n!$. The reproducing kernel e_a for B is given by $e_a(z) := e^{z\bar{a}}$, $z, a \in \mathbb{C}^n$.

The following notation will be used in this paper. Let T be a linear operator in a Hilbert space H . $D(T)$, $\text{Ker } T$, $R(\lambda, T)$, T^* denote the domain, the null space, the resolvent and the adjoint of T , respectively. T is closed if $D(T)$ is complete with respect to the graph norm $\|x\|_g := (\|Tx\|^2 + \|x\|^2)^{1/2}$. We write $T_1 \subseteq T_2$ if $D(T_1) \subseteq D(T_2)$ and $T_1 x = T_2 x$, $x \in D(T_1)$. \bar{T} denotes the closure of T . All other notions or notations will be defined in the text.

1. General results. In this section we discuss some general properties of Toeplitz operators and relations between the above definitions. They are far from being complete. Surely much more properties remain to be discovered. We have collected here only a few simple ones. Let us start with the following

PROPOSITION 1.1. For any measurable φ we have $T_\varphi \subseteq \Pi_\varphi \subseteq \tilde{T}_\varphi$.

PROOF. The inclusion $T_\varphi \subseteq \Pi_\varphi$ is obvious by the definitions. Now suppose that $f \in D(\Pi_\varphi)$. Then

$$(1.1) \quad \varphi f = \Pi(\varphi f) + r,$$

where $\Pi(\varphi f)(z) := \int \varphi(a) f(a) \bar{e}_z(a) d\mu(a)$ and $r := \varphi f - \Pi(\varphi f)$. It is clear that $r \bar{e}_z \in L^1(\mu)$ and

$$\int r \bar{e}_z d\mu = \int \varphi f \bar{e}_z d\mu - (\Pi(\varphi f), e_z) = 0.$$

We claim that

$$\int r(a) \bar{a}^k d\mu(a) = 0, \quad k = 0, 1, \dots$$

In fact, we know that

$$(1.2) \quad \int r(a) e^{z\bar{a}} d\mu(a) \equiv 0, \quad z \in \mathbb{C}^n.$$

Using the Lebesgue dominated convergence theorem one can differentiate at zero under the integral sign in (1.2). Hence

$$0 = D^k [\int r \bar{e}_z d\mu] |_{z=0} = \int r(a) \bar{a}^k d\mu(a),$$

$k \in \mathbb{N}^n$. Moreover, by (1.1) we have $\Pi_\varphi f = \tilde{T}_\varphi f$ and the proof is complete.

REMARK. The above proof shows that $D(\Pi_\varphi)$ can be described as follows: $D(\Pi_\varphi) = \{f \in B: \varphi f = F + r, \text{ where } F \in B \text{ and } \int r \bar{e}_z d\mu = 0 \text{ for all } z \in \mathbb{C}^n\}$.

It turns out that for holomorphic φ all the above definitions coincide.

PROPOSITION 1.2. If φ is an entire function and $P \subset D(T_\varphi)$ then $T_\varphi = \tilde{T}_\varphi$.

PROOF. Since $T_\varphi \subseteq \tilde{T}_\varphi$ we have to show the opposite inclusion. Let $f \in D(\tilde{T}_\varphi)$. Then $\varphi f = g + r$, where $g \in B$ and $\int r \bar{p} d\mu = 0$ for any $p \in P$. But r is also an entire function and we can write $r = \sum_k a_k z^k$. Thus for $p = f_s$ we have

$$0 = \int r \bar{f}_s d\mu = \lim_{R \rightarrow \infty} \int_{|z| < R} r \bar{f}_s d\mu = a_s, \quad s \in \mathbb{N}^n,$$

which completes the proof.

COROLLARY 1.3. If φ is an entire function and $P \subset D(T_\varphi)$, then $T_\varphi = \Pi_\varphi = \tilde{T}_\varphi$.

PROOF. Obvious.

Later we shall give an example of φ for which $T_\varphi \subsetneq \tilde{T}_\varphi$.

The next result explains more or less natural appearance of the definitions of \tilde{T}_φ and Π_φ . First denote by E the span of $\{e_z: z \in \mathbb{C}^n\}$.

PROPOSITION 1.4. If $P \subset D(T_\varphi)$ (resp. $E \subset D(T_\varphi)$) and $T := T_\varphi|_P$ (resp. $T_1 = T_\varphi|_E$), then $T^* = \tilde{T}_\varphi$ (resp. $T_1^* = \Pi_\varphi$).



Proof. (i) $\tilde{T}_{\bar{\varphi}} \subset T^*$. Let $y \in D(\tilde{T}_{\bar{\varphi}})$. Then $\bar{\varphi}y = g + r$, $g \in B$ and $\int r \bar{p} d\mu = 0$ for all $p \in P$. We have

$$(T_p, y) = \int p \varphi \bar{y} d\mu = (p, \bar{\varphi}y) = (p, g) = (p, \tilde{T}_{\bar{\varphi}}y).$$

Thus $y \in D(T^*)$ and $T^*y = \tilde{T}_{\bar{\varphi}}y$.

(ii) $T^* \subset \tilde{T}_{\bar{\varphi}}$. For $q \in D(T^*)$ and arbitrary $p \in P$ we have $(\varphi p, q) = (p, \Phi)$, where $\Phi \in B$. It follows that $\int (\bar{\varphi}q - \Phi) \bar{p} d\mu = 0$. Since p is arbitrary q must be in $D(\tilde{T}_{\bar{\varphi}})$ and $T^*q = \Phi = \tilde{T}_{\bar{\varphi}}q$. Both (i) and (ii) give the desired equality. By a similar reasoning one proves that $T_{\bar{\varphi}}^* = \Pi_{\bar{\varphi}}$ and the proof is complete.

COROLLARY 1.5. If $P \subset D(T_{\varphi})$, then

- (a) $\tilde{T}_{\bar{\varphi}}$ is closed,
- (b) $\tilde{T}_{\bar{\varphi}}^* = \bar{T}$ (the closure),
- (c) $\overline{T_{\bar{\varphi}}|_P} = \tilde{T}_{\bar{\varphi}}^* \subseteq \tilde{T}_{\bar{\varphi}}$,
- (d) if $\bar{T} = T_{\varphi}$ and $T_{\bar{\varphi}}^* = T_{\varphi}$, then $T_{\varphi} = \Pi_{\bar{\varphi}}^* = \tilde{T}_{\bar{\varphi}}^*$.

Proof. (a) is obvious by Proposition 1.4.

(b) Obvious.

(c) Since $T \subset \tilde{T}_{\bar{\varphi}}$, by (b) we have $\tilde{T}_{\bar{\varphi}} = T^* \supset \tilde{T}_{\bar{\varphi}}^* = \overline{T_{\bar{\varphi}}|_P}$.

(d) $\bar{T} = T_{\varphi} = T_{\bar{\varphi}}^* \supset \Pi_{\bar{\varphi}}^* \supset \tilde{T}_{\bar{\varphi}}^* = \bar{T}$.

In the next section we shall see a similar relation between T_{φ}^* and $\Pi_{\bar{\varphi}}$.

Remarks. 1) If φ is an entire function, then $T_{\varphi} = \tilde{T}_{\bar{\varphi}}$ and so T_{φ} must be closed. On the other hand, if $D(T_{\varphi})$ is dense in B then T_{φ} is closable. Indeed, since $D(T_{\varphi}) \subset D(T_{\varphi}^*)$ this is obvious.

2) If $\psi \in L^2(0, 2\pi)$ and if we define the symbol $\tilde{\psi}(re^{i\theta}) := \psi(e^{i\theta})$, then applying the Lemma of [8] we see that $T_{\tilde{\psi}}$ is bounded in B (for $n = 1$). However, for a homogeneous symbol $\varphi > 0$, i.e. $\varphi(\lambda z) = \lambda^k \varphi(z)$, $k \in \mathbb{N}$, $k \neq 0$, $\lambda > 0$, one can check that T_{φ} is unbounded in B (because the Berezin transform $\bar{\varphi}$ of φ is unbounded, see [8]).

2. Subnormal Toeplitz operators. One of the main testing examples in the theory of bounded subnormal operators are analytic Toeplitz operators. It seems that similar role in the theory of unbounded subnormal operators should be played by T_{φ} with entire φ . Recall that a densely defined operator S in H is said to be *subnormal* if there exists a normal operator N in a larger Hilbert space $K \supset H$ such that $S \subset N$. It is obvious that Toeplitz operators with nonconstant entire symbols are unbounded and subnormal. By Corollary 1.3 we know that $T_{\varphi} = \Pi_{\bar{\varphi}} = \tilde{T}_{\bar{\varphi}}$ provided $P \subset D(T_{\varphi})$. It turns out that for polynomial symbols φ we have $\overline{T_{\varphi}|_P} = T_{\varphi}$.

PROPOSITION 2.1. If $\varphi \in P$ and $n = 1$, then $\overline{T_{\varphi}|_P} = T_{\varphi}$.

Proof (induction on $\deg \varphi$). Without any loss of generality we may assume that $\varphi(0) = 0$. If $\deg \varphi = 1$ then the assertion can be checked directly by the definition. Suppose that it holds for all polynomials of $\deg \leq N - 1$. Let $\varphi \in P$ and $\deg \varphi = N$. We have $\varphi(z) = zq(z)$, where $\deg q = N - 1$. Let $f \in D(T_{\varphi})$. Then $zf \in D(T_q)$. Choose a sequence of polynomials p_k such that $p_k \rightarrow zf$ and $T_q p_k \rightarrow T_q zf$. Since $p_k \rightarrow zf$ we have $p_k(0) \rightarrow 0$ and the sequence of polynomials $\tilde{p}_k = p_k - p_k(0)$ also satisfies

$$\tilde{p}_k = z r_k, \quad r_k \in P, \quad \tilde{p}_k \rightarrow zf, \quad T_q \tilde{p}_k \rightarrow T_q zf = \varphi f.$$

By the definition of T_z we have $\|zg\| \geq \delta \|g\|$, for some $\delta > 0$. Hence r_k must be convergent to f and $T_{\varphi} r_k = T_q z r_k = T_q \tilde{p}_k \rightarrow \varphi f$. The proof is complete.

If φ is a bounded holomorphic function in the unit disc, then the classical Toeplitz operator in the Hardy space is bounded and $T_{\varphi}^* = T_{\bar{\varphi}}$. Since there are no nontrivial bounded entire functions, the problem of computing T_{φ}^* is far more nontrivial in our case. Nevertheless, we have the following result.

THEOREM 2.2. Let φ be an entire function for which Π_{φ} is densely defined. Suppose that for any $h \in D(\Pi_{\varphi})$ there exists $\varepsilon > 0$ such that

$$(+)\quad \sum_s \frac{\|\varphi^{(s)} h\|^2}{s!} (1 + \varepsilon)^{|s|} < \infty,$$

where $\varphi^{(s)}(z) = D_{z_1}^{s_1} \dots D_{z_n}^{s_n} \varphi(z)$. Then $\Pi_{\varphi}^* = \Pi_{\bar{\varphi}}$.

Proof. Let $h \in D(\Pi_{\varphi})$ and $f \in D(\Pi_{\bar{\varphi}})$. We have

$$(\varphi h, f) = \int \varphi(a) h(a) \bar{f}(a) d\mu(a) = \iint \varphi(a) h(z) e^{z\bar{a}} \bar{f}(a) d\mu(z) d\mu(a).$$

If we knew that $\varphi(a) h(z) \bar{f}(a) e^{z\bar{a}}$ belongs to $L^1(\mu \otimes \mu)$, then we could change the order of integration (by Fubini's theorem) and write

$$(\varphi h, f) = \int h(z) \int \varphi(a) e^{z\bar{a}} \bar{f}(a) d\mu(a) d\mu(z) = (h, \Pi_{\bar{\varphi}} f).$$

Since $h \in D(\Pi_{\varphi})$ and $f \in D(\Pi_{\bar{\varphi}})$ are arbitrary, we get the desired equality.

What is left is to show that $\varphi(a) h(z) e^{z\bar{a}} \bar{f}(a) \in L^1(\mu \otimes \mu)$. Writing $a = z + w$ we have

$$\|\varphi(a) h(z) e_z(a) \bar{f}(a)\|_{L^1(\mu \otimes \mu)} = \iint |h(z) \varphi(z+w) \bar{f}(z+w) e^{-w\bar{z}}| d\mu(z) d\mu(w);$$

we used here the equality $d\mu(z+w)/d\mu(w) = e^{-2\operatorname{Re}z\bar{w} - |z|^2}$. The last integral is (by the Schwarz inequality) less than

$$\int \|h(\cdot) \varphi(\cdot + w)\| \cdot \|\bar{f}(\cdot + w) e^{-w(\cdot)}\| d\mu(w) =: I.$$

But $\int |\bar{f}(z+w) e^{-w\bar{z}}|^2 d\mu(z) = e^{|\bar{w}|^2} \|f\|^2$ (see [5] or [1]). Thus

$$I = \|f\| \int \|h(\cdot) \varphi(\cdot + w)\| e^{-|\bar{w}|^2/2} dV(w).$$

Note that by (+) we know that

$$h(z)\varphi(z+w) = h(z) \sum_s \frac{1}{s!} \varphi^{(s)}(z) w^s \in B,$$

for every $w \in \mathbb{C}^n$.

Applying the Schwarz inequality we have

$$|h(z)\varphi(z+w)|^2 \leq \sum_s \frac{(1+\varepsilon)^{|s|}}{s!} |\varphi^{(s)}(z)h(z)|^2 \sum_s \frac{|w|^{2s}}{s!(1+\varepsilon)^{|s|}}.$$

It follows that

$$\|h(\cdot)\varphi(\cdot+w)\|^2 \leq M e^{w^{2/(1+\varepsilon)}}, \quad M = \sum_s \frac{(1+\varepsilon)^{|s|}}{s!} \|\varphi^{(s)}h\|^2.$$

Hence $I < \infty$. The proof is complete.

EXAMPLE 2.3. Let $\varphi(z) = \sum_{k=1}^N p_k(z) e^{z\lambda_k}$, where $p_k \in P$ and $\lambda_k \in \mathbb{C}^n$. If $f \in D(T_\varphi)$, then (+) holds for f .

Indeed, for $|s| \geq \max_{1 \leq k \leq N} \deg p_k = K$ we have

$$\varphi^{(s)}(z) = \sum_{k=1}^N \sum_{t \leq s} \binom{s}{t} p_k^{(t)}(z) \lambda_k^{s-t} e^{z\lambda_k}.$$

Hence for any $\varepsilon > 0$

$$\begin{aligned} \sum_s \frac{\|\varphi^{(s)}f\|^2}{s!} (1+\varepsilon)^{|s|} &\leq \left(\max_{\substack{1 \leq k \leq N \\ |t| \leq K}} \|p_k^{(t)}(\cdot) e^{(\cdot)\lambda_k} f(\cdot)\| \right)^2 \\ &\times \sum_s \sum_{k=1}^N \left[\sum_{|t|=0}^K |\lambda_k|^{s-t} \binom{s}{t} \right]^2 \frac{(1+\varepsilon)^{|s|}}{s!} \\ &\leq C \sum_{k=1}^N \sum_s \frac{(|\lambda_{1k}|+1)^{2s_1}}{s_1!} \dots \frac{(|\lambda_{nk}|+1)^{2s_n}}{s_n!} (1+\varepsilon)^{|s|} < \infty, \end{aligned}$$

for some finite $C > 0$.

COROLLARY 2.4. For any polynomial p we have $\Pi_{\bar{p}} = \tilde{T}_p$.

Proof. Combining Propositions 1.4 and 2.1 we know that $(\overline{T_p|_P})^* = \tilde{T}_p$ and $\overline{T_p|_P} = T_p$. Applying Theorem 2.2 we get the desired equality.

COROLLARY 2.5. If an entire function φ satisfies the condition (+), then

- (i) $\overline{T_\varphi|_E} = T_\varphi$,
- (ii) $\Pi_{|\varphi|^2}$ is selfadjoint provided $P \subset D(T_\varphi)$.

Proof. (i) As we know $(T_\varphi|_E)^* = \Pi_{\bar{\varphi}} = T_\varphi^*$. Hence (taking adjoints) $\overline{T_\varphi|_E} = \tilde{T}_\varphi = T_\varphi$.

(ii) Since $\Pi_{\bar{\varphi}} = T_\varphi$, we have $T_\varphi^* T_\varphi = \Pi_{\bar{\varphi}} \Pi_{\bar{\varphi}} = \Pi_{|\varphi|^2}$, and the result follows from the general theory [9] because T_φ is closed.

COROLLARY 2.6. If φ satisfies the condition (+) and $W = \{f \in B: f/\varphi \text{ is entire}\}$, then $W \cap \text{Ker} \Pi_{\bar{\varphi}} = \{0\}$.

Proof. If $f \in W \cap \text{Ker} \Pi_{\bar{\varphi}}$ then $f = \varphi h$. Hence $\|\varphi h\|^2 = (h, T_\varphi^* f) = (h, \Pi_{\bar{\varphi}} f) = 0$.

3. Selfadjoint Toeplitz operators. The question whether a given Toeplitz operator T_φ is selfadjoint, for a real-valued function φ , is in general not an easy one. We are going to show that for special real symbols Toeplitz operators are indeed selfadjoint. We start with the following general result.

PROPOSITION 3.1. Let φ be a real-valued measurable function. Suppose that $D(T_\varphi)$ is dense in B . Then T_φ has equal deficiency indices provided that $\varphi(z) \equiv \varphi(\bar{z})$.

Proof. Let $C: L^2(\mu) \rightarrow L^2(\mu)$ be given by $Cf(z) = \overline{f(\bar{z})}$. By direct computation we check that $CT_\varphi = T_\varphi C$. Since T_φ is symmetric the result follows by a general theorem of J. von Neumann [9].

COROLLARY 3.2. If φ is real-valued and $D(T_\varphi)$ is dense in B then T_φ has a selfadjoint extension provided that $\varphi(z) \equiv \varphi(\bar{z})$.

It is well known that for $\varphi(z) = \text{Re} z$ ($n=1$), T_φ is selfadjoint [1]. Therefore one could ask whether for any polynomial p the operator $T_{\text{Re} p}$ is selfadjoint in B . However, this turns out to be true only for polynomials of degree not exceeding 2. As we shall see later $T_{\text{Re} z^3}$ is not selfadjoint in B .

PROPOSITION 3.3. If q is a polynomial, then $T_{\text{Re} q}$ is closed.

Proof. Direct computation.

THEOREM 3.4. Let p be a polynomial in \mathbb{C} of degree two. Then $T_{\text{Re} p}$ is a selfadjoint operator in B .

Proof. The idea of the proof is to apply the theorem of Nelson on analytic vectors of symmetric operators [9]. Namely, we claim that every f_n is an analytic vector for $T_{\text{Re} p}$.

Let $p(z) = a_1 z + a_2 z^2$. Surely we may and do assume that $p(0) = 0$. Set $|p| := \max(|a_1|, |a_2|)$ and $A := T_{\text{Re} p}$. We have

$$(3.1) \quad \|A^k f_n\| \leq 2^k |p|^k [(n+1) \dots (n+2k)]^{1/2}, \quad n = 0, 1, \dots; k = 1, 2, \dots$$

One can check (3.1) easily by induction on k . Hence the series $\sum_k \|A^k f_n\| t^k/k!$ is convergent for $0 < t < 1/(4|p|)$ and the proof is complete.

COROLLARY 3.5. Let $p(z_1, \dots, z_n) = p_1(z_1) + \dots + p_n(z_n)$ where $\deg p_k \leq 2$. Then $T_{\text{Re} p}$ is selfadjoint on $D(T_{\text{Re} p_1}) \otimes \dots \otimes D(T_{\text{Re} p_n})$.

Proof. Combining Th. VIII.33 from [9] and the last theorem we obtain the desired conclusion.

EXAMPLE 3.6. Now we shall prove that despite the above result the operator $T = T_{R_{\mathbb{R}z^3}}$ is not selfadjoint. Namely, we shall see that the equation

$$(3.2) \quad (T-i)u = 0$$

has a nontrivial solution in B .

Suppose that $u = \sum_k u_k f_k$. Assume that all the coefficients of u of the form u_{3k+1} , u_{3k+2} vanish for $k = 0, 1, \dots$. Then (3.2) is equivalent to the infinite system of linear equations

$$(3.3) \quad [(3k-2)(3k-1)3k]^{1/2} u_{3k-3} + [(3k+1)(3k+2)(3k+3)]^{1/2} u_{3k+3} = iu_{3k}, \quad k = 1, 2, \dots$$

Set $v_k = u_{3k}$ and $a_p = [(3p+1)(3p+2)(3p+3)]^{1/2}$. Then by (3.3) we have

$$(3.4) \quad a_{k-1} v_{k-1} + a_k v_{k+1} = i v_k, \quad k = 1, 2, \dots$$

However, now (3.4) is equivalent to the equation

$$(3.5) \quad (A-i)v = 0,$$

where A is the operator corresponding to the Jacobi type matrix

$$\begin{bmatrix} 0 & a_0 & 0 & \dots & 0 & \dots \\ a_0 & 0 & a_1 & \dots & 0 & \dots \\ 0 & a_1 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}$$

Since $\sum_k a_k^{-1} < \infty$ and $a_{k-1} a_{k+1} \leq a_k^2$, applying Th. 1.5, VII from [2] we conclude that (3.5) has a nontrivial solution in l^2 . This gives us the desired conclusion.

Remark. The last example also shows that $T_{R_{\mathbb{R}z^3}} \not\subseteq \tilde{T}_{R_{\mathbb{R}z^3}}$ and $T_{R_{\mathbb{R}z^3}} \not\subseteq H_{R_{\mathbb{R}z^3}}$. This is clear by Proposition 1.4.

We end this section with the next result concerning selfadjoint T_φ . It is related to Hankel operators. Recall that for a measurable function φ the Hankel operator $H_\varphi: B \rightarrow L^2(\mu)$ is given by $H_\varphi u = (I-Q)M_\varphi u$, $u \in B$, where $M_\varphi u = \varphi u$.

PROPOSITION 3.7. Let φ be a real-valued function such that H_φ is a bounded operator. Then T_φ is selfadjoint in B .

Proof. The decomposition $L^2(\mu) = B \oplus B^\perp$ yields the decomposition of M_φ as the operator matrix

$$\begin{bmatrix} T_\varphi & H_\varphi^* \\ H_\varphi & (I-Q)M_\varphi(I-Q) \end{bmatrix}$$

Since φ is real M_φ is selfadjoint. Hence

$$\begin{bmatrix} T_\varphi & 0 \\ 0 & (I-Q)M_\varphi(I-Q) \end{bmatrix} = M_\varphi \begin{bmatrix} 0 & H_\varphi^* \\ H_\varphi & 0 \end{bmatrix}$$

as the difference of a selfadjoint operator and a bounded selfadjoint one must be selfadjoint. The proof is complete.

Remark. In our previous work [8] we found a sufficient condition on φ which guaranteed the boundedness of H_φ . As noted by J. Peetre (in a letter to the author) this condition is equivalent to $|\varphi(z) - \varphi(w)| \leq C(1 + |z - w|)$.

4. Spectral properties of some Toeplitz operators. In this section, we discuss some spectral properties of T_φ for special symbols φ . In case φ is a bounded continuous function in \mathbb{C}^n such that

$$\lim_{R \rightarrow \infty} \sup_{\substack{|z-w| \leq 1 \\ |z| \geq R}} |\varphi(z) - \varphi(w)| = 0,$$

it was shown in [5] that the essential spectrum of T_φ is

$$\sigma_e(T_\varphi) = \bigcap_{R > 0} \text{closure} \{ \varphi(z) : |z| \geq R \}.$$

We do not have so precise a result for unbounded T_φ but nevertheless the essential spectrum of T_φ is contained in the above intersection, under certain assumptions on φ .

First let us recall what we mean by the essential spectrum $\sigma_e(T)$ of a closed densely defined operator T in a Hilbert space H .

DEFINITION 4.1.

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : \text{there exists a sequence } u_n \in D(T) \text{ with } \|u_n\| = 1, u_n \rightarrow 0 \text{ and } (T - \lambda I)u_n \rightarrow 0 \text{ in } H \}$$

(see [6]).

Before we proceed further let us recall the definition of the Berezin transform $\tilde{\varphi}$ of φ [3]:

$$\tilde{\varphi}(a) = \pi^{-n} \int \varphi(z) e^{-|z-a|^2} dV(z).$$

Let $\Gamma = \{ \varphi \text{ measurable} : H_\varphi \text{ is compact} \}$.

PROPOSITION 4.2. If $\varphi \in \Gamma \cap L^1_{\text{loc}}$, then

$$\sigma_e(T_\varphi) \subseteq \bigcap_{R > 0} \text{closure} \{ \varphi(z) : |z| \geq R \}.$$

Proof. First note that T_φ is closed. This is obvious by the equality $M_\varphi f = T_\varphi f + H_\varphi f$, $f \in B$. If $\lambda \notin \bigcap_{R > 0} \text{closure} \{ \dots \}$ then there exist $R > 0$ such that

$$|\lambda - \varphi(z)| \geq \varepsilon \quad \text{for } |z| \geq R.$$

Let

$$\psi_R(z) := \begin{cases} |\varphi - \lambda| + \varepsilon, & |z| \leq R, \\ 0, & |z| > R. \end{cases}$$

Put $\Phi_R(z) := \varphi(z) - \lambda + \psi_R(z)$. Then $|\Phi_R(z)| \geq \varepsilon$ for all $z \in \mathbb{C}^n$. Since $\varphi \in L^1_{loc}$ it follows that $\psi_R(a) \rightarrow 0$ as $|a| \rightarrow \infty$. Hence T_{ψ_R} is compact (see [5] or [8]). Thus

$$\sigma_e(T_{\varphi-\lambda}) = \sigma_e(T_{\Phi_R}).$$

Now we have

$$T_{\Phi_R}^{-1} T_{\Phi_R} - I = Q M_{\Phi_R}^{-1} (Q - I) M_{\Phi_R} Q,$$

where I stands for the identity operator in B or $L^2(\mu)$ respectively.

It is easy to see that $\Phi_R \in \Gamma$ and the operator on the right hand side of the last equality is compact. Therefore $0 \notin \sigma_e(T_{\Phi_R})$ and this completes the proof.

The next result concerns the point spectrum of T_φ for a certain class of symbols. Namely, suppose that φ can be written as

$$(4.1) \quad \varphi(z) = \sum_{|k| \leq M} a_k(|z|) z^k = \sum_{|k| \leq M} A_k(z).$$

Assume that

$$(*) \quad P \subset \bigcap_{|k| \leq M} D(T_{A_k}) = D(T_\varphi).$$

Let $f = \sum_i b_i f_i \in D(T_\varphi)$. Define the sequence $F_m = \sum_{|i| \leq m} b_i f_i$. By (*) and direct computation one can check that $T_\varphi F_m$ is weakly convergent to $T_\varphi f$.

PROPOSITION 4.3. *Let φ be given by (4.1). If φ satisfies the assumption (*) then*

$$\sigma_p(T_\varphi) \subseteq \bigcup_s \{A_{0ss}\},$$

where

$$A_{klp} = (p!)^{-1/2} \int a_k(r_1, \dots, r_n) r_1^{k_1+l_1+p_1} \dots r_n^{k_n+l_n+p_n} d\mu(r_1, \dots, r_n),$$

$$d\mu(r_1, \dots, r_n) = \pi^{-n} e^{-|r|^2} r_1 \dots r_n dr_1 \dots dr_n, \quad k, l, p \in \mathbb{N}^n.$$

Proof (for $n = 2$ but the same method works in general). If $T_\varphi f = \lambda f$ then $(T_\varphi f, e_z) = \lambda f(z)$, $z \in \mathbb{C}^2$. Let $f = \sum_i b_i f_i$. Using the polar coordinates an easy computation shows that

$$\sum_p \left(\sum_{k+l=p} b_i A_{klp} \right) f_p(z) = \lambda \sum_p b_p f_p(z).$$

If $b_{00} \neq 0$ it follows that $\lambda = A_{000}$. If $b_{00} = 0$ but $b_{10} \neq 0$ then $\lambda = A_{0(1,0)(1,0)}$. In case $b_{00} = b_{10} = 0$ but $b_{01} \neq 0$ we have $\lambda = A_{0(0,1)(0,1)}$. Repeating this reasoning we get the desired inclusion.

Remark. The above inclusion gives no information on whether $\sigma_p(T_\varphi)$ is not empty.

In the theory of partial differential operators the ones with compact resolvent play an essential role [10]. Now we shall prove two results concerning Toeplitz operators in B with that property. In fact, the second result deals with a general closed operator A in B .

THEOREM 4.4. *Let $\varphi \geq c > -\infty$ be a real function such that $\bar{T}_\varphi^* = \bar{T}_\varphi$. Suppose that $\lim_{|z| \rightarrow \infty} \varphi(z) = \infty$. If $e_z \in D(T_\varphi)$ for all $z \in \mathbb{C}^n$, then the resolvent $R(\lambda, T_\varphi)$ is compact.*

Proof. We may and do assume that $\varphi \geq 0$. By Th. XIII.64 from [10] it is enough to show that $\mu_N(\bar{T}_\varphi) \rightarrow \infty$ as $N \rightarrow \infty$. Here

$$\mu_N(\bar{T}_\varphi) = \sup_{y_1, \dots, y_{N-1}} \inf_{\substack{f \in D(\bar{T}_\varphi) \\ \|f\| = 1, f \perp [y_1, \dots, y_{N-1}]^\perp}} (\bar{T}_\varphi f, f).$$

Let $b > 0$ be an arbitrary number. Choose R so large that $|\varphi(z)| \geq b$ for $|z| \geq R$. Fix $1 > \varepsilon > 0$. Note that the set

$$Z = \{f \text{ entire: } \|f\|_{K(0,R)} := \sup_{|z| \leq R} |f(z)| \leq e^{R^2/2}\}$$

is compact in the space $\text{Hol}(\mathbb{C}^n)$ of all entire functions in \mathbb{C}^n . Hence it is also compact in the space $C(K(0, R))$ of all continuous functions on the ball $K(0, R)$. It follows that Z is an equicontinuous family of functions in $C(K(0, R))$. Therefore there exists $\delta > 0$ such that for the above ε we have

$$|x - y| \leq \delta \text{ implies that } |f(x) - f(y)| < \varepsilon$$

for all $f \in Z$.

Now for this δ we find a set of points $z_1, \dots, z_{N-1} \in K(0, R)$ such that every point of $K(0, R)$ is within δ of some z_i . Put $y_s(a) = e_{z_s}(a)$, $s = 1, \dots, N-1$, in the definition of $\mu_N(\bar{T}_\varphi)$. Note that $f \in D(\bar{T}_\varphi)$, $\|f\| = 1$, $f \perp [y_1, \dots, y_{N-1}]^\perp$ implies that $f \in Z$ and $\|f\|_{K(0,R)} \leq \varepsilon$. Hence (everywhere below the infimum is taken over the same set)

$$\begin{aligned} \mu_N(\bar{T}_\varphi) &\geq \inf_{\substack{\|f\|=1, f \in D(\bar{T}_\varphi) \\ f(z_s)=0, s=1, \dots, N-1}} (T_\varphi f, f) = \inf_f \left[\int_{K(0,R)} \varphi |f|^2 d\mu + \int_{\mathbb{C}^n \setminus K(0,R)} \varphi |f|^2 d\mu \right] \\ &\geq \inf_f \int_{\mathbb{C}^n \setminus K(0,R)} \varphi |f|^2 d\mu \geq b \inf_f \left(1 - \int_{K(0,R)} |f|^2 d\mu \right) \geq b(1 - \varepsilon^2). \end{aligned}$$

Since b is arbitrary the proof is complete.

The above result combined with the results of the previous section gives us some examples of Toeplitz operators with compact resolvent. However, the assumption $\bar{T}_\varphi^* = \bar{T}_\varphi$ puts a severe restriction on φ . That it is not necessary is the content of our last result of this paper.

LEMMA 4.5. Let A be a closed operator in B which satisfies the inequality

$$(E) \quad \operatorname{Re}(Af, f) \geq c \|\operatorname{grad} f\|^2, \quad f \in D(A), \quad c > 0,$$

where $\|\operatorname{grad} f\|^2 = \sum_{i=1}^n \int |\partial f / \partial z_i|^2 d\mu$. Then $R(\lambda, A)$ is compact.

Proof. For the sake of simplicity we shall give the proof only for $n = 1$. However, the same method works for $n > 1$. It suffices to show that the canonical injection of $(D(A), \|\cdot\|_g)$ considered with the graph norm into B is compact. Let $S = \{h \in D(A): \|h\|_g \leq 1\}$. We have to show that S is compact in B . Take a sequence $\{f_k\} \subset S$. Applying (E) we deduce that

$$(i) \quad \|df_k/dz\|^2 \leq 1/(2c), \quad k = 1, 2, \dots$$

Without losing any generality we may assume that $f_k \rightarrow f$ (weakly) in B (passing to a convergent subsequence if necessary). Write $f_k(z) = \sum_s a_{ks} z^s$, $f(z) = \sum_s a_s z^s$. Then $a_{ks} \rightarrow a_s$ for all $s = 0, 1, \dots$. Now (i) implies that

$$1/(2c) \geq \|df_k/dz\|^2 = \sum_s |a_{sk}|^2 s^2 (s-1)!, \quad k = 1, 2, \dots$$

Hence

$$\begin{aligned} \lim_k [\|f_k\|^2 - \|f\|^2] &\leq \lim_k \sum_{s=0}^N [|a_{sk}|^2 - |a_s|^2] s! \\ &\quad + \lim_k \sum_{s=N+1}^{\infty} \frac{|a_{sk}|^2}{s} s^2 (s-1)! + \sum_{s=N+1}^{\infty} |a_s|^2 s! \\ &\leq \frac{1}{2c(N+1)} + \sum_{s=N+1}^{\infty} |a_s|^2 s! \rightarrow 0 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

Thus $\lim_k \|f_k\| = \|f\|$ and we conclude that $f_k \rightarrow f$. The proof is complete.

Note (November 1990). After this work had been accepted for publication Professor Harold Shapiro informed us about his two joint papers with D. J. Newman:

I. *Certain Hilbert spaces of entire functions*, Bull. Amer. Math. Soc. 72 (1966), 971–977.

II. *Fischer spaces of entire functions*, in: Entire Functions and Related Parts of Analysis, J. Korevar (ed.), Proc. Sympos. Pure Math. 11, Amer. Math. Soc., Providence 1968.

The second work (II) contains a result (see point 5, p. 367) which is exactly our Example after Theorem 2.2.

We thank him for making these papers available to us.

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