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A method of solving a cocycle functional equation and applications

by

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Abstract. The problem of isomorphism in a class of finite extensions of adding machines is studied. This leads to a functional equation of the form

(*) $$f\circ T-f=\psi\circ S-\varphi$$

where $\varphi,\psi,f\colon X\to H$ are measurable, $T\colon (X,\mathscr{B},m)\circlearrowleft$ is an ergodic adding machine and S commutes with T. A new method of solving (*) is provided. Some applications are given.

Introduction. Let $T\colon (X,\mathscr{B},m)\circlearrowleft$ be an ergodic automorphism of a Lebesgue space. Given a compact metric abelian group H and a measurable map $\varphi\colon X\to H$ (called from now on a *cocycle*) we define $T_\varphi\colon (X\times H, m\times m_H)\circlearrowleft$ (m_H is the Haar measure of H), the *H-extension* of T, by

$$T_\varphi(x,h)=\big(Tx,\ \varphi(x)+h\big).$$

Two cocycles $\varphi,\varphi'\colon X\to H$ are said to be *equivalent* whenever there exist a measurable $f\colon X\to H$, an $S\in C(T)$ ($C(T)$ being the centralizer of T, i.e. the semigroup of all $S\colon (X,\mathscr{B},m)\circlearrowleft$ commuting with T) and a continuous isomorphism $v\colon H\to H$ such that

(A) $$f(Tx)-f(x)=\varphi(Sx)-v\big(\varphi'(x)\big),\qquad m\text{-a.e. } x\in X.$$

Assuming that $T\colon (X,\mathscr{B},m)\circlearrowleft$ has discrete spectrum, (A) has a measurable solution f iff T_φ and $T_{\varphi'}$ are metrically isomorphic [6]. Many other problems concerning group extensions are equivalent to the existence of a solution of a functional equation of the form

(B) $$f\circ T-f=\varphi$$

with given φ (for instance ergodicity of T_φ [7], the centralizer of T_φ [3], [2], weak isomorphism problem [4]). In this paper we take up the study of (B) when $T\colon (X,\mathscr{B},m)\circlearrowleft$ is an adding machine (i.e. up to isomorphism T has rational pure point spectrum with the group of eigenvalues $\mathrm{Sp}(T)=G\{n_t\colon t\geqslant 0\}$, $n_t|n_{t+1}$, $t\geqslant 0$, generated by the n_t-roots of unity), H is a finite abelian group and $\varphi\colon X\to H$ has only a finite number of points of discontinuity (a special Toeplitz cocycle).

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The paper consists of two sections. The first is devoted to the presentation of a method giving a necessary and sufficient condition for (B) to have a measurable solution. In §2 we present some applications. The method of solving (B) is based on the existence in X of a sequence of T-towers ξ_t, $\xi_t = (D_0^t, \ldots, D_{n_t-1}^t)$, $t \geq 0$, such that $\xi_t \nearrow \varepsilon$ (ε is the partition of X into points). If $f\colon X \to H$ is a measurable solution of (B), then there exist ξ_t-measurable functions f_t, $t \geq 0$, such that the sequence $\{f_t\}$ satisfies the Cauchy condition in measure. The main result of §1 (Theorem 1) says that the functions f_t can be chosen in such a way that

$$\sum_{t=0}^{\infty} m\{x\colon f_t(x) \neq f_{t+1}(x)\} < \infty.$$

The last condition implies that

$$m(\limsup A_t) = 0,$$

where $A_t = \{x \in X\colon f_t(x) \neq f_{t+1}(x)\}$. Therefore $\{f_t\}$ is convergent m-a.e. Our method of proving Theorem 1 is based on methods used in [8].

In §2 we describe the centralizer $C(T_\varphi)$, where $\varphi\colon X \to H$ is a regular Morse cocycle. This result is a generalization of [2], where $C(T_\varphi)$ was described if $H = \mathbf{Z}_2$. The next problem we deal with is concerned with $\varphi\colon X \to \mathbf{Z}_2$ admitting an approximation with speed $o(1/n^{1+\varepsilon})$, $\varepsilon \geq 0$. Is it possible to modify this cocycle by adding a coboundary to get a cocycle with one point of discontinuity (a *Morse cocycle*), i.e. are there $\psi, f\colon X \to \mathbf{Z}_2$ such that

$$\varphi(x) + \psi(x) = f(Tx) + f(x), \qquad m\text{-a.e. } x \in X$$

(of course, except for some trivial cases we cannot get ψ continuous)? For $\varepsilon \geq 1$ the positive answer has been found in [5] and then for every $\varepsilon > 0$ in [1]. In [1] the authors stated the question whether the same is valid for $\varepsilon = 0$. It seems to A. Fieldsteel (Math. Rev. 89f:28038, p. 3158) that this is also true. Here, we construct a cocycle $\varphi\colon X \to \mathbf{Z}_2$ with speed of approximation $o(1/n)$ which is not equivalent to any Morse cocycle. We finish this paper by proving that for cocycles $\varphi\colon X \to \mathbf{Z}_2$ having only one point of discontinuity with a condition of regularity there exists $S \in C(T)$ such that the functional equation

$$\varphi \circ S^{i_1} + \ldots + \varphi \circ S^{i_k} + \varphi \circ U = f \circ T + f$$

cannot be solved for any $i_1 < \ldots < i_k$, $k \geq 2$, $U \in C(T)$. This gives a partial (and positive) answer to a question stated in [4]. The relevance of the existence of φ with this property has been exhibited in [4], namely it leads to some counterexample machinery in the class of loosely Bernoulli transformations (i.e. in the class of induced automorphisms of irrational rotations).

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§1. Description of the method. Let (X, T) be an *adding machine*, i.e. T has rational pure point spectrum with the group of eigenvalues $\mathrm{Sp}(T) = G\{n_t\colon t \geq 0\}$, $n_t | n_{t+1}$, $t \geq 0$, where $G\{n_t\colon t \geq 0\}$ denotes the group of all n_t-roots of unity. The set X can be identified with the group of all n_t-adic integers, i.e.

$$X = \{x = (j_t)_{t=0}^{\infty}\colon 0 \leq j_t \leq n_t - 1,\ j_{t+1} \equiv j_t \ (\mathrm{mod}\, n_t),\ t \geq 0\},$$

and T with the translation by the unit element $\hat{1} = (1, 1, \ldots)$. Denote by m the normalized Haar measure on X defined on the algebra of all borelian subsets of X. Putting

$$D_j^t = \{x = (j_u)_{u=0}^{\infty}\colon j_t = j\}, \qquad j = 0, \ldots, n_t - 1,$$

we see that $D_j^t \cap D_s^t = \emptyset$ if $j \neq s$, $\bigcup_{j=0}^{n_t-1} D_j^t = X$ and $T(D_j^t) = D_{j+1}^t$, $j = 0, \ldots, n_t - 2$, $T(D_{n_t-1}^t) = D_0^t$. Therefore the partitions $\xi_t = (D_0^t, \ldots, D_{n_t-1}^t)$ are T-towers and $\xi_t \nearrow \varepsilon$, where ε is the partition of X into points. The sets D_j^t will be called *levels* of ξ_t. Finally, put $\lambda_t = n_t/n_{t-1}$, $t \geq 1$, $\lambda_0 = n_0$.

Let H be a finite abelian group. A measurable function $\varphi\colon X \to H$ is called an *H-valued cocycle*. A cocycle φ is a *coboundary* if there exists a measurable function $f\colon X \to H$ satisfying

(1) $$f(Tx) - f(x) = \varphi(x)$$

for a.e. $x \in X$. A cocycle φ is *Toeplitz* if the set S_t of all levels of $\xi_t = (D_0^t, \ldots, D_{n_t-1}^t)$ on which φ is not constant a.e. satisfies the condition $\mathrm{card}\, S_t/n_t \to 0$ as $t \to \infty$ (here $\mathrm{card}\, A$ is the cardinality of the set A). By a *block* B over H we mean a finite sequence $B[0] \ldots B[n-1]$, $B[i] \in H$, $0 \leq i \leq n-1$. The number n is called the *length* of B and is denoted by $|B|$. If $0 \leq i \leq k \leq n-1$ then $B[i, k]$ denotes the block $B[i] \ldots B[k]$. Given $h \in H$ define a block $B + h$ by $(B+h)[i] = B[i] + h$, $i = 0, \ldots, n-1$. If $C = C[0] \ldots C[m-1]$ is another block over H then $B \times C$ is the concatenation of the blocks $B + C[j]$, $0 \leq j \leq m-1$, i.e.

$$B \times C = (B + C[0]) \ldots (B + C[m-1]).$$

Assuming $|B| = |C| = n$ one can define a block $B + C$ by

$$(B+C)[i] = B[i] + C[i], \qquad i = 0, \ldots, n-1.$$

By the distance $d(B, C)$ between such blocks we mean the number

$$d(B, C) = \mathrm{card}\{0 \leq i \leq n-1\colon B[i] \neq C[i]\}/n.$$

A Toeplitz cocycle φ determines a sequence of blocks $\{A_t\}_{t=0}^{\infty}$ such that $|A_t| = n_t$, $A_t[j] = \varphi|_{D_j^t} \in H$ if $j \notin S_t$ and $A_t[j] = $ "$-$" for the remaining $j \in \{0, \ldots, n_t - 1\}$. We say that a place j, $0 \leq j \leq n_t - 1$, in A_t is *filled* if $A_t[j] \in H$, i.e. if φ is constant on D_j^t, and that it is *empty* (or the place j is a *hole*) if $A_t[j] = $ "$-$". If $j \in S_t$ then the function φ admits at least two values on subsets of D_j^t with positive measures (see Fig. 1).

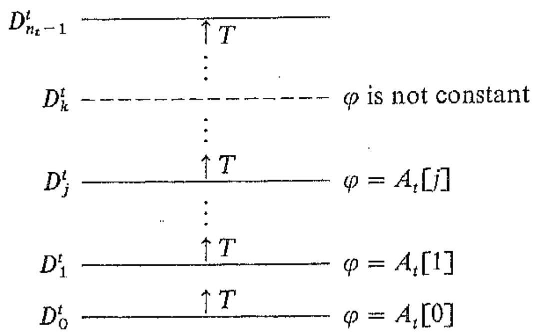

Fig. 1

It follows from the definition of the blocks A_t, $t \geqslant 0$, that A_{t+1} is obtained as the λ_{t+1}-fold concatenation $A_t \ldots A_t$, in which some holes are filled. A Toeplitz cocycle φ *has k holes* ($k \geqslant 1$) if $|S_t| = k$ for every $t \geqslant 0$.

Assume that $\varphi\colon X \to H$ is a Toeplitz cocycle having k holes and consider the equation (1), where f is an unknown measurable function. Now, we are in a position to describe a method of solving (1). Suppose that $S_t = \{j_0 < \ldots < j_{k-1}\}$. Take blocks $\alpha^t = \alpha^t[0] \ldots \alpha^t[k-1]$, $\alpha^t[s] \in H$, $s = 0, \ldots, k-1$, and $h_t \in H$. Define cocycles φ_t, $t \geqslant 0$, as follows:

$$\varphi_t(x) = \begin{cases} \varphi(x) & \text{if } x \notin \bigcup_{j \in S_t} D^t_j, \\ \alpha^t[s] & \text{if } x \in D^t_{j_s},\ s = 0, \ldots, k-1. \end{cases} \tag{2}$$

The cocycles φ_t allow us to define functions $f_t = f_t(\alpha^t, h_t)\colon X \to H$ by

$$f_t(x) = \begin{cases} h_t, & x \in D^t_0, \\ h_t + \varphi_t \circ T^{-j} x + \ldots + \varphi_t \circ T^{-1} x, & x \in D^t_j,\ j = 1, \ldots, n_t - 1. \end{cases} \tag{3}$$

In the sequel we will consider special elements $h_t \in H$. Let

$$M(\alpha^t, \alpha^{t-1}, h', h) = m\{x \in X\colon f_{t-1}(\alpha^{t-1}, h')(x) \neq f_t(\alpha^t, h)(x)\}, \qquad h, h' \in H.$$

Put $h_0 = 0$ and next choose inductively $h_1, h_2, \ldots$ as follows:

$$M(\alpha^t, \alpha^{t-1}, h_{t-1}, h_t) = \min_{h \in H} M(\alpha^t, \alpha^{t-1}, h_{t-1}, h). \tag{4}$$

Then $f_t(\alpha^t, h_t)$ depends only on α^t and we will write $f_t(\alpha^t)$ instead of $f_t(\alpha^t, h_t)$.

Let φ be a Toeplitz cocycle having k holes and assume that

$$\sum_{t=0}^{\infty} 1/\lambda_t < \infty. \tag{5}$$

LEMMA 1. *Let $f\colon X \to H$ be a measurable function such that*

$$f(Tx) - f(x) = \varphi(x) \quad \textit{for a.e. } x \in X.$$

Then there exist blocks α^t, $t \geqslant 0$, of length k such that $f_t(\alpha^t) \to f$ in measure.

Proof. There exist ξ_t-measurable functions $g_t\colon X \to H$ such that $g_t \to f$ in measure, because $\xi_t \nearrow \varepsilon$. Let us divide the tower ξ_t into λ_{t+1} columns $K_t(j)$, $j = 0, \ldots, \lambda_{t+1} - 1$, where $K_t(j) = \bigcup_{r=0}^{n_t-1} D^{t+1}_{jn_t+r}$ (see Fig. 2).

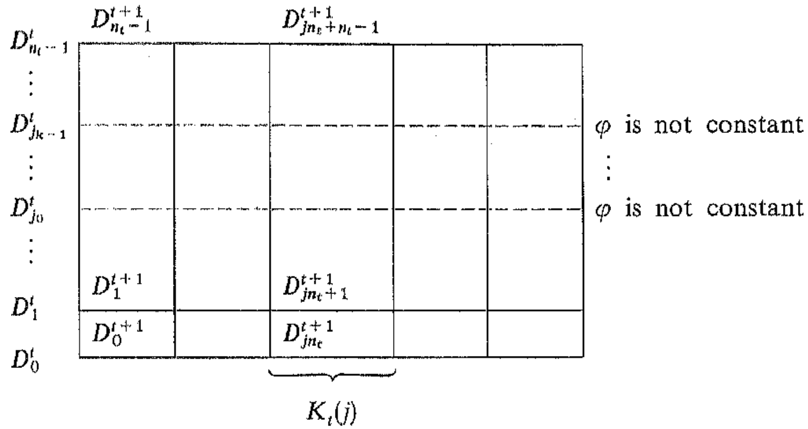

Fig. 2

Let J_t be the set of those j, $0 \leqslant j \leqslant \lambda_{t+1} - 1$, such that φ is constant on each level $D^{t+1}_{jn_t+r}$, $r = 0, \ldots, n_t - 1$. Since φ has k holes on ξ_{t+1} we have $|J_t| \geqslant \lambda_{t+1} - k$. For every $j \in J_t$ the cocycle φ determines a block B^t_j with $|B^t_j| = k$ such that $B^t_j[s] = \varphi|_{D^{t+1}_{jn_t+j_s}}$, $s = 0, \ldots, k-1$. There exists a block α^t, $|\alpha^t| = k$, for which the set $\bar{J}_t = \{j \in J_t\colon B^t_j = \alpha^t\}$ satisfies

$$|\bar{J}_t|/\lambda_{t+1} \geqslant \varrho > 0. \tag{6}$$

Put $E^t_r = \bigcup_{j \in \bar{J}_t} D^{t+1}_{jn_t+r}$, $r = 0, \ldots, n_t - 1$. We have $T(E^t_r) = E^t_{r+1}$, $r = 0, \ldots, n_t - 2$; $T(E^t_{n_t-1}) = E^t_0$. Hence

$$m\Big(\bigcup_{r=0}^{n_t-1} E^t_r\Big) = |\bar{J}_t|/\lambda_{t+1}, \qquad m(E^t_0) = |\bar{J}_t|/n_{t+1}, \tag{7}$$

and the functions φ and g_t are constant on each E^t_r, $0 \leqslant r \leqslant n_t - 1$. Let $h'_t \in H$ be such that

$$m\{x \in E^t_0\colon f(x) = h'_t\} = \max_{h \in H} m\{x \in E^t_0\colon f(x) = h\}.$$

Hence putting $F_0^t = \{x \in E_0^t\colon f(x) = h_t'\}$ we obtain

(8) $$m(F_0^t) \geqslant m(E_0^t)/|H|.$$

If $x \in T^r(F_0^t) =: F_r^t$, then

(9) $$f(x) = f(T^{-r}x) + \varphi(T^{-r}x) + \ldots + \varphi(T^{-1}x)$$
$$= h_t' + \varphi(T^{-r}x) + \ldots + \varphi(T^{-1}x) = f_t(x) = f_t(\alpha^t, h_t)(x)$$

because f satisfies (1), $r = 0, \ldots, n_t - 1$. Further, $T^r(E_0^t) = E_r^t$, which implies

(10) $$m(F_r^t) = m(F_0^t) \geqslant m(E_0^t)/|H| = m(E_r^t), \quad 0 \leqslant r \leqslant n_t - 1.$$

Now using (6)–(10) we have

$$m\{x\colon f_t(x) \neq g_t(x)\} = n_t^{-1}\operatorname{card}\{0 \leqslant r \leqslant n_t - 1\colon f_t|_{D_r^t} \neq g_t|_{D_r^t}\}$$
$$= \frac{\lambda_{t+1}}{|\bar{J}_t|} \cdot \frac{|\bar{J}_t|}{n_{t+1}} \operatorname{card}\{0 \leqslant r \leqslant n_t - 1\colon f_t|_{D_r^t} \neq g_t|_{D_r^t}\}$$
$$= \frac{\lambda_{t+1}}{|\bar{J}_t|} m(E_0^t)\operatorname{card}\{\ldots\} \leqslant \varrho^{-1}|H| m(F_0^t)\operatorname{card}\{\ldots\}$$
$$= \varrho^{-1}|H| m\{x \in \bigcup_{r=0}^{n_t-1} F_r^t\colon f_t(x) \neq g_t(x)\} = \varrho^{-1}|H| m\{x \in \bigcup_{r=0}^{n_t-1} F_r^t\colon f(x) \neq g_t(x)\}$$
$$\leqslant \varrho^{-1}|H| m\{x \in X\colon f(x) \neq g_t(x)\}.$$

Thus the condition $g_t \to f$ in measure implies $m\{x \in X\colon f_t(x) \neq g_t(x)\} \to 0$, which gives $f_t \to f$ in measure. The proof of the lemma is finished.

THEOREM 1. *A measurable solution $f\colon X \to H$ of (1) exists if and only if there exists a sequence of blocks $\{\alpha^t\}$ of length k over H such that the functions $f_t = f_t(\alpha^t)$ satisfy*

(11) $$\sum_{t=0}^{\infty} m\{x \in X\colon f_t(x) \neq f_{t+1}(x)\} < \infty.$$

Proof. *Sufficiency.* For every $t \geqslant 0$ and $r > 0$,

$$m\{x\colon f_t(x) \neq f_{t+r}(x)\} \leqslant m\Big(\bigcup_{i=0}^{r-1}\{x\colon f_{t+i}(x) \neq f_{t+i+1}(x)\}\Big)$$
$$\leqslant \sum_{i=0}^{r-1} m\{x\colon f_{t+i}(x) \neq f_{t+i+1}(x)\}.$$

The above inequality and (11) imply that the sequence $\{f_t\}$ satisfies the Cauchy condition for the convergence in measure, so $\{f_t\}$ converges in measure to a measurable function $f\colon X \to H$. It follows from the definition of φ_t and f_t that

$$f_t(Tx) - f_t(x) = \varphi_t(x) = \varphi(x)$$

for $x \in \bigcup_{j \notin S_t} D_j^t$. Using this equality and the property $m(\bigcup_{j \notin S_t} D_j^t) \nearrow 1$ we obtain (1).

Necessity. Suppose that f satisfies (1). There exist ξ_t-measurable functions $g_t\colon X \to H$, $t \geqslant 0$, such that $g_t \to f$ in measure, because $\xi_t \nearrow \varepsilon$. Using Lemma 1 we can assume g_t is of the form $g_t = f_t(\alpha^t)$, where the α^t are blocks over H, $|\alpha^t| = k$, $t \geqslant 0$. Let B_t be the block obtained from A_t by putting $\alpha^t[s]$ in the places $A_t[j]$, $j \in S_t$. If $C = C[0]C[1]\ldots C[n-1]$ is a block over H, then denote by $\check{C}$ the block

$$0,\ C[0],\ C[0]+C[1],\ \ldots,\ C[0]+C[1]+\ldots+C[n-2].$$

It follows from (2) and (3) that the members of $C_t = \check{B}_t + h_t$ are the successive values of f_t on $D_0^t, D_1^t, \ldots, D_{n_t-1}^t$. Now we will modify the functions f_t to obtain other ξ_t-measurable functions $\bar{f}_t$ such that

(12) $$\sum_{t=0}^{\infty} m\{x\colon \bar{f}_t(x) \neq f_{t+1}(x)\} < \infty, \quad \sum_{t=0}^{\infty} m\{x\colon f_t(x) \neq \bar{f}_t(x)\} < \infty.$$

From now on we use the following notation:

$$0_n = 0\ldots0 \quad (n\text{-fold repetition}),\ n \geqslant 1.$$

Since $f_t \to f$ in measure, we have

(13) $$\sup_{r \geqslant 1} m\{x\colon f_{t+r}(x) \neq f_t(x)\} \to 0.$$

The values of f_t on $D_0^{t+r}, \ldots, D_{n_{t+r}-1}^{t+r}$ form the block $C_t \times 0_{\lambda_{t+1}\ldots\lambda_{t+r}}$ so the condition (13) can be rewritten as

(14) $$\sup_{r \geqslant 1} d(C_{t+r}, C_t \times 0_{\lambda_{t+1}\ldots\lambda_{t+r}}) \to 0.$$

Define blocks N_{t+1}, $|N_{t+1}| = \lambda_t\lambda_{t+1}$, in the following way: For $0 \leqslant l \leqslant \lambda_t\lambda_{t+1} - 1$ consider the numbers

(15) $$d(h) = d\big(C_{t+1}[ln_{t-1}, (l+1)n_{t-1} - 1],$$
$$(C_t \times 0_{\lambda_{t+1}})[ln_{t-1}, (l+1)n_{t-1} - 1] + h\big), \qquad h \in H.$$

Now, choose $h_0 \in H$ such that $d(h_0) = \min_{h \in H} d(h)$ and put $N_{t+1}[l] = h_0$. It is not hard to see that at least $\lambda_t - k$ of the subblocks $C_t[qn_{t-1}, (q+1)n_{t-1} - 1]$, $q = 0, \ldots, \lambda_t - 1$, of C_t have the form

(16) $$\check{B}_{t-1} + h_q',$$

where $h_q' \in H$. Thus at least $\lambda_{t+1}\lambda_t - \lambda_{t+1}k$ corresponding subblocks of $C_t \times 0_{\lambda_{t+1}}$ have the same form. Moreover, the same subblocks of C_{t+1} have the form (16) too. Hence the distance d in (15) is 0 for at least $\lambda_{t+1}\lambda_t - \lambda_{t+1}k$ of l's. As a consequence we have

$$d(C_{t+1}-C_t\times 0_{\lambda_{t+1}},\ 0_{n_{t-1}}\times N_{t+1})\leqslant k\lambda_{t+1}n_{t-1}/n_{t+1}=k/\lambda_t$$

and using (5) we obtain

$$\sum_{t=0}^{\infty} d(C_{t+1}-C_t\times 0_{\lambda_{t+1}},\ 0_{n_{t-1}}\times N_{t+1})<\infty. \tag{17}$$

Further we will need the identity

$$\begin{aligned}C_{t+r}-C_t\times 0_{\lambda_{t+1}\ldots\lambda_{t+r}} &= C_{t+r}-C_{t+r-1}\times 0_{\lambda_{t+r}}+(C_{t+r-1}-C_{t+r-2}\times 0_{\lambda_{t+r-1}})\times 0_{\lambda_{t+r}}+\ldots\\ &\quad +(C_{t+2}-C_{t+1}\times 0_{\lambda_{t+2}})\times 0_{\lambda_{t+3}\ldots\lambda_{t+r}}+(C_{t+1}-C_t\times 0_{\lambda_{t+1}})\times 0_{\lambda_{t+2}\ldots\lambda_{t+r}}.\end{aligned} \tag{18}$$

Putting

$$d_t(r)=d(C_{t+r}-C_t\times 0_{\lambda_{t+1}\ldots\lambda_{t+r}},\ 0_{n_{t+r}})$$

and using (17), (18) and the inequalities

$$|d(A,\ C)-d(B,\ C)|\leqslant d(A,\ B),\qquad A,\ B,\ C \text{ blocks of the same length},$$

$$d(A_1+B_1,\ A_2+B_2)\leqslant d(A_1,\ A_2)+d(B_1,\ B_2),$$

we have

$$\begin{aligned}d_t(r) &= d(0_{n_{t+r-2}}\times N_{t+r}+0_{n_{t+r-3}}\times N_{t+r-1}\times 0_{\lambda_{t+r}}+\ldots+0_{n_t}\times N_{t+2}\\ &\quad \times 0_{\lambda_{t+3}\ldots\lambda_{t+r}}+0_{n_{t-1}}\times N_{t+1}\times 0_{\lambda_{t+2}\ldots\lambda_{t+r}},\ 0_{n_{t+r}})+\varepsilon_t(r)=\bar d_t(r)+\varepsilon_t(r),\end{aligned}$$

where $\sup_{r\geqslant 1}\varepsilon_t(r)\to 0$. For even r,

$$\begin{aligned}\bar d_t(r) &= d(0_{n_{t+r-2}}\times N_{t+r}+0_{n_{t+r-4}}\times N_{t+r-2}\times 0_{\lambda_{t+r-1}\lambda_{t+r}}+\ldots+0_{n_t}\times N_{t+2}\\ &\quad \times 0_{\lambda_{t+3}\ldots\lambda_{t+r}},\ -[0_{n_{t+r-3}}\times N_{t+r-1}\times 0_{\lambda_{t+r}}]\\ &\quad -\ldots-[0_{n_{t-1}}\times N_{t+1}\times 0_{\lambda_{t+2}\ldots\lambda_{t+r}}])=:d(\mathrm{I},\ \mathrm{II}).\end{aligned}$$

Now, we apply the identity

$$(A\times 0_s)+(0_q\times B)=A\times B,\qquad |A|=q,\ |B|=s. \tag{19}$$

We obtain

$$0_{n_{t+r-2}}\times N_{t+r}+0_{n_{t+r-4}}\times N_{t+r-2}\times 0_{\lambda_{t+r}\lambda_{t+r-1}}=0_{n_{t+r-4}}\times N_{t+r-2}\times N_{t+r}$$

and applying (19) many times we have

$$\mathrm{I}=0_{n_t}\times N_{t+2}\times\ldots\times N_{t+r},\qquad \mathrm{II}=-[0_{n_{t-1}}\times N_{t+1}\times\ldots\times N_{t+r-1}\times 0_{\lambda_{t+r}}].$$

Hence

$$\bar d_t(r)=d(0_{\lambda_t}\times N_{t+2}\times\ldots\times N_{t+r},\ -[N_{t+1}\times\ldots\times N_{t+r-1}\times 0_{\lambda_{t+r}}]). \tag{20}$$

The condition (14) gives $\sup_{r\geqslant 1}d_t(r)\to 0$ as $t\to\infty$ and similarly

$$\sup_{r\geqslant 1}\bar d_t(r)\to 0\qquad \text{as } t\to\infty. \tag{21}$$

Now, it follows from (19), (20) and from [8] that there exist blocks a_s, $\bar a_s$, $|a_{s+1}|=\lambda_{2s}$, $|\bar a_s|=\lambda_{2s-1}$, $s\geqslant 0$, satisfying

$$\sum_{s=0}^{\infty} d(-N_{2s},\ \bar a_s\times a_{s+1})<\infty,\qquad \sum_{s=0}^{\infty} d(N_{2s-1},\ a_s\times \bar a_s)<\infty.$$

Since $\bar d_t(1)=d(N_{t+1},\ 0_{\lambda_t\lambda_{t+1}})\to 0$ as $t\to\infty$, we have

$$d(a_s\times\bar a_s,\ 0_{\lambda_{2(s-1)}\lambda_{2s-1}})\to 0\quad\text{and}\quad d(\bar a_s\times a_{s+1},\ 0_{\lambda_{2s-1}\lambda_{2s}})\to 0\qquad \text{as } s\to\infty.$$

The above implies that there exists $h\in H$ such that

$$d(a_s+h,\ 0_{\lambda_{2(s-1)}})\to 0\quad\text{and}\quad d(\bar a_s-h,\ 0_{\lambda_{2s-1}})\to 0\qquad\text{as } s\to\infty.$$

Again we obtain

$$\sum_s d\big(-N_{2s},\ (\bar a_s-h)\times(a_{s+1}+h)\big)<\infty,\qquad \sum_s d\big(N_{2s-1},\ (a_s\times h)\times(\bar a_s-h)\big)<\infty.$$

Put

$$R_t=\begin{cases}-a_{s+1}-h & \text{if } t=2s,\\ \bar a_s-h & \text{if } t=2s-1.\end{cases}$$

Then

$$\sum_{t=0}^{\infty} d\big(N_{t+1},\ (-R_t)\times R_{t+1}\big)<\infty. \tag{22}$$

Put

$$P_t=C_t+(0_{n_{t-1}}\times R_t),\qquad t\geqslant 0, \tag{23}$$

and define a ξ_t-measurable function $\bar f_t\colon X\to H$ by

$$\bar f_t(x)=P_t[j],\qquad x\in D_j^t,\ j=0,\ \ldots,\ n_t-1.$$

We have $m\{x\colon \bar f_t(x)\neq\bar f_{t+1}(x)\}=d(P_{t+1},\ P_t\times 0_{\lambda_{t+1}})$. Next, (22), (23) and (16) imply

$$\sum_{t=0}^{\infty} d(P_{t+1},\ P_t\times 0_{\lambda_{t+1}})<\infty, \tag{24}$$

which gives $\sum_{t=0}^{\infty} m\{x\colon \bar f_t(x)\neq \bar f_{t+1}(x)\}<\infty$. To finish the proof it suffices to show $\sum_{t=0}^{\infty} m\{x\colon \bar f_t(x)\neq f_t(x)\}<\infty$.

If $j\in J_t$ then $C_{t+1}[jn_t,\ (j+1)n_t-1]$ has the form $C_t+h_t^j$, $h_t^j\in H$. Denote by D_t the block of length $|\bar J_t|$ with members h_t^j, $j\in\bar J_t$, and let $\bar R_t=R_{t+1}|_{\bar J_t}$. Then $P_{t+1}[jn_t,\ (j+1)n_t-1]=C_t+h_t^j+\bar R_t[j]$, $j\in\bar J_t$. Further

$$d(P_{t+1}, P_t\times 0_{\lambda_{t+1}}) \geqslant \frac{1}{n_{t+1}}\sum_{j\in \bar{J}_t} \operatorname{card}\{0\leqslant r\leqslant n_t-1:\ P_{t+1}[jn_t+r]\neq P_t[r]\}$$
$$= \frac{1}{n_{t+1}}\sum_{j\in \bar{J}_t} \operatorname{card}\{0\leqslant r\leqslant n_t-1:\ C_t[r]+h_t^j+\bar{R}_t[j]\neq P_t[r]\}$$
$$= \frac{|\bar{J}_t|}{\lambda_{t+1}}\cdot\frac{1}{n_t|\bar{J}_t|}\sum_{j\in \bar{J}_t}\operatorname{card}\{\ldots\}$$
$$= \frac{|\bar{J}_t|}{\lambda_{t+1}} d\big(C_t\times D_t+(0_{n_t}\times \bar{R}_t),\ P_t\times 0_{|\bar{J}_t|}\big) \geqslant \varrho \min d(C_t,\ P_t+h).$$

But $d(C_t, P_t) = d(0_{n_t}, 0_{n_{t-1}}\times R_t) = d(R_t, 0_{\lambda_t})\to 0$ as $t\to\infty$. Thus

$$\min_{h\in H} d(C_t, P_t+h) = d(C_t, P_t)$$

for t large enough. As a consequence we obtain

$$d(P_{t+1}, P_t\times 0_{\lambda_{t+1}}) \geqslant \varrho d(C_t, P_t).$$

Therefore in view of (24) we have $\sum_{t=0}^{\infty} d(C_t, P_t) < \infty$. This means that

$$\sum_{t=0}^{\infty} m\{x:\ f_t(x)\neq \bar{f}_t(x)\} < \infty.$$

This completes the proof of the theorem.

COROLLARY. *Suppose that the equation*

$$f(Tx)-f(x) = \varphi(x),\qquad x\in X,$$

can be solved and let C_t, $t\geqslant 0$, be the blocks determining the functions f_t constructed in the proof of Theorem 1. If $I_t^{(1)}$, $I_t^{(2)}$ are disjoint subsets of $\{0,\ldots,\lambda_{t+1}-1\}$, $t\geqslant 0$, then

$$\sum_{t=0}^{\infty}\min\left\{\frac{|I_t^{(1)}|}{\lambda_{t+1}}, \frac{|I_t^{(2)}|}{\lambda_{t+2}}\right\}\min\{d(C_{t+1}^{(j)}, C_{t+1}^{(j')}+h):\ j\in I_t^{(1)},\ j'\in I_t^{(2)},\ h\in H\} < \infty,$$

where $C_{t+1}^{(j)} = C_{t+1}[jn_t, (j+1)n_t-1]$, $j = 0,\ldots,\lambda_{t+1}-1$.

Proof. We have

$$d(C_{t+1}, C_t\times 0_{\lambda_{t+1}}) = \lambda_{t+1}^{-1}\sum_{j=0}^{\lambda_{t+1}-1} d(C_{t+1}^{(j)}, C_t)$$
$$\geqslant \lambda_{t+1}^{-1}\Big[\sum_{j\in I_t^{(1)}} d(C_{t+1}^{(j)}, C_t)+\sum_{j'\in I_t^{(2)}} d(C_{t+1}^{(j')}, C_t)\Big]$$
$$\geqslant \lambda_{t+1}^{-1}\min\{|I_t^{(1)}|, |I_t^{(2)}|\}\min_{j,j'} d(C_{t+1}^{(j)}, C_{t+1}^{(j')}).$$

§ 2. Applications.

We start with two definitions.

DEFINITION 1. We say that $\varphi\colon X\to H$ is a *Morse cocycle* if φ has exactly one hole for every $t\geqslant 0$, i.e. $S_t = \{D_{r_t}^t\}$.

Of course, $y_0 = (r_t)_{t=0}^{\infty}$ is an n_t-adic integer.

DEFINITION 2. A Morse cocycle $\varphi\colon X\to H$ is said to be *regular* if there exists $\varrho > 0$ such that

$$n_t m\{x\in D_{r_t}^t:\ \varphi(x) = h\}\geqslant \varrho$$

for all $h\in H$ and $t\geqslant 0$.

Now suppose that $\varphi\colon X\to H$ is a Morse cocycle. Without loss of generality we can assume that $r_t = n_t-1$ for every $t\geqslant 0$. Let

$$A_t = A_t[0]\ldots A_t[n_t-2]\,\underline{\text{hole}}$$

be the block which defines φ on $D_0^t,\ldots,D_{n_t-2}^t$. To obtain A_{t+1} we use the block

$$\beta^{t+1} = \beta^{t+1}[0]\ldots\beta^{t+1}[\lambda_{t+1}-2]\,\underline{\text{hole}} \tag{25}$$

in such a way that

$$A_{t+1} = B_t\underline{\beta^{t+1}[0]}\,B_t\underline{\beta^{t+1}[1]}\ldots B_t\underline{\beta^{t+1}[\lambda_{t+1}-2]}\,B_t\underline{\text{hole}},$$

where $B_t = A_t[0]\ldots A_t[n_t-2]$ with B_t appearing λ_{t+1} times. If we put $\beta^0 = A_0$ then the sequence of blocks $\{\beta^t\}_{t=0}^{\infty}$ of the form (25) determines the Morse cocycle φ completely.

There is another way of describing the Morse cocycles. Given a (completely filled) sequence of blocks $\{b^t\}_{t=0}^{\infty}$ over H with $|b^t| = \lambda_t$ and $b^t[0] = 0$. Define a new sequence of blocks $\{c_t\}$ by

$$c_t = b^0\times\ldots\times b^t,\qquad t\geqslant 0,$$

and put $A_t[j] = c_t[j+1]-c_t[j]$, $j = 0,\ldots,n_t-2$. Then the blocks A_t, $t\geqslant 0$, determine a Morse cocycle φ. It is easy to see that each Morse cocycle φ can be obtained in this way.

The regularity of φ means that

$$\lambda_{t+1}^{-1}\operatorname{card}\{0\leqslant j\leqslant\lambda_{t+2}-2:\ b^{t+1}[j+1]-b^{t+1}[j] = h\}\geqslant\varrho \tag{26}$$

for all $h\in H$ and $t\geqslant 0$.

Consider the metric centralizer $C(T)$ of T. It is well known that $C(T)$ can be identified with X. Each $y\in X$ determines $S = S_y\in C(T)$ by $S(x) = x+y$, $x\in X$, and if $y = (j_t)_{t=0}^{\infty}$, $0\leqslant j_t\leqslant n_t-1$, $j_t\equiv j_{t+1}\ (\operatorname{mod} n_t)$, then $S = \lim_t T^{j_t}$ in the weak topology of $C(T)$. It is proved in [6] that every $S\in C(T_\varphi)$ is determined by a triple (S, f, v) satisfying

$$f(Tx)-f(x) = \varphi(Sx)-v(\varphi(x)),\qquad x\in X, \tag{27}$$

where $S \in C(T)$, $f\colon X \to H$ is a measurable function and $v\colon H \to H$ is a group automorphism. We say that $S \in C(T)$ *can be lifted* to $C(T_\varphi)$ if there exist $f\colon X \to H$ and $v\colon H \to H$ satisfying (27).

Let $y = (j_t)_{t=0}^{\infty}$ be an irrational n_t-adic integer and $v\colon H \to H$ an automorphism. Put $S = S_y$, $\psi(x) = \varphi(Sx) - v(\varphi(x))$ and consider the equation

$$f(Tx) - f(x) = \psi(x), \quad x \in X. \tag{28}$$

Let $j_{t+1} = j_t + q_{t+1} n_t$, $0 \leqslant q_{t+1} \leqslant \lambda_{t+1} - 1$, $t \geqslant 0$. Then $j_t \nearrow \infty$ and $(n_t - j_t) \nearrow \infty$, so we can assume $0 < q_{t+1} < \lambda_{t+1} - 1$. The cocycle ψ has two holes on ξ_t. We illustrate it in Fig. 3.

$D^t_{n_t-1}$ - - - - - - - - ψ is not constant

$D^t_{n_t-2}$ ——— $\psi = c_t[j_t-1] - c_t[j_t-2] - v(c_t[n_t-1] - c_t[n_t-2])$

$D^t_{n_t-j_t}$ ——— $\psi = c_t[1] - c_t[0] - v(c_t[n_t-j_t+1] - c_t[n_t-j_t])$

$D^t_{n_t-j_t-1}$ - - - - - - - - ψ is not constant

$D^t_{n_t-j_t-2}$ ——— $\psi = c_t[n_t-1] - c_t[n_t-2] - v(c_t[n_t-j_t-1] - c_t[n_t-j_t-2])$

D^t_1 ——— $\psi = c_t[j_t+2] - c_t[j_t+1] - v(c_t[2] - c_t[1])$

D^t_0 ——— $\psi = c_t[j_t+1] - c_t[j_t] - v(c_t[1] - c_t[0])$

Fig. 3

Now, recall that if A, B are blocks and $h, h' \in H$, then by $(A+h)(B+h')$ we denote the concatenation of the blocks $A+h$ and $B+h'$. If $v\colon H \to H$ is an automorphism, then $v(A)$ denotes the block $v(A[0]) \ldots v(A[n-1])$ (here $A = A[0] \ldots A[n-1]$).

THEOREM 2. *Let $\varphi\colon X \to H$ be a regular Morse cocycle determined by a sequence of blocks $\{b^t\}$, $|b^t| = \lambda_t$, $b^t[0] = 0$, $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$. An element $S_y = S \in C(T)$ can be lifted to $C(T_\varphi)$ if and only if there exists a group automorphism $v\colon H \to H$ such that*

(i) $$\sum_{t=0}^{\infty} \min\{j_t/n_t, 1 - j_t/n_t\} < \infty, \qquad y = (j_t)_{t=0}^{\infty} = \sum_{t=0}^{\infty} q_t n_{t-1},$$

(ii) $$\sum_{t=0}^{\infty} d(\gamma_t, v(b^t)) < \infty,$$

where $\gamma_t = (b^t + r_t)(b^t + s_t)[q'_t, q'_t + \lambda_t - 1]$, r_t, s_t are elements of H and $q'_0 = q_0$, $q'_t = q_t$ if $j_{t-1} \leqslant n_{t-1} - j_{t-1} - 1$ and $q'_t \equiv q_t + 1 \pmod{\lambda_t}$ otherwise, $t \geqslant 1$.

Proof. We can assume that y is an irrational n_t-adic integer, because every rational y can be lifted to $C(T_\varphi)$. $S_y = S$ can be lifted iff there exists a function $f\colon X \to H$ satisfying (28). According to Theorem 1 this is equivalent to the convergence of the series $\sum_{t=0}^{\infty} m\{x\colon f_t(x) \neq f_{t+1}(x)\}$, where f_t, $t \geqslant 0$, are functions constructed in §1. Analysing the cocycle ψ (see Fig. 3) we see that

(29) the values of f_t on $D^t_0, \ldots, D^t_{n_t-1}$ coincide with the members of the block $(c_t + r'_t)(c_t + s'_t)[j_t, n_t + j_t - 1] - v(c_t)$,

where $r'_t, s'_t \in H$. By a simple computation we obtain

$$\begin{aligned} (30)\quad & m\{x\colon f_t(x) \neq f_{t+1}(x)\} \\ &= \Big(1 - \frac{j_t}{n_t}\Big) d\big((b^{t+1} + r'_{t+1} - r'_t)(b^{t+1} + s'_{t+1} - r'_t)[q_{t+1}, \lambda_{t+1} + q_{t+1} - 1], v(b^{t+1})\big) \\ &\quad + \frac{j_t}{n_t} d\big((b^{t+1} + r'_{t+1} - s'_t)(b^{t+1} + s'_{t+1} - s'_t)[q_{t+1} + 1, \lambda_{t+1} + q_{t+1}], v(b^{t+1})\big) \\ &=: \Big(1 - \frac{j_t}{n_t}\Big) \cdot \mathrm{III} + \frac{j_t}{n_t} \cdot \mathrm{IV}. \end{aligned}$$

If $j_t/n_t < 1/2$ then putting

$$r_{t+1} = r'_{t+1} - r'_t, \quad s_{t+1} = s'_{t+1} - r'_t \tag{31}$$

we have

$$d\big((b^{t+1} + r_{t+1})(b^{t+1} + s_{t+1})[q_{t+1}, \lambda_{t+1} + q_{t+1} - 1], v(b^{t+1})\big) \leqslant 2m\{x\colon f_t(x) \neq f_{t+1}(x)\}.$$

If $j_t/n_t \geqslant 1/2$ then for

$$r_{t+1} = r'_{t+1} - s'_t, \quad s_{t+1} = s'_{t+1} - s'_t \tag{32}$$

we have

$$d\big((b^{t+1} + s_{t+1})(b^{t+1} + r_{t+1})[q_{t+1} + 1, \lambda_{t+1} + q_{t+1}], v(b^{t+1})\big) \leqslant 2m\{x\colon f_t(x) \neq f_{t+1}(x)\}.$$

Now, Theorem 1 implies (ii). (30) yields the inequality

$$m\{x\colon f_t(x) \neq f_{t+1}(x)\} \geqslant \min\{j_t/n_t, 1 - j_t/n_t\}(\mathrm{III} + \mathrm{IV}). \tag{33}$$

It is not hard to see that the condition of regularity implies $\mathrm{III} + \mathrm{IV} \geqslant \frac{1}{2}\varrho$. Therefore (33) gives (i).

On the other hand, if (i) and (ii) are satisfied then we put $r'_0 = r_0$, $s'_0 = s_0$ and define r'_t, s'_t, $t \geqslant 1$, by (31) and (32). Next define f_t by (29). It follows from (30) that

$$m\{x\colon f_t(x) \neq f_{t+1}(x)\} \leqslant \begin{cases} j_t/n_t + \mathrm{III} & \text{if } j_t/n_t < 1/2, \\ 1 - j_t/n_t + \mathrm{IV} & \text{if } j_t/n_t \geqslant 1/2. \end{cases}$$

Then the above inequality and (i) and (ii) imply

$$\sum_{t=0}^{\infty} m\{x\colon f_t(x) \neq f_{t+1}(x)\} < \infty.$$

This completes the proof of the theorem.

The next theorem gives a negative answer to a question contained in [1]. Here we assume $H = \mathbf{Z}_2$. For completeness we recall the following definitions.

DEFINITION 3. Let $f\colon \mathbf{N} \to \mathbf{R}$ be a real function. We say that a cocycle $\varphi\colon X \to \mathbf{Z}_2$ admits an *approximation with speed* $o(f(n))$ if for some subsequence $\{t_k\}$ there exist sets F_k consisting of some levels of the partitions ξ_{t_k} such that

$$m(\varphi^{-1}(1)\Delta F_k) = o(f(n_{t_k})).$$

DEFINITION 4. A cocycle φ is called *ergodic* if the $\mathbf{Z}_2$-extension T_φ is ergodic.

THEOREM 3. *There exists an ergodic Toeplitz cocycle $\varphi\colon X \to \mathbf{Z}_2$ admitting an approximation with speed $o(1/n)$ such that for every Morse cocycle $\psi\colon X \to \mathbf{Z}_2$, φ and ψ are not equivalent.*

Proof. Put $q_t = [\frac{1}{3}\lambda_t]$, $p_t = [\frac{2}{3}\lambda_t]$ (here $[a]$ denotes the integer part of a), $j_t = \sum_{r=0}^{t} q_r n_{r-1}$, $l_t = \sum_{r=0}^{t} p_r n_{r-1}$, $n_{-1} = 1$, $t \geqslant 0$.

A cocycle φ will be defined stepwise. On step t, φ will be defined (and constant) on the levels D_i^t except $i = j_t$ and $i = l_t$. Put $x_t = [t^{-1}\lambda_{t+1}]$. For $t = 2$ we define $\varphi|_{D_i^t} = 0$, $i \neq l_2, j_2$. Now suppose that φ is defined on the levels D_i^t (except $i = l_t, j_t$). To define φ on step $t+1$ it suffices to define φ on the levels $D_{j_t+rn_t}^{t+1}$, $D_{l_t+sn_t}^{t+1}$, where $r, s = 0, \ldots, \lambda_{t+1}-1$, $r \neq q_{t+1}$, $s \neq p_{t+1}$, since on the remaining levels φ was defined earlier. We put

$$\varphi|_{D_{j_t+rn_t}^{t+1}} = \begin{cases} 1 & \text{for } r \in \{0, \ldots, q_{t+1}-1\} \cup \{q_{t+1}+x_t, \ldots, \lambda_{t+1}-1\}, \\ 0 & \text{for } r \in \{q_{t+1}+1, \ldots, q_{t+1}+x_t-1\}, \end{cases}$$

$$\varphi|_{D_{l_t+sn_t}^{t+1}} = \begin{cases} 1 & \text{for } s \in \{0, \ldots, p_{t+1}-1\} \cup \{p_{t+1}+x_t, \ldots, \lambda_{t+1}-1\}, \\ 0 & \text{for } s \in \{p_{t+1}+1, \ldots, p_{t+1}+x_t-1\}. \end{cases}$$

To illustrate φ on ξ_{t+1} let us make some remarks. It follows from the definition of q_t, p_t, j_t and l_t that

$$p_t > q_t, \quad l_t > j_t, \quad q_t/\lambda_t \to 1/3, \quad j_t/n_t \to 1/3,$$

$$(p_t - q_t)/t \to 1/3, \quad (l_t - j_t)/n_t \to 1/3.$$

Further, $q_{t+1}+x_t < p_{t+1}$, $p_{t+1}+x_t < \lambda_{t+1}-1$ for large t because $\sum_{t=0}^{\infty} 1/\lambda_t < \infty$.

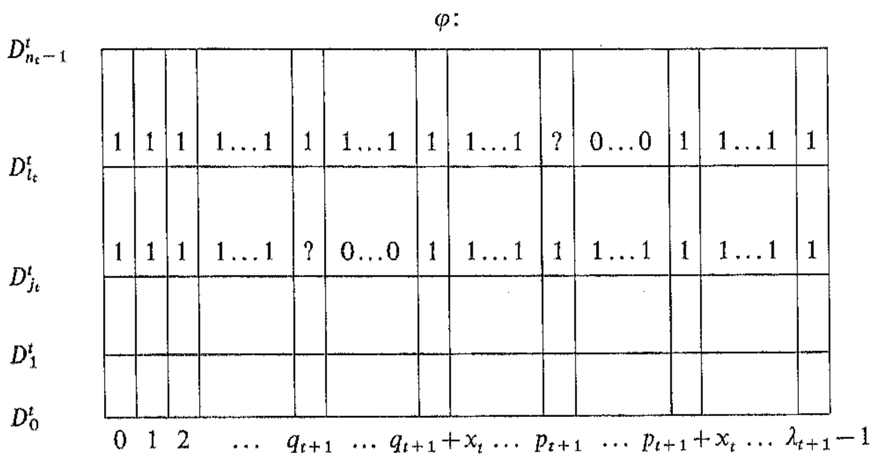

Fig. 4

It is easy to see that

$$m(\varphi^{-1}(1)\Delta F_t) \leqslant 2x_t \frac{1}{n_{t+1}} = \frac{2}{t}\cdot\frac{1}{n_t},$$

where $F_t = \bigcup_{i \in I_t} D_i^t$. Here $I_t = \{l_t, j_t\} \cup \{i\colon i \neq l_t, j_t, \ \varphi(D_i^t) = 1\}$. Therefore the cocycle φ admits an approximation with speed $o(1/n)$. The ergodicity of φ will be proved later.

Now assume that there is a Morse cocycle η (with respect to the partition $\xi_t = (D_0^t, \ldots, D_{n_t-1}^t)$) which is equivalent to φ. Take $S \in C(T)$ such that there is a measurable function $f\colon X \to \mathbf{Z}_2$ satisfying

$$f \circ T - f = \psi,$$

where $\psi = \varphi + \eta \circ S$. Put $\bar{\eta} = \eta \circ S$. Then $\bar{\eta}$ has a hole $(k_t)_{t=0}^{\infty} \in X$. Let $\{\alpha^{t+1}\}$ be a sequence of blocks of length 3 and $f_{t+1} = f_t(\alpha^{t+1})$, $t \geqslant 0$, the functions constructed in § 1. Consider two functional equations

$$g(Tx) - g(x) = \varphi(x), \tag{34}$$

$$\bar{g}(Tx) - \bar{g}(x) = \bar{\eta}(x), \quad x \in X. \tag{35}$$

Fix $t \geqslant 0$ and construct two functions g_{t+1} and $\bar{g}_{t+1}$ corresponding to (34) and (35) as in § 1 using the blocks α^{t+1} (we use α^{t+1} on $D_{j_{t+1}}^{t+1}$ and $D_{l_{t+1}}^{t+1}$ for g_{t+1} and α^{t+1} on $D_{k_{t+1}}^{t+1}$ for $\bar{g}_{t+1}$). It is clear that

$$f_{t+1} = g_{t+1} + \bar{g}_{t+1}.$$

Let E, F be the blocks of length n_{t+1} determined by the functions g_{t+1} and $\bar{g}_{t+1}$ respectively. Let E_j denote the block $E[jn_t, (j+1)n_t - 1]$;

$j = 0, \ldots, \lambda_{t+1}-1$. Consider the block $E_0 = E[0, n_t-1]$. This block determines three blocks A_1, A_2, A_3 of length j_t, l_t-j_t, n_t-l_t respectively such that E_0 is the concatenation of A_1, A_2, A_3: $E_0 = A_1A_2A_3$. Set

$$I_1 = \{0, \ldots, q_{t+1}-1\} \cup \{q_{t+1}+x_t+1, \ldots, p_{t+1}-1\}$$
$$\cup \{p_{t+1}+x_t, \ldots, p_{t+1}+\lambda_{t+1}-1\},$$
$$I_2 = \{q_{t+1}+1, \ldots, q_{t+1}+x_t-1\},$$
$$I_3 = \{p_{t+1}+1, \ldots, p_{t+1}+x_t-1\}.$$

It follows clearly from the definition of the cocycles φ and g_{t+1} that there are elements $r_j \in \mathbf{Z}_2$ such that

$$j \in I_1 \Rightarrow E_j = A_1A_2A_3+r_j,$$
$$j \in I_2 \Rightarrow E_j = A_1\tilde{A}_2\tilde{A}_3+r_j,$$
$$j \in I_3 \Rightarrow E_j = A_1\tilde{A}_2A_3+r_j,$$

where $\tilde{B}$ denotes the block $B+1$. Since the cocycle $\bar{\eta}$ has exactly one hole, there exist blocks $B_1, B_2, |B_1| = k_t, |B_2| = n_t-k_t$, such that for each $j = 0, \ldots, \lambda_{t+1}-1$, $F_j = F[jn_t, (j+1)n_t-1]$ is one of the blocks $B_1B_2, \tilde{B}_1B_2, B_1\tilde{B}_2, \tilde{B}_1\tilde{B}_2$. Let $\bar{H}$ be the block determined by the cocycle f_{t+1}. Of course $\bar{H}_j = E_j+F_j$, $j \geq 0$. We show that for $l = 2$ or $l = 3$

$$\min\{d(H_{j_1}, H_{j_2}+r)\colon j_1 \in I_1,\ j_2 \in I_l,\ r \in \mathbf{Z}_2\} \geq 1/6. \tag{36}$$

To this end consider two cases.

(a) $k_t \leq [n_t/2] =: u$. In this case we take $l = 3$. Put $A_2'' = A[u+1-j_t, l_t-j_t]$. By the definition of φ we have $\bar{H}_j[u+1, n_t-1] = A_2''A_3+\bar{r}_j$ for $j \in I_1$ and $\bar{H}_j[u+1, n_t-1] = A_2''\tilde{A}_3+\bar{r}_j$ for $j \in I_3$. Here $\bar{r}_j \in \mathbf{Z}_2$. Since

$$d(A_2''A_3, A_2''\tilde{A}_3) = (n_t-l_t)/n_t \geq 1/3,$$
$$d(A_2''A_3, \tilde{A}_2''A_3) = (l_t-u)/n_t \geq 2/3-1/2 = 1/6,$$

for $j_1 \in I_1$, $j_2 \in I_3$ and $r \in \mathbf{Z}_2$ we obtain

$$d(\bar{H}_{j_1}, \bar{H}_{j_2}+r) \geq 1/6.$$

(b) $k_t > u$. In this case we put $l = 2$. Similarly to (a) (considering I_2 instead of I_3 and $\bar{H}_j[0, u]$ instead of $\bar{H}_j[u+1, n_t-1]$) we show that (36) is true again. Since

$$|I_2|/\lambda_{t+1} = |I_3|/\lambda_{t+1} \geq 1/t-2/\lambda_{t+1}, \quad |I_1|/\lambda_{t+1} > 1/2,$$

by (36) we obtain a contradiction with Corollary 1. A cocycle η does not exist.

It remains to show the ergodicity of φ. But this is clear by Corollary 1, because for $j_1 \in I_1, j_2 \in I_2, r \in \mathbf{Z}_2$ we have $d(E_{j_1}, E_{j_2}+r) \geq 1/3$, which means that there is no measurable function $f\colon X \to \mathbf{Z}_2$ such that $f \circ T - f = \varphi$.

THEOREM 4. *For each regular Morse cocycle $\varphi\colon X \to \mathbf{Z}_2$ there exists an element $S \in C(T)$ such that for each $n \in \mathbf{N}$, $U \in C(T)$ there is no measurable function $f\colon X \to \mathbf{Z}_2$ satisfying the equation*

$$\varphi \circ S^{i_1} + \ldots + \varphi \circ S^{i_n} + \varphi \circ U = f \circ T - f,$$

where $i_1 < \ldots < i_n$.

Proof. We show that there is no measurable solution f of the equation

$$\varphi + \varphi \circ S + \varphi \circ S^2 + \ldots + \varphi \circ S^n + \varphi \circ U = f \circ T - f, \quad n \geq 2. \tag{37}$$

The proof of the general case is the same. Let $\{a_t\}$ be an arbitrary sequence of positive integers in which every number 1, 2, ... appears infinitely many times. Considering some subsequences of the towers $\xi_t = (D_0^t, \ldots, D_{n_t-1}^t)$ if necessary, we may assume that

$$\lambda_t/a_t \geq 4, \quad t \geq 0.$$

Put

$$q_t = \left(1-\frac{1}{n_t}\right)\frac{\lambda_t}{a_t} - 1, \quad t \geq 0.$$

Then $1 \leq q_t \leq \lambda_t - 1$. Set $l_t = \sum_{r=0}^{t} q_r n_{r-1}$, $t \geq 0$. It is not hard to see that

$$l_t \leq \frac{1}{a_t}(n_t-1), \quad \frac{l_t}{n_t} \geq \frac{1}{a_t}\left(1-\frac{1}{n_t}\right) - \frac{2}{\lambda_t}, \quad t \geq 0. \tag{38}$$

Denote the weak limit of $\{T^{l_t}\}$ by S. Assume that for some $n \geq 1$ and $U \in C(T)$ there exists a measurable f satisfying (37). Choose $0 < \varepsilon < 1/2$. Fix t (sufficiently large) such that $a_t = n+1$. Let $g = (j_t)_{t=0}^{\infty} \in X$ be an n_t-adic number such that $U = \lim_t T^{j_t}$. Consider the cocycle

$$\psi = \varphi + \varphi \circ S + \ldots + \varphi \circ S^n + \varphi \circ U.$$

This cocycle has $n+2$ holes: ψ is not constant (a.e.) on the levels $D_{n_t-1}^t$, $D_{n_t-1-l_t}^t, \ldots, D_{n_t-1-nl_t}^t, D_{n_t-1-j_t}^t$. By the definition of l_t we obtain $n_t-1-nl_t > 0$, which means that the level $D_{n_t-1-nl_t}^t$ is the lowest from among $D_{n_t-1-il_t}^t$, $i = 0, \ldots, n$. The distance between the levels $D_{n_t-1-il_t}^t$ and $D_{n_t-1-jl_t}^t$, $i \neq j$, is at least

$$n_t/(2(n+1)) \tag{39}$$

(this is a consequence of (38) and the fact that $\sum 1/\lambda_t < \infty$). Let $s = [\frac{1}{2}l_t]$. By the above we can choose $r = n_t-1$ or $r = n_t-1-nl_t$ such that ψ is constant a.e. on the levels $D_{r-s}^t, D_{r-s+1}^t, \ldots, D_{r+s}^t$ except the level D_r^t.

Now consider the equation $\psi = f \circ T - f$. It follows from (37) that

$$\delta_{r+1} = n_t \min_{a \in \mathbf{Z}_2} m\{x \in D_{r+i}^t\colon f(x) = a\}$$

are the same for $i = -s, -s+1, \ldots, 0$ and the same for $i = 1, \ldots, s$. Since f is measurable and $\xi_t \nearrow \varepsilon$ and $s/n_t \geqslant 1/(4(n+1))$ (this is true by (39)), we can find $1 \leqslant i_1 \leqslant s$, $r-s \leqslant i_2 \leqslant 0$ such that

$$\delta_{r+i_1} < \varepsilon, \quad \delta_{r+i_2} < \varepsilon$$

(if t is sufficiently large). Therefore $\delta_r < \varepsilon$, $\delta_{r+1} < \varepsilon$. So the equality $\psi = f \circ T - f$ gives

$$\min_{a \in \mathbf{Z}_2} m\{x \in D_r^t \colon \psi(x) = a\} \leqslant 2\varepsilon.$$

But this is impossible, because ε was arbitrarily small and the cocycle φ is regular.

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