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Complex interpolation and L^p spaces

by

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Abstract. A variant of the first Calderón interpolation method is defined in connection with an analytic functional, for complex interpolation families in the sense of Coifman, Cwikel, Rochberg, Sagher and Weiss. Some of its interpolation, duality and reiteration properties are described and applied to the identification of the interpolated space for a family of L^p spaces.

§ 1. Introduction. In this paper we consider interpolation spaces for interpolation families of Banach spaces in the sense of [4]. Our method is a natural extension of that of Schechter [7] and Lions [5].

Throughout the paper we shall use the notation $A \equiv B$ to indicate a two-sided inequality, that is, there exist two constants C and C' such that $CA \leqslant B \leqslant C'A$.

Let D denote the disc $\{|z| < 1\}$ and Γ its boundary, and let $\{B(\gamma);\ \gamma \in \Gamma\}$ be a complex interpolation family (c.i.f.) on Γ with $\mathscr{V}$ as the containing Banach space and $\mathscr{B}$ as the log-intersection space, in the sense of [4]. That is:

(a) The complex Banach spaces $B(\gamma)$ are continuously embedded in $\mathscr{V}$ ($\|\cdot\|_\gamma$ will be the norm on $B(\gamma)$ and $\|\cdot\|_{\mathscr{V}}$ the norm on $\mathscr{V}$),

(b) for every $b \in \bigcap_{\gamma\in\Gamma} B(\gamma)$, $\gamma \in \Gamma \to \|b\|_\gamma$ is a measurable function on Γ,

(c) $\mathscr{B} = \{b \in \bigcap_{\gamma\in\Gamma} b(\gamma);\ \int_\Gamma \log^+ \|b\|_\gamma\, d\gamma < \infty\}$, and there exists a measurable function $K(\gamma)$ on Γ such that

$$\int_\Gamma \log^+ K(\gamma)\, d\gamma < \infty \quad \text{and} \quad \|b\|_{\mathscr{V}} \leqslant K(\gamma)\|b\|_\gamma \quad \text{a.e. } \gamma\ (b \in \mathscr{B}).$$

In [4], for every $z \in D$, the Banach space $B[z] = \{f(z);\ f \in \mathscr{F}\}$ is defined with the norm $\|b\|_z = \inf\{\|f\|_{\mathscr{F}};\ f(z) = b\}$, where $\mathscr{F} = \mathscr{F}(B(\cdot), \Gamma)$ is a Banach space of $\mathscr{V}$-valued analytic functions f on D with a.e. nontangential boundary values $f(\gamma) = \mathscr{V}\text{-}\lim_{\xi\to\gamma} f(\xi)$, that can be described as the completion of the space

$$\mathscr{G} = \{g = \sum_{j=1}^N \varphi_j(\cdot) b_j;\ b_j \in \mathscr{B},\ \varphi_j \in N^+(D),\ \operatorname*{ess\,sup}_{\gamma\in\Gamma} \|g(\gamma)\|_\gamma < \infty\},$$

with the norm $\|f\|_{\mathscr{F}} = \operatorname{ess\,sup}_{\gamma\in\Gamma} \|f(\gamma)\|_\gamma$.

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Let T be an analytic functional on D. Although many of the results still hold true in the general case (see [2], [3]), in this paper T will always have "finite support". That is, it has a representation of the form:

$$T(\varphi)=\frac{1}{2\pi i}\int_{\Sigma}\varphi(\xi)h_T(\xi)\,d\xi \qquad \forall\,\varphi\in H(D) \tag{1}$$

where $h_T = P_T/Q_T$ is a rational function with poles z_j in D and $\Sigma\subset D$ is a circle $\xi = re^{it}$ $(0\leqslant t<2\pi)$ containing these poles. Hence, T can be represented as

$$T=\sum_{j=0}^{n}\sum_{l=0}^{m(j)} a_{jl}\,\delta^{(l)}(z_j). \tag{2}$$

We can associate to T the analytic functional S with $h_S = Q_T^{-1}$, that is,

$$S(\varphi)=\frac{1}{2\pi i}\int_{\Sigma}\frac{\varphi(\xi)}{Q_T(\xi)}\,d\xi. \tag{3}$$

Let R be an analytic functional on the band $0<\operatorname{Re} z<1$. For interpolation pairs (A_0, A_1) of Banach spaces, the interpolation space $[A_0, A_1]_R$ has been defined as an extension of the Calderón method $[A_0, A_1]_\theta$ ([7] and [5]). Here $R(f)$ is considered instead of the evaluation $f(\theta)$.

In the same way we define, for the c.i.f. $\{B(\gamma)\}$, the spaces

$$B[T]=\{T(f);\ f\in\mathscr{F}\},\quad\text{with}\quad \|b\|_{[T]}=\inf\{\|f\|_{\mathscr{F}};\ f\in\mathscr{F},\ T(f)=b\}.$$

If $T=\delta'(z_0)$, these spaces are related to those defined in [6] as

$$B_{z_0}^{(2)}=\{(u, v);\ \exists\,F\in\mathscr{F},\ F(z_0)=u,\ F'(z_0)(1-|z_0|^2)=v\}$$

and, in fact, we shall use similar techniques to those developed in that paper.

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The space $B[T]$ is a Banach space and, as in [4] for $B[z]$ $(T=\delta(z))$, the following theorem is easily proved.

THEOREM 1.1. *Let $\{A(\gamma)\}$ and $\{B(\gamma)\}$ be two c.i.f. on Γ, with the containing spaces $\mathscr{U}$ and $\mathscr{V}$, and the log-intersection spaces $\mathscr{A}$ and $\mathscr{B}$, respectively. Let $L\colon \mathscr{U}\to\mathscr{V}$ be a bounded linear operator and suppose that $L(\mathscr{A})\subseteq\bigcap_{\gamma\in\Gamma}B(\gamma)$ with*

$$\|La\|_\gamma\leqslant M(\gamma)\|a\|_\gamma \qquad \text{a.e. } \gamma\in\Gamma$$

where $\log M(\gamma)$ is integrable. Then $L\colon A[GT]\to B[T]$ is bounded with norm less than or equal to 1, where

$$G(z)=\exp\Big(\int_{\Gamma}-\log M(\gamma)\,dH_z(\gamma)\Big)$$

and $H_z(\gamma)=\dfrac{1}{2\pi}\dfrac{e^{i\gamma}+z}{e^{i\gamma}-z}$ is the Herglotz kernel.

Observe that using that $T=G^{-1}GT$ we find that $A[T]$ is contained in $A[GT]$ with norm less than or equal to $\sup_{z\in\operatorname{supp}T}|G(z)|$ and therefore by the previous theorem we see that $L\colon A[T]\to B[T]$ is bounded with norm less than or equal to $\sup_{z\in\operatorname{supp}T}|G(z)|$.

The following proposition for the case of a couple of Banach spaces is due to Schechter (see [7]). The proof in our case is completely similar and we only include a sketch of it.

PROPOSITION 1.2. *For T as in (2), $B[T]=\sum_{j=0}^{n}B[\delta^{(m(j))}(z_j)]$ with equivalent norms.*

Proof. The proof is a consequence of the following facts:

a) For any bounded analytic function ω on D, $B[\omega T]$ is continuously embedded in $B[T]$ with norm less than or equal to $\sup_{z\in D}|\omega(z)|$.

b) Let $\omega(\xi)=\xi-z$. Then $B[\delta(z)]=B[\omega\delta'(z)]$ is continuously embedded in $B[\delta'(z)]$ with norm less than or equal to $1+|z|$. Similarly, for $m\leqslant n$, $B[\delta^{(m)}(z)]$ is continuously embedded in $B[\delta^{(n)}(z)]$.

c) There is a bounded analytic function $\omega=\omega_{jl}$ $(0\leqslant j\leqslant n$ and $0\leqslant l\leqslant m(j))$ such that $\delta^{(l)}(z_j)=\omega T$. ∎

§2. Duality and reiteration. The following "fundamental inequality" will be useful in the sequel.

PROPOSITION 2.1. *There exist a compact set $K\subset D$ and a constant $C>0$ such that*

$$\|T(f)\|_{[T]}\leqslant C\sup_{z\in K}\exp\Big(\int_{\Gamma}\log\|f(\gamma)\|_\gamma\,dP_z(\gamma)\Big)\qquad(f\in\mathscr{F}),$$

where $P_z(\gamma)$ is the Poisson kernel.

Proof. We can suppose $T=\delta^{(n)}(z)$.

First, let us see that there is a compact set $K\subset D$ such that $B[\omega T]$ is continuously embedded in $B[T]$ with norm less than or equal to $C\sup_{z\in K}|\omega(z)|$, for any bounded analytic function $\omega(z)$ on D.

If $b\in B[\omega T]$, let $f\in\mathscr{F}$ with $\|f\|_{\mathscr{F}}\leqslant\|b\|_{[\omega T]}+\varepsilon$ and $T(\omega f)=b$. Then $\|b\|_{[T]}\leqslant\sum_{p=0}^{n}C(n, p)|\omega^{(n-p)}(z)|\,\|f^{(p)}(z)\|_{[T]}$, and using the Cauchy inequalities on a circle K with center z together with b) of Proposition 1.2 we get $\|b\|_{[T]}\leqslant C\sup_{\xi\in K}|\omega(\xi)|\,\|f\|_{\mathscr{F}}$.

Now let $\omega(z)=\exp(\int_\Gamma\log\|f(\gamma)\|_\gamma\,dH_z(\gamma))$. Then

$$\begin{aligned}\|b\|_{[T]}=\|\omega T(\omega^{-1}f)\|_{[T]}&\leqslant C\sup_{\xi\in K}|\omega(\xi)|\,\|\omega T(\omega^{-1}f)\|_{[\omega T]}\\&\leqslant C\sup_{\xi\in K}|\omega(\xi)|\,\|\omega^{-1}f\|_{\mathscr{F}}\end{aligned}$$

and we get the desired inequality. ∎

To characterize the dual of the space $B[T]$ we shall consider the class $\mathscr{W}(T)$ of functions $H\colon D\to\mathscr{B}^*$ (the algebraic dual of $\mathscr{B}$) with the following properties:

(a) For each $b\in\mathscr{B}$, $\langle b, H(\cdot)\rangle \in N^+(D)$.

(b) $\|H\|_{\mathscr{W}} = \sup_{0\neq b\in\mathscr{B}} \operatorname{ess\,sup}_{\gamma\in\Gamma} |\langle b, H(\gamma)\rangle|/\|b\|_\gamma < \infty$.

(c) $S((f, H)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(f, H)(\gamma)}{Q_T(e^{i\gamma})} e^{i\gamma}\, d\gamma = 0$ whenever $T(f) = 0$, $f\in\mathscr{F}$.

Here, for any $f\in\mathscr{F}$, the "boundary values" $(f, H)(\gamma)$ are defined in the following way:

Let $f = \mathscr{F}\text{-}\lim_n g_n$ with $g_n\in\mathscr{G}$. We have

$$\lim_{n,m}\int_\Gamma |\langle g_n(\gamma)-g_m(\gamma), H(\gamma)\rangle|\, d\gamma = 0$$

and we can assume that $\lim_n \langle g_n(\gamma), H(\gamma)\rangle = (f, H)(\gamma)$ exists a.e., and that the limit is independent of the particular sequence $(g_n)_n$.

$\mathscr{G}$ being dense in $\mathscr{F}$, $\mathscr{B}$ is dense in $B[T]$. Therefore, a continuous linear form $u\in B[T]'$ is determined by its values on $\mathscr{B}$. In the next theorem, the analytic functional S defined in (3) is used to determine those $u\in\mathscr{B}^*$ that belong to $B[T]'$.

THEOREM 2.2. *$u\in\mathscr{B}^*$ belongs to $B[T]'$ if and only if there exists an $H\in\mathscr{W}(T)$ such that*

(4) $$u(b)T(\varphi) = S(\langle b, H\rangle\varphi) \quad (b\in\mathscr{B},\ \varphi\in H(D)).$$

Moreover, $\|u\|_{B[T]'} \equiv \inf\{\|H\|_{\mathscr{W}};\ H \text{ as in (4)}\}$.

Proof. From condition (4), we see that $u(T(g)) = S(\langle g, H\rangle)$ for $g\in\mathscr{G}$. Hence,

$$|u(T(g))| \leqslant C\sup_{z\in\Sigma} |\langle g(z), H(z)\rangle| \leqslant C\|H\|_{\mathscr{W}}\|g\|_{\mathscr{F}}.$$

Let now $f\in\mathscr{F}$ and $g_n\in\mathscr{G}$ such that $f = \lim_n g_n$ and $S((f, H)) = \lim_n S(\langle g_n, H\rangle)$. It is clear that $u(T(f)) = S((f, H))$ defines $u\in B[T]'$, and $\|u\|_{B[T]'} \leqslant C\|H\|_{\mathscr{W}}$.

Conversely, let $u\in B[T]'$. From Proposition 2.1 and Jensen's inequality, we have

(5) $$|u(T(g))| \leqslant C\int_\Gamma \|g(\gamma)\|_\gamma\, d\gamma \quad (g\in\mathscr{G}).$$

The Hahn–Banach theorem allows us to extend $u\circ T$ from $\mathscr{G}$ to a continuous (with respect to the above integral norm) linear form w on the space of functions $g(\gamma) = \sum_{j=1}^m \varphi_j(\gamma)b_j$, with $b_j\in\mathscr{B}$ and $\varphi_j(\gamma)$ measurable on Γ, such that the integral norm is finite.

For each $b\in\mathscr{B}$ we define, on the space $L(b) = L^1(\Gamma, \|b\|_\gamma\, d\gamma)$, the continuous linear functional $\varphi\to w(\varphi b)$. Then there exists a unique measurable function $H(b,\cdot)$ on Γ such that

$$|H(b,\gamma)| \leqslant C\|b\|_\gamma \quad (\gamma\in\Gamma), \qquad w(\varphi b) = \int_\Gamma \varphi(\gamma)H(b,\gamma)\, d\gamma \quad (\varphi\in L(b)).$$

For any $\varphi\in H^\infty(D)$ and $\varphi_b(z) = \exp(-\int_\Gamma \log\|b\|_\gamma\, dH_z(\gamma))$,

$$w(\varphi\varphi_b b) = u(b)T(\varphi\varphi_b) = \int_\Gamma \varphi(\gamma)\varphi_b(\gamma)H(b,\gamma)\, d\gamma.$$

The function $\Psi_b(z) = \int_\Gamma Q_T(\gamma)\varphi_b(\gamma)H(b,\gamma)\, dP_z(\gamma)$ belongs to $H^\infty(D)$, because $Q_T T = 0$ implies $\hat{\Psi}(-n) = 0$ $(n = 1, 2, \ldots)$, and on Γ we have the nontangential limit values $\Psi_b(\gamma) = Q_T(\gamma)\varphi_b(\gamma)H(b,\gamma)$.

If we define $\langle b, H(z)\rangle = \Psi_b(z)/(\varphi_b(z)z)$ we have $H\in\mathscr{W}(T)$ and

$$S(\langle b, H\rangle\varphi\varphi_b) = \int_\Gamma H(b,\gamma)\varphi(\gamma)\varphi_b(\gamma)\, d\gamma = u(b)T(\varphi\varphi_b)$$

for any $\varphi\in H^\infty(D)$. Thus, we get (4) by a density argument. ∎

Remark I. In the case of an interpolation pair, M. Schechter [7] described $[A_0, A_1]'_R$ as the space of all elements $u\in(A_0\cap A_1)^*$ with the property

(6) $$uR = h'R \quad \text{for some } h\in\mathscr{F}(A'_0, A'_1),$$

where $\mathscr{F}(A'_0, A'_1)$, as $\mathscr{G}(A_0, A_1)$ and $\mathscr{F}(A_0, A_1)$, is defined in [1].

But observe that although for $x = R(g)$ with $g = \sum_{j=1}^N a_j\varphi_j$ in $\mathscr{G}(A_0, A_1)$ the function $\langle g(z), h'(z)\rangle$ is well defined and (6) implies $u(x) = R(\langle g, h'\rangle)$, when $f\in\mathscr{F}(A_0, A_1)$ the function $\langle f, h'\rangle$ has to be defined as $\langle f, h'\rangle = \lim_n\langle g_n, h'\rangle$ for $f = \lim_n g_n$ and $g_n\in\mathscr{G}(A_0, A_1)$, and a second condition, corresponding to (c) in the definition of $\mathscr{W}(T)$, is needed. This condition is the following:

(7) $$R(\langle f, h'\rangle) = 0 \quad \text{whenever } T(f) = 0.$$

If $\mathscr{F}(A'_0, A'_1; R)$ is the space of all functions $h\in\mathscr{F}(A'_0, A'_1)$ satisfying (7), we have

$$[A_0, A_1]'_R = \{u\in(A_0\cap A_1)^*;\ uR = h'R \text{ for some } h\in\mathscr{F}(A'_0, A'_1; R)\}.$$

Let now $\alpha\colon \Gamma\to[0,1]$ be a measurable function, (A_0, A_1) an interpolation pair of Banach spaces and $A_\theta = [A_0, A_1]_\theta$. Then $A(\gamma) = A_{\alpha(\gamma)}$, $\gamma\in\Gamma$, defines a c.i.f. and it is known that $A[z] = A_{\alpha(z)}$ (see [4]), for $\alpha(z) = \int_\Gamma \alpha(\gamma)\, dP_z(\gamma)$.

To identify $A[T]$ we consider $\omega(z) = \alpha(z) + i\tilde{\alpha}(z)$, an analytic function $\omega\colon D\to\Omega$ $(\Omega = \{z;\ 0 < \operatorname{Re} z < 1\})$, and we define the analytic functional R on Ω as $R(\varphi) = T(\varphi\circ\omega)$.

PROPOSITION 2.3. *$[A_0, A_1]_R$ is continuously embedded into $A[T]$, with norm less than or equal to* 1. ∎

The converse may not be true. In fact, if $T = \delta'(z)$ and $w'(z) = 0$ then $S = 0$ and the result fails.

We shall suppose, from now on, that $w'(z) \neq 0$ for z in the support of T.

In the following theorem we impose the condition of $A_0\cap A_1$ being dense in $A_{\alpha_0}\cap A_{\alpha_1}$. This condition could seem superfluous due to the fact that in [7] it is said that under the natural hypothesis of $A_0\cap A_1$ being dense in both A_0

and A_1 this always happens. However, the proof of this is done by duality and as we mention in Remark I, it seems that an extra condition for the dual space is needed and with it the above mentioned density cannot be deduced.

THEOREM 2.4. *$A[T]=[A_0, A_1]_R$ with equivalent norms in the following cases:*

(a) *The function α attains the values $\alpha_0=\sup_{\gamma\in\Gamma}\alpha(\gamma)$ and $\alpha_1=\inf_{\gamma\in\Gamma}\alpha(\gamma)$, and $A_0\cap A_1$ is dense in $A_{\alpha_0}\cap A_{\alpha_1}$.*

(b) *There is a simply connected domain Ω containing the support of T whose boundary Σ is a simple rectifiable closed curve in D such that $A_0\cap A_1$ is dense in $A_{\alpha_0}\cap A_{\alpha_1}$, with $\alpha_0=\sup_{\xi\in\Sigma}\alpha(\xi)$ and $\alpha_1=\inf_{\xi\in\Sigma}\alpha(\xi)$, and $\mathscr{A}$ is dense in $\bigcap_{\xi\in\Sigma}A[\xi]$ in the following sense: For each $b\in\bigcap_{\xi\in\Sigma}A[\xi]$ there is a sequence $(a_n)_n$ in $\mathscr{A}$ such that*

$$\lim_n\int_\Sigma\|a_n-b\|_\xi\,d\mu_z(\xi)=0,$$

where $d\mu_z$ is the harmonic measure on Σ at $z\in\Omega$.

Proof. Let $T=\delta^{(n)}(z)$.

(a) We have to see that $A[T]\subseteq[A_0, A_1]_R$. First, we observe that $\mathscr{A}=A_{\alpha_0}\cap A_{\alpha_1}$ is contained in $[A_0, A_1]_{\alpha(z_0)}$ (see [4]). Now, we have

$$\begin{aligned}R(\varphi)=T(\varphi\circ\omega)&=\delta^{(n)}(\varphi\circ\omega)(z_0)\\&=\sum_j\varphi^{(j)}(\alpha(z_0))c_j(\omega)=\Big(\sum_j c_j(\omega)\delta^{(j)}_{\alpha(z_0)}\Big)(\varphi),\end{aligned}$$

and by Proposition 1.2, $[A_0, A_1]_{\alpha(z_0)}$ is contained in $[A_0, A_1]_{\delta^{(j)}_{\alpha(z_0)}}$, for every $j\geqslant 0$. Therefore $\mathscr{A}$ is contained in $[A_0, A_1]_R$. Moreover, since $\mathscr{A}$ is dense in both $A[T]$ and $[A_0, A_1]_R$, it suffices to prove that

$$\|a\|_{[A_0,A_1]_R}\leqslant C\|a\|_{[T]}\quad(a\in\mathscr{A}),$$

i.e.,

$$\|g^{(n)}(z)\|_{[A_0,A_1]_R}\leqslant C\|g\|_{\mathscr{F}}\quad(g\in\mathscr{G}=\mathscr{G}(\{A(\gamma)\})).\tag{8}$$

Using reiteration, we show that $A_{\alpha_0}\cap A_{\alpha_1}$ is continuously embedded in $[A_0, A_1]_{\alpha(\gamma)}=A(\gamma)$ with norm less than or equal to 1 for any $\gamma\in\Gamma$, and that $g=\sum_{j=1}^N\varphi_j a_j\in\mathscr{G}$ can be approximated by functions of the same class but with $a_j\in A_0\cap A_1$. Thus, to prove (8) we can assume that this condition is fulfilled.

Let $u\in[A_0, A_1]_R'$ with $u(g^{(n)}(z))=\|g^{(n)}(z)\|_{[A_0,A_1]_R}$. Then there exists an $h\in\mathscr{F}(A_0', A_1'; R)$ such that $h'R=uR$, i.e. $R(\langle a, h'\rangle\varphi)=u(a)R(\varphi)$ for any $a\in A_0\cap A_1$. Since $R(\Psi)=\delta^{(n)}(z)(\Psi\circ\omega)$ and $\omega'(z)\neq 0$ we have $h'(\omega(z))=u$ and $h^{(j)}(\omega(z))=0$ for $2\leqslant j\leqslant n+1$. Therefore,

$$\begin{aligned}\|g^{(n)}(z)\|_{[A_0,A_1]_R}&=\langle h'(\omega(z)), g^{(n)}(z)\rangle=\langle h'\circ\omega, g\rangle^{(n)}(z)\\&\leqslant C\operatorname*{ess\,sup}_{\gamma\in\Gamma}|\langle h'(\omega(\gamma)), g(\gamma)\rangle|\leqslant\|h\|_{\mathscr{F}}\|g\|_{\mathscr{F}}\end{aligned}$$

and (8) follows.

(b) Let $B(\xi)=[A_0, A_1]_{\alpha(\xi)}$, $\xi\in\Sigma$. We have to show that $A[T]=B[T]$. For each $a\in A[T]$, let $f\in\mathscr{F}(A(\cdot), \Gamma)$ with $T(f)=x$. Then $f|_\Omega\in\mathscr{F}(B(\cdot), \Sigma)$ and $\|f|_\Omega\|_{\mathscr{F}(B(\cdot),\Sigma)}\leqslant\|f\|_{\mathscr{F}(A(\cdot),\Gamma)}$. Thus $x\in B[T]$ and $\|x\|_{B[T]}\leqslant\|x\|_{A[T]}$.

For the converse we shall use duality. If $T=\delta^{(n)}(z_0)$, then $S=T$ in Theorem 2.2 and hence, using this result and the Cauchy formula for the nth derivative on Ω, we find that there exists $H\in\mathscr{W}$ such that

$$\begin{aligned}\|T(g)\|_{A[T]}=T(\langle g, H\rangle)&\leqslant C\left|\int_\Sigma\frac{\langle g(\xi), H(\xi)\rangle}{(\xi-z)^{n+1}}d\xi\right|\\&\leqslant C\int_\Sigma\|g(\xi)\|_{B(\xi)}\,d\xi,\end{aligned}\tag{9}$$

where $g=\sum_{j=1}^N\varphi_j a_j$ with $a_j\in\mathscr{A}$ and $\varphi_j\in N^+(\Omega)$.

Let $f\in\mathscr{F}(B(\cdot), \Sigma)$ with $T(f)=x$ and $g_n=\sum_{j=1}^{N(n)}\varphi_{jn}b_{jn}\in\mathscr{G}(B(\cdot), \Sigma)$ with $f\in\mathscr{F}$-$\lim_n g_n$. We can assume $b_{jn}\in\mathscr{B}$ and $\varphi_{jn}\in H^\infty(\Omega)$ (the proof is similar to the case $T=\delta(z_0)$ [4]). Now, given $\varepsilon>0$, take $a_{jn}\in\mathscr{A}$ such that

$$\int_\Sigma\|a_{jn}-b_{jn}\|_{B(\xi)}\,d\xi\leqslant\varepsilon/(N(n)\|\varphi_j\|_\infty).$$

When $\varepsilon=1/k$, we have, for the corresponding $G_{nk}=\sum\varphi_{jn}a_{jn}$,

$$\int_\Sigma\|G_{nk}(\xi)-g_n(\xi)\|_{B(\xi)}\,d\xi\leqslant 1/k$$

and, from (9),

$$\begin{aligned}\|T(G_{nk})-T(G_{mh})\|_{A[T]}&\leqslant C\int_\Sigma\|G_{nk}(\xi)-G_{mh}(\xi)\|_{B(\xi)}\,d\xi\\&\leqslant C\Big(1/k+\int_\Sigma\|g_n(\xi)-g_m(\xi)\|_{B(\xi)}\,d\xi+1/h\Big)\\&\leqslant C(1/k+1/h+\|g_n-g_m\|_{\mathscr{F}(B(\cdot),\Sigma)}).\end{aligned}$$

Let $a\in A[T]$ be the limit of the Cauchy sequence $(T(G_{nk}))_k$. Then

$$\|T(G_{nk})-T(g_n)\|_{B[T]}\leqslant C\int_\Sigma\|G_{nk}(\xi)-g_n(\xi)\|_{B(\xi)}\,d\xi\leqslant 1/k,$$

and if $x=B[T]$-$\lim_n T(g_n)$, then $a=x=B[T]$-$\lim_k T(G_{nk})$. Therefore, $x\in A[T]$. ∎

Remark II. In case (b) of Theorem 2.4, the condition that $\mathscr{A}$ is dense in $\bigcap_{\xi\in\Sigma}A[\xi]$ is not surprising. It is the hypothesis needed in the reiteration theorem of [4]. If we define $B(\xi)=A_{\alpha(\xi)}$, $\xi\in\Sigma$, we can expect, from (a), that $B[T]=[A_0, A_1]_R$. Hence, $[A_0, A_1]_R=A[T]$ if and only if $A[T]=B[T]$ and this is a reiteration result.

§3. Interpolation of L^p spaces. Some of the results that we obtain in this section are closely related to those in [6].

Let $1\leqslant p(\gamma)\leqslant\infty$ be a measurable function on Γ and, for a fixed σ-finite measure space (Ω, Σ, μ), consider the interpolation family of Banach spaces

$$L^{p(\cdot)}=\{L^{p(\gamma)}(\mu);\ \gamma\in\Gamma\}.$$

We want to extend to this family the following results, using now our method of interpolation.

(A) $[L^{p_0}(\mu), L^{p_1}(\mu)]_\theta = L^p(\mu)$, with $1/p = (1-\theta)/p_0 + \theta/p_1$ $(0 < \theta < 1)$.

(B) $[L^{p(\cdot)}][z] = [L^\infty(\mu), L^1(\mu)]_{\alpha(z)} = L^{p(z)}(\mu)$, where $\alpha(z) = 1/p(z) = \int_\Gamma p(\gamma)^{-1} \times dP_z(\gamma)$ (see [4]).

We begin with the case $T = \delta'(\theta)$.

THEOREM 3.1. *$f \in [L^{p_0}(\mu), L^{p_1}(\mu)]_{\delta'(\theta)}$ if and only if $f = f_0 + f_1 \log|f_1|$ with f_0, f_1 in $L^p(\mu)$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ $(0 < \theta < 1)$. Furthermore,*

$$\|f\|_{[T]} \equiv \inf\left\{\left\|f_0 + f_1 \log\|f_1\|_p\right\|_p + \frac{p_0 p_1}{p}|p_0 - p_1|^{-1}\|f_1\|_p;\ f = f_0 + f_1 \log|f_1|\right\}.$$

Proof. (a) For any $f \in [L^{p_0}(\mu), L^{p_1}(\mu)]_{\delta'(\theta)}$ and $F \in \mathscr{F}(L^{p_0}, L^{p_1})$ such that $F'(\theta) = f$, consider the set $A = \{\omega \in \Omega;\ F(\theta, \omega) = 0\}$ and write

$$F(\xi, \omega) = G(\xi, \omega)(\xi - \theta)1_A(\omega) + F(\xi, \omega)1_{A^c}(\omega).$$

We have $F'(\theta, \cdot) = G(\theta, \cdot)1_A(\cdot) + f\,1_{A^c}(\cdot)$. In view of (A), it is easily seen that the first term is in $L^p(\mu)$ since $G(\xi, \cdot)1_A$ is in $\mathscr{F}(L^{p_0}, L^{p_1})$ as a function of ξ.

To see that the second term is of the type $f_1 \log|f_1|$ with $f_1 \in L^p$, we consider the function

$$H(\xi, \omega) = \|F(\theta)\|_p \frac{F(\theta, \omega)}{|F(\theta, \omega)|}\left(\frac{|F(\theta, \omega)|}{\|F(\theta)\|_p}\right)^{((1-\xi)p_1 + \xi p_0)p/(p_0 p_1)} 1_{A^c}(\omega).$$

(This function plays an important role in [6].)

As a function of ξ, $H \in \mathscr{F}(L^{p_0}, L^{p_1})$, $\|H\|_{\mathscr{F}} = \|F(\theta)\|_p = \|F(\theta)\|_{[\theta]} \leqslant \|F\|_{\mathscr{F}}$ and $H'(\theta, \cdot) = MF(\theta)(\log|F(\theta)| - \log\|F(\theta)\|_p)1_{A^c}$ with $M = p(p_0 - p_1)/(p_0 p_1)$.

Define

$$f_0^* = \big(F(\xi)1_{A^c} - H(\xi)\big)'_{\xi=\theta} = f\,1_{A^c} - MF(\theta)(\log|F(\theta)| - \log\|F(\theta)\|_p)1_{A^c}.$$

Since $F(\theta, \cdot)1_{A^c} - H(\theta, \cdot) = 0$, we see that f_0^* belongs to L^p. Moreover,

$$\|f_0^*\|_p \leqslant \max\left(\frac{1}{1-\theta}, \frac{1}{\theta}\right)\|F1_{A^c} - H\|_{\mathscr{F}} \leqslant 2\max\left(\frac{1}{1-\theta}, \frac{1}{\theta}\right)\|F\|_{\mathscr{F}}.$$

Therefore,

$$f = F'(\theta) = G(\theta)1_A + f\,1_{A^c} = G(\theta)1_A + f_0^* + H'(\theta) = f_0 + f_1 \log|f_1|$$

with $f_1 = MF(\theta)1_{A^c}$ and $f_0 = G(\theta)1_A + f_0^* - f_1 \log\|f_1\|_p$, both in $L^p(\mu)$.

To estimate the norms observe that

$$\left\|f_0 + f_1 \log\|f_1\|_p\right\|_p = \|G(\theta)1_A + f_0^*\|_p \leqslant 3\max\left(\frac{1}{1-\theta}, \frac{1}{\theta}\right)\|F\|_{\mathscr{F}}$$

whenever $F'(\theta) = f$.

To prove the converse result and the corresponding norm inequality consider f_0 and f_1 in L^p, $f = f_0 + f_1 \log|f_1|$ and

$$(10)\qquad F(\xi) = \|f_1\|_p \frac{f_1}{|f_1|}\left(\frac{|f_1|}{\|f_1\|_p}\right)^{((1-\xi)p_1 + \xi p_0)p/(p_0 p_1)}.$$

A straightforward calculation shows that this function belongs to $\mathscr{F}(L^{p_0}, L^{p_1})$, $\|F\|_{\mathscr{F}} = \|f_1\|_p$, and $F'(\theta) \in [L^{p_0}, L^{p_1}]_{\delta'(\theta)}$. But $F'(\theta) = Mf_1(\log|f_1| - \log\|f_1\|_p)$, and hence $f = f_0 + f_1 \log\|f_1\|_p + M^{-1}F'(\theta)$ belongs to $[L^{p_0}, L^{p_1}]_{\delta'(\theta)}$. The norm inequality follows from this decomposition of f.

The case p_0 or $p_1 = \infty$ follows as usual. ∎

Remark III. (a) Substitution of F' and H' by the higher order derivatives and induction give the result for $T = \delta^{(n)}(\theta)$:

$$[L^{p_0}, L^{p_1}]_{\delta^{(n)}(\theta)} = \left\{f;\ f = \sum_{j=0}^{n} f_j(\log|f_j|)^j, \text{ with } f_j \in L^p(\mu)\ (j = 0, \ldots, n)\right\}.$$

To prove it, let us see that if $f \in \mathscr{F}$ and $F(\theta, \cdot) = 0$, then $F^{(n)}(\theta) \in L^{p(\cdot)}[\delta^{(n-1)}(\theta)]$.

Let $F \in \mathscr{F}$ such that $F^{(n)}(\theta) = f$ and consider, as before, the decomposition $F(\xi, \cdot) = G(\xi, \cdot)(\xi - \theta)1_A(\cdot) + F1_{A^c}(\cdot)$ and the function H. Then

$$F^{(n)}(\theta, \cdot) = nG^{(n-1)}(\theta, \cdot)1_A(\cdot) + (F1_{A^c}(\cdot) - H)^{(n)}(\theta, \cdot) + H^{(n)}(\theta, \cdot).$$

The first two terms on the right are in $L^{p(\cdot)}[\delta^{(n-1)}(\theta)]$ and the last one is of the form $f_n(\log|f_n|)^n$ with $f_n \in L^p(\mu)$. So the first part of the proof follows by induction.

Conversely, let $f \in \sum_{j=1}^{n} f_j(\log|f_j|)^j$ and consider $F(\xi)$ as in (10) with f_n instead of f_1. We have

$$f_n(\log|f_n|)^n = F^{(n)}(\theta) + g$$

with $g = \sum_{j=1}^{n-1} f_j^*(\log|f_j^*|)^j$ and $f_j^* \in L^p(\mu)$. As before, an induction argument finishes things off.

(b) It is easy to see that the convergence to 0 of a sequence $(f^m)_m$ in $[L^{p_0}(\mu), L^{p_1}(\mu)]_{\delta^{(n)}(\theta)}$ means that there are decompositions $f^m = \sum_{j=0}^{n} f_j^m(\log|f_j^m|)^j$ with $\lim_m f_j^m = 0$ in $L^p(\mu)$, for $j = 1, \ldots, n$.

Using now (a), (b) above and also 1.2 and 2.4 we get the following result, for T as in (1) and $w'(z_j) \neq 0$ $(j = 1, \ldots, n)$:

THEOREM 3.2. *$f \in L^{p(\cdot)}[T]$ if and only if there exists a family of functions*

$$f_{k,j} \in L^{p(z_j)}(\mu) \quad (j = 1, \ldots, n \text{ and } k = 0, \ldots, m(j))$$

such that

$$f = \sum_{j=1}^{n} \sum_{k=0}^{m(j)} f_{k,j}(\log|f_{k,j}|)^k.$$

Furthermore, $\lim_m f^m = 0$ *in* $L^{p(\cdot)}[T]$ *if and only if we have a decomposition* $f^m = \sum_j \sum_k f^m_{k,j}(\log|f^m_{k,j}|)^k$ *such that* $\lim_m f^m_{k,j} = 0$ *in* $L^{p(z_j)}(\mu)$ *for all* $j = 1, \ldots, n$ *and* $k = 0, \ldots, m(j)$. ∎

We use now Orlicz spaces to indentify the space $L^{p(\cdot)}[T]$.

We begin with $T = \delta'(z_0)$ so that every $f \in L^{p(\cdot)}[\delta'(z_0)]$ is of the form $f = g + h\log|h|$ with $g, h \in L^{p(z_0)}(\mu)$.

Let $\varphi(x) = x(1+|\log x|)$ for $x \geqslant 0$, a continuous increasing function such that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. If we define $\phi(t) = \varphi^{-1}(t)^p$ ($p = p(z_0)$, $1 \leqslant p < \infty$) then ϕ has the same properties as φ and also ϕ is convex in $[0, 1]$ and satisfies the Δ_2-condition.

We shall consider the Orlicz space

$$L_\phi(\mu) = \{f;\ f \text{ measurable and } \int_\Omega \phi(|f|)\,d\mu < \infty\}.$$

$L_\phi(\mu)$ is an F-space (see [8]) with a basis of neighbourhoods of 0 given by

$$\{f;\ \int_\Omega \phi(|f|)\,d\mu \leqslant r\} \qquad (r > 0).$$

From the indentity $\phi(x(1+|\log x|)) = x^p$ we get $\phi(t) \leqslant t^p$ and hence $L^p(\mu)$ is continuously embedded in $L_\phi(\mu)$.

THEOREM 3.3. $L^{p(\cdot)}[\delta'(z_0)] = L_\phi(\mu)$ *with equivalent norms.*

Proof. Let $f \in L_\phi(\mu)$, $f \geqslant 0$, and consider $g = \varphi^{-1}(f)$. Then $f = g(1+|\log g|)$ with $g = \phi(|f|)^{1/p} \in L^p(\mu)$, and $f \in L^{p(\cdot)}[\delta'(z_0)]$.

Conversely, take $f = g + h\log|h|$ in $L^{p(\cdot)}[\delta'(z_0)]$ ($g, h \in L^p(\mu)$) and observe that

$$\int_\Omega \phi\big(|h|(1+|\log|h||)\big)\,d\mu = \int_\Omega |h|^p\,d\mu$$

and that $L^p(\mu)$ is continuously embedded in $L_\phi(\mu)$. The proof now follows using Remark III (b). ∎

Remark IV. Define $\varphi_m(x) = x(m+|\log x|)^m$ and $\phi_m(t) = \varphi_m^{-1}(t)^{1/p}$. Then the above argument can be generalized to get

$$L^{p(\cdot)}[\delta^{(m)}(z_0)] = L_{\phi_m}(\mu), \quad \text{for all } m \in \mathbf{N},$$

$$L^{p(\cdot)}[T] = \sum_{j=1}^{n} L_{\phi_{m(j)}}(\mu).$$

Remark V. If $\phi_n(x(n+|\log x|)^n) = x^p$, it is easily seen that ϕ_n is equivalent, at 0 and ∞, to the function $\varphi_n(x) = (x(n+|\log x|)^{-n})^p$. Hence $f \in [L^{p_0}(\mu), L^{p_1}(\mu)]_{\delta^{(n)}(z_0)}$ if and only if

$$f(n+|\log|f||)^{-n} \in L^p(\mu),$$

and $\lim_m f_m = 0$ in $[L^{p_0}(\mu), L^{p_1}(\mu)]_{\delta^{(n)}(z_0)}$ is equivalent to $\lim_m f_m(n+|\log|f_m||)^{-n} = 0$ in the $L^p(\mu)$-norm.

Remark VI. All the previous results hold true for vector-valued functions, that is, in the case of the interpolation family

$$L_B^{p(\cdot)} = \{L_B^{p(\gamma)}(\mu);\ \gamma \in \Gamma\},$$

with B a Banach space. To see this, it is enough to change the absolute value $|\cdot|$ to the corresponding norm $\|\cdot\|_B$.

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