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Interpolation of Orlicz-valued function spaces and U.M.D. property

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Abstract. We show that a variant of an interpolation functor introduced by J. Gustavsson and J. Peetre commutes with L^p. As a consequence, reflexive Orlicz spaces as well some other function spaces modeled on Orlicz spaces (e.g. Orlicz–Besov–Hardy–Sobolev spaces and the Schatten–Orlicz class) have the U.M.D. property.

1. Introduction. Let E be a Banach space, let $(\Omega, \mathscr{A}, P)$ be a probability space and let $(\mathscr{A}_k)_{k\in\mathbf{Z}}$ be an increasing sequence of sub-σ-algebras which generates $\mathscr{A}$. A sequence of E-valued random variables $(d_k)_{k\geqslant 1}$, with $d_0 = 0$, adapted to $(\mathscr{A}_k)_k$ (i.e., d_k is $\mathscr{A}_k$-measurable) is a sequence of *martingale differences* if $\mathscr{E}(d_k|\mathscr{A}_{k-1}) = 0$. We say that an E-valued sequence of martingale differences $(d_k)_{k\geqslant 1}$ is *unconditional* in $L^p(\mathbf{R}, E)$, $1 < p < \infty$, if for all sequences $(\varepsilon_k)_k$ with $\varepsilon_k = \pm 1$,

$$\Big\|\sum_{k=1}^{n} \varepsilon_k d_k\Big\|_{L^p(\mathbf{R},E)} \leqslant C\Big\|\sum_{k=1}^{n} d_k\Big\|_{L^p(\mathbf{R},E)}, \qquad n \geqslant 1,$$

where the constant C is independent of n. It is well known that all $\mathbf{R}$-valued martingale differences are unconditional, but this is not the case for a general Banach space E. Thus, we say that a Banach space has the *U.M.D. property* (Unconditional Martingale Differences property) or is a *U.M.D. space* if *all* E-valued sequences of martingale differences are unconditional.

It turns out that the U.M.D. property is also a necessary and sufficient condition on a Banach space E in order that the Hilbert transform induces a bounded linear mapping H from $L^p(\mathbf{R}, E)$ into itself. Moreover, Banach spaces with the U.M.D. property have a geometrical characterization, namely they are the ζ-convex spaces (see Burkholder [5]).

From the scalar case and Fubini's theorem it follows that the spaces $L^q(X, \mu)$, where μ is a σ-finite measure on X and $1 < q < \infty$, are U.M.D. spaces. In particular, it follows that the sequence spaces l^q and the weighted spaces L^q_w are U.M.D., $1 < q < \infty$. The Schatten class S_p is also a U.M.D. space (see Bourgain [2]).

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It is also well known that the U.M.D. property is stable by interpolation, i.e., if (E_0, E_1) is a compatible couple of U.M.D. Banach spaces then the intepolation spaces $[E_0, E_1]_\theta$ and $(E_0, E_1)_{\theta,q}$ are also U.M.D. spaces, whenever $0<\theta<1$ and $1<q<\infty$. Thus, Lorentz spaces $L^{p,q}$ and Schatten–Lorentz classes $S_{p,q}$ are also U.M.D. spaces, for $1<p,q<\infty$.

In [6], F. Cobos proved that reflexive Lorentz–Zygmund spaces $L_{pq}(\log L)^\gamma$ are U.M.D. spaces by stressing the use of the real interpolation method with a *function parameter*. As a follow-up of that paper, F. Cobos and the first-named author remarked in [7] that reflexive Besov spaces and Hardy–Sobolev spaces, with a function parameter, are also U.M.D. spaces.

In the present paper we shall focus our attention on Orlicz spaces. It is already known that reflexive Orlicz sequence spaces have the U.M.D. property (this follows from a result on the structure of these spaces ([11])). We shall show that this remains true for general reflexive Orlicz spaces.

Our main tool will be a variant of the interpolation method introduced by J. Gustavsson and J. Peetre in [11]. We begin by showing that this variant commutes with L^p. The U.M.D. property for the reflexive Orlicz spaces will then follow after a characterization of these spaces as interpolation spaces between L^p-spaces. It is interesting to point out that the Gustavsson–Peetre interpolation method is modeled on spaces of unconditional sequences.

The present paper reports the definitive version of results presented to the 26° Seminário Brasileiro de Análise (Rio de Janeiro, Nov. 1987) (see [9]).

The authors are indebted to F. Cobos, of Madrid, for his criticism on an early version of this paper.

2. The Gustavsson–Peetre interpolation method. We now introduce our variant of the Gustavsson–Peetre interpolation method. It will depend on function parameters.

2.1. *The function parameters.* The function parameter we shall use will be taken in Peetre's class $\mathscr{P}^{+-}$ (see [11]), i.e., the class of pseudoconcave positive functions ϱ on $\mathbf{R}_+$ which satisfy

(1) $$\bar{\varrho}(t) = \sup_{s>0} \varrho(st)/\varrho(s) = o(\max(1, t)).$$

For a function parameter $\varrho \in \mathscr{P}^{+-}$, we can replace (1) by $\bar{\varrho}(t) = o(\max(t^\varepsilon, t^{1-\varepsilon}))$, provided $\varepsilon>0$ is small enough. Consequently, we have

$$\int_0^\infty \min(1, 1/t)\,\bar{\varrho}(t)\,dt/t < \infty,$$

which implies

(2) $$\sum_{n=0}^{\infty} \varrho(2^n)/2^n + \sum_{n=0}^{\infty} \varrho(2^{-n}) < \infty.$$

We shall also consider the sequence $(\tilde{r}_n(\cdot))_{n\in\mathbf{Z}}$ defined in $[0, 1]$ by

$$\tilde{r}_n = \begin{cases} r_{2n} & \text{if } n \geqslant 0, \\ r_{2|n|-1} & \text{if } n < 0, \end{cases}$$

where $(r_n(\cdot))_{n\in\mathbf{Z}_+}$ is the sequence of Rademacher functions.

2.2. *The interpolation spaces.* Let (E_0, E_1) be a Banach couple. The space $\langle E_0, E_1\rangle_{\varrho,p}$ is the linear space of all $x\in E_0+E_1$ such that there is a sequence $(u_n)_{n\in\mathbf{Z}}$ in $E_0\cap E_1$ which satisfies

(1) $$x = \sum_{n=-\infty}^{\infty} u_n \quad (\text{convergence in } E_0+E_1),$$

(2) $$\sup_J \int_0^1 \Big\| \sum_{n\in J} \tilde{r}_n(t) \frac{2^{kn}}{\varrho(2^n)} u_n \Big\|_{E_k}^p dt < \infty,$$

where the supremum is taken over all *finite* subsets of $\mathbf{Z}$. Moreover, we shall also assume that, for $k=0, 1$, the sequence $(\tilde{r}_n(\cdot)\,2^{kn}u_n/\varrho(2^n))_{n\in\mathbf{Z}}$ is $L^p([0, 1], E_k)$-summable.

We equip the space $\langle E_0, E_1\rangle_{\varrho,p}$ with the norm

(3) $$\|x\|_{\langle E_0,E_1\rangle_{\varrho,p}} = \inf \max_{k=0,1} \sup_J \Big\| \sum_{n\in J} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} u_n \Big\|_{L^p([0,1],E_k)},$$

where the infimum is taken over all admissible sequences $(u_n)_{n\in\mathbf{Z}}$.

2.3. THEOREM. *Let (E_0, E_1) be a Banach couple, $\varrho\in\mathscr{P}^{+-}$ and $1\leqslant p<\infty$. Then $\langle E_0, E_1\rangle_{\varrho,p}$ is a Banach space.*

Proof. Step 1. Write $\lambda_{\varrho,p} = \lambda_{\varrho,p}(E_0, E_1)$ for the linear space of all sequences $(u_n)_{n\in\mathbf{Z}}$ in $E_0\cap E_1$ such that

$$\|(u_n)_{n\in\mathbf{Z}}\|_{\lambda_{\varrho,p}} = \max_{k=0,1} \sup_J \Big\| \sum_{n\in J} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} u_n \Big\|_{L^p([0,1],E_k)} < \infty,$$

and $(\tilde{r}_n(\cdot)\,2^{kn}u_n/\varrho(2^n))_{n\in\mathbf{Z}}$ is $L^p([0, 1], E_k)$-summable. It is not hard to show that $\lambda_{\varrho,p}$ is complete under the norm $\|\cdot\|_{\lambda_{\varrho,p}}$. In fact, let $(U^n)_{n\in\mathbf{N}} = ((u_j^n)_{j\in\mathbf{Z}})_{n\in\mathbf{N}}$ be an absolutely summable sequence in $\lambda_{\varrho,p}$. Then, setting $M_j = \max\{\varrho(2^j), \varrho(2^j)/2^j\}$ for each $j\in\mathbf{Z}$, we have

$$\sum_{n=1}^{\infty} \|u_j^n\|_{E_0\cap E_1} \leqslant M_j\Big(\sum_{n=1}^{\infty} \Big\|\frac{1}{\varrho(2^j)}u_j^n\Big\|_{E_0} + \Big\|\frac{2^j}{\varrho(2^j)}u_j^n\Big\|_{E_1}\Big)$$

$$\leqslant M_j\Big(\sum_{n=1}^{\infty} \max_{k=0,1} \sup_J \Big\|\sum_{m\in J} \tilde r_m(\cdot)\frac{2^{km}}{\varrho(2^m)}u_m^n\Big\|_{L^p([0,1],E_k)}\Big)$$

$$\leqslant M_j\big(\sum_{n=1}^{\infty} \|U^n\|_{\lambda_{\varrho,p}}\big),$$

i.e. $(u_j^n)_{n\in\mathbf{N}}$ is $E_0\cap E_1$-summable. Next, setting $u_j = \sum_{n=1}^{\infty} u_j^n$ and $U = (u_j)_{j\in\mathbf{Z}}$, we have $U\in\lambda_{\varrho,p}$. Indeed,

$$\|U\|_{\lambda_{\varrho,p}} = \max_{k=0,1}\sup_J \Big\|\sum_{j\in J}\tilde r_j(\cdot)\frac{2^{kj}}{\varrho(2^j)}u_j\Big\|_{L^p([0,1],E_k)}$$
$$= \max_{k=0,1}\sup_J \Big\|\sum_{j\in J}\tilde r_j(\cdot)\frac{2^{kj}}{\varrho(2^j)}\sum_{n=1}^{\infty}u_j^n\Big\|_{L^p([0,1],E_k)}$$
$$\leqslant \sum_{n=1}^{\infty}\max_{k=0,1}\sup_J \Big\|\sum_{j\in J}\tilde r_j(\cdot)\frac{2^{kj}}{\varrho(2^j)}u_j^n\Big\|_{L^p([0,1],E_k)} = \sum_{n=1}^{\infty}\|U^n\|_{\lambda_{\varrho,p}} < \infty.$$

Now, in order to show that $U = \sum_{n=1}^{\infty} U^n$ in $\lambda_{\varrho,p}$, given $\varepsilon > 0$, let $n(\varepsilon)$ be large enough so that

$$\sum_{n=n(\varepsilon)+1}^{\infty} \|U^n\|_{\lambda_{\varrho,p}} < \varepsilon.$$

Therefore,

$$\Big\|U - \sum_{n=0}^{n(\varepsilon)} U^n\Big\|_{\lambda_{\varrho,p}} = \Big\|\Big(\sum_{n=n(\varepsilon)+1}^{\infty} u_j^n\Big)_j\Big\|_{\lambda_{\varrho,p}} \leqslant \sum_{n=n(\varepsilon)+1}^{\infty}\|U^n\|_{\lambda_{\varrho,p}} < \varepsilon.$$

It remains to prove the L^p-summability of $\big(\tilde r_j(\cdot)2^{jk}u_j/\varrho(2^j)\big)_{j\in\mathbf{Z}}$. Then, since $U^n = (u_j^n)_{j\in\mathbf{Z}}\in\lambda_{\varrho,p}$, for each $n\in\mathbf{N}$, there exists a finite subset $J_\varepsilon^n \subset \mathbf{Z}$ such that

$$\Big\|\sum_{j\in J}\tilde r_j(\cdot)\frac{2^{kj}}{\varrho(2^j)}u_j^n\Big\|_{L^p([0,1],E_k)} \leqslant \varepsilon/2^n$$

for all finite subsets $J\subset\mathbf{Z}$ with $J\cap J_\varepsilon^n = \emptyset$. Next, for $n(\varepsilon)$ as above, set $J_\varepsilon = \bigcup_{n=1}^{n(\varepsilon)} J_\varepsilon^n$. Hence, for all finite subsets $J\subset\mathbf{Z}$ with $J\cap J_\varepsilon = \emptyset$, we have

$$\Big\|\sum_{j\in J}\tilde r_j(\cdot)\frac{2^{kj}}{\varrho(2^j)}u_j\Big\|_{L^p([0,1],E_k)} \leqslant \sum_{n=1}^{\infty}\Big\|\sum_{j\in J}\tilde r_j(\cdot)\frac{2^{kj}}{\varrho(2^j)}u_j^n\Big\|_{L^p([0,1],E_k)}$$
$$\leqslant \sum_{n=1}^{n(\varepsilon)}\ldots + \sum_{n=n(\varepsilon)+1}^{\infty}\ldots$$
$$\leqslant \sum_{n=1}^{n(\varepsilon)}\varepsilon/2^n + \sum_{n=n(\varepsilon)+1}^{\infty}\|U^n\|_{\lambda_{\varrho,p}} < \varepsilon.$$

Step 2. Consider the subspace $\mathcal{N}$ of all sequences $u = (u_n)_{n\in\mathbf{Z}}$ in $\lambda_{\varrho,p}$ such that $\sum_{n\in\mathbf{Z}} u_n = 0$ in E_0+E_1. We shall show that $\mathcal{N}$ is a closed subspace of $\lambda_{\varrho,p}$.

Let $(U^j)_{j\in\mathbf{N}} = ((u_n^j)_{n\in\mathbf{Z}})_{j\in\mathbf{N}}$ be a sequence in $\mathcal{N}$ and $U = (u_n)_{n\in\mathbf{Z}}\in\lambda_{\varrho,p}$ be the $\lambda_{\varrho,p}$-limit of $(U^j)_{j\in\mathbf{N}}$. We have to show that $U\in\mathcal{N}$. Thus, given $\varepsilon>0$, there exists a j_0 such that for all $n\in\mathbf{Z}$, we have

$$\text{(1)}\qquad \Big\|\frac{2^{kn}}{\varrho(2^n)}(u_n^{j_0}-u_n)\Big\|_{E_k} \leqslant \varepsilon,\quad k = 0,1.$$

On the other hand, there exists a finite subset $J_\varepsilon^{j_0}\subset\mathbf{Z}$ such that for all finite subsets J with $J\supset J_\varepsilon^{j_0}$, we have

$$\text{(2)}\qquad \Big\|\sum_{n\in J}u_n^{j_0}\Big\|_{E_0+E_1} < \varepsilon.$$

Consequently,

$$\Big\|\sum_{n\in J}u_n\Big\|_{E_0+E_1} \leqslant \Big\|\sum_{n\in J}(u_n^{j_0}-u_n)\Big\|_{E_0+E_1} + \Big\|\sum_{n\in J}u_n^{j_0}\Big\|_{E_0+E_1}$$
$$\leqslant \sum_{n\in J_+}\frac{\varrho(2^n)}{2^n}\cdot\frac{2^n}{\varrho(2^n)}\|u_n^{j_0}-u_n\|_{E_1} + \sum_{n\in J_-}\varrho(2^n)\frac{1}{\varrho(2^n)}\|u_n^{j_0}-u_n\|_{E_0}+\varepsilon$$
$$\leqslant \varepsilon\Big(\sum_{n=-\infty}^{-1}\varrho(2^n) + \sum_{n=0}^{\infty}\varrho(2^n)/2^n + 1\Big),$$

where J_+ and J_- have the obvious meaning. Therefore, since $\varrho\in\mathcal{P}^{+-}$ and ε is arbitrary, we get $\sum_{n\in\mathbf{Z}}u_n = 0$. Consequently, $\mathcal{N}$ is a closed subspace of $\lambda_{\varrho,p}$ and $\lambda_{\varrho,p}/\mathcal{N}$ is a Banach space. Finally, since it can be shown that $\lambda_{\varrho,p}/\mathcal{N}$ is isometric and isomorphic to $\langle E_0, E_1\rangle_{\varrho,p}$, the assertion follows.

To show that $\langle E_0, E_1\rangle_{\varrho,p}$ is an intermediate space between E_0 and E_1 we need two lemmas.

2.4. Lemma. *For all Banach spaces E, we have $\langle E, E\rangle_{\varrho,p} = E$.*

Proof. The embedding $E\hookrightarrow\langle E, E\rangle_{\varrho,p}$ is immediate. On the other hand, we have $\langle E, E\rangle_{\varrho,p}\subset E$. It remains to prove that the embedding is bounded. Let $x\in\langle E, E\rangle_{\varrho,p}$ and let (u_n) be an admissible sequence in E for x such that

$$\sup_J\Big\|\sum_{n\in J}\tilde r_n(\cdot)\frac{2^{kn}}{\varrho(2^n)}u_n\Big\|_{L^p([0,1],E)} \leqslant 2\|x\|_{\langle E,E\rangle_{\varrho,p}}.$$

Hence, for each n in $\mathbf{Z}$ and $k = 0, 1$, we have

$$\Big\|\frac{2^{kn}}{\varrho(2^n)}u_n\Big\|_E \leqslant 2\|x\|_{\langle E,E\rangle_{\varrho,p}},$$

and, recalling 2.1(2), it follows that

$$\|x\|_E \leqslant \sum_{n=-\infty}^{\infty}\|u_n\|_E = \sum_{n=0}^{\infty}\|u_n\|_E + \sum_{n=-\infty}^{-1}\|u_n\|_E$$
$$\leqslant 2\|x\|_{\langle E,E\rangle_{\varrho,p}}\Big(\sum_{n=0}^{\infty}\varrho(2^n)/2^n + \sum_{n=-\infty}^{-1}\varrho(2^n)\Big) \leqslant C\|x\|_{\langle E,E\rangle_{\varrho,p}}.$$

2.5. Lemma. *Let (E_0, E_1) and (F_0, F_1) be Banach couples such that $E_k \hookrightarrow F_k$, $k = 0, 1$. Then $\langle E_0, E_1\rangle_{\varrho,p} \hookrightarrow \langle F_0, F_1\rangle_{\varrho,p}$.*

Proof. Immediate.

2.6. Theorem. *If (E_0, E_1) is a Banach couple we have*

$$E_0 \cap E_1 \hookrightarrow \langle E_0, E_1\rangle_{\varrho,p} \hookrightarrow E_0+E_1,$$

i.e., $\langle E_0, E_1\rangle_{\varrho,p}$ is an intermediate space between E_0 and E_1.

Proof. The left embedding is clear. Now, since $E_k \hookrightarrow E_0+E_1$, Lemmas 2.4 and 2.5 yield the right embedding.

One basic fact is that the functor $(E_0, E_1) \mapsto \langle E_0, E_1\rangle_{\varrho,p}$ is an interpolation functor.

2.7. Theorem. *Let (E_0, E_1) and (F_0, F_1) be Banach couples, $\varrho \in \mathscr{P}^{+-}$ and $1 \leqslant p < \infty$. Then, for all $T \in L(E_k, F_k)$, $k = 0, 1$, we have $T \in L(\langle E_0, E_1\rangle_{\varrho,p}, \langle F_0, F_1\rangle_{\varrho,p})$. Moreover, $\|T\| \leqslant \max[\|T\|_0, \|T\|_1]$.*

Proof. Let $x \in \langle E_0, E_1\rangle_{\varrho,p}$, let $(u_n)_{n\in\mathbf{Z}}$ be an admissible sequence for x, and set $M_k = \|T\|_{L(E_k,F_k)}$, $k = 0, 1$.

Since the convergence in 2.2(1) is in E_0+E_1, we have $Tx = \sum_{n\in\mathbf{Z}} Tu_n$. Hence, setting $v_n = Tu_n$, we see that $\sum_{n\in\mathbf{Z}} v_n$ converges in F_0+F_1 and is a representation of Tx. Next, for $k = 0, 1$, we get

$$\Big\|\sum_{n\in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} v_n\Big\|_{L^p([0,1],F_k)} = \Big\|T\Big(\sum_{n\in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n\Big)\Big\|_{L^p([0,1],F_k)}$$
$$\leqslant M_k \Big\|\sum_{n\in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n\Big\|_{L^p([0,1],E_k)},$$

and consequently $\|Tx\|_{\langle E_0,E_1\rangle_{\varrho,p}} \leqslant C\|x\|_{\langle E_0,E_1\rangle_{\varrho,p}}$. The proof is complete.

Actually, the spaces $\langle E_0, E_1\rangle_{\varrho,p}$ do not depend on p. This fact will be a consequence of a result due to J.-P. Kahane.

2.8. Lemma (Kahane, see [13, p. 74]). *For every $1 < r < \infty$, for any Banach space E and every finite sequence $(u_j)_{1\leqslant j\leqslant n}$ in E, we have*

$$\int_0^1 \Big\|\sum_{j=1}^n r_j(t)u_j\Big\|_E dt \approx \Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t)u_j\Big\|_E^r dt\Big)^{1/r}.$$

2.9. Theorem. *Let (E_0, E_1) be a Banach couple, $\varrho \in \mathscr{P}^{+-}$ and $1 \leqslant p,q < \infty$. Then*

$$\langle E_0, E_1\rangle_{\varrho,p} = \langle E_0, E_1\rangle_{\varrho,q},$$

with norm equivalence.

Proof. This follows at once from Kahane's theorem.

Next, we state a density theorem. It will be the sole result, in this section, which depends on the L^p-summability requirement in Definition 2.2.

2.10. Theorem. *Let (E_0, E_1) be a Banach couple, $\varrho \in \mathscr{P}^{+-}$ and $1 \leqslant p < \infty$. Then $E_0 \cap E_1$ is dense in $\langle E_0, E_1\rangle_{\varrho,p}$.*

Proof. Let $x \in \langle E_0, E_1\rangle_{\varrho,p}$ and let (u_n) be an admissible sequence in $E_0 \cap E_1$ for x. Then, given $\varepsilon > 0$, there exists J_ε such that

$$\Big\|\sum_{n\in J'} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n\Big\|_{L^p([0,1],E_k)} < \varepsilon, \qquad k = 0, 1,$$

for all finite sets $J' \subset \mathbf{Z}$ with $J' \cap J_\varepsilon = \varnothing$. Next, setting $x_j = \sum_{n=-j}^{j} u_n$, we see that $x_j \in E_0 \cap E_1$ and

$$x - x_j = \sum_{|n|>j} u_n.$$

For $j > j_0 = \max\{|n| \mid n \in J_\varepsilon\}$, set $J_j = \{n \mid |n| \leqslant j\}$. Then

$$\|x - x_j\|_{\langle E_0,E_1\rangle_{\varrho,p}} \leqslant \max_{k=0,1} \sup_{J\cap J_j=\varnothing} \Big\|\sum_{n\in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n\Big\|_{L^p([0,1],E_k)}.$$

But $J \cap J_j = \varnothing$ implies $J \cap J_\varepsilon = \varnothing$. Consequently, $\|x-x_j\|_{\langle E_0,E_1\rangle_{\varrho,p}} < \varepsilon$.

To close this section, we compare the Lions–Peetre interpolation method with a function parameter with the variant of the Gustavsson–Peetre interpolation method.

2.11. Let (E_0, E_1) be a Banach couple, $1 \leqslant q \leqslant \infty$ and $\varrho \in \mathscr{P}^{+-}$. We say that $x \in E_0+E_1$ belongs to the *Lions–Peetre interpolation space* $(E_0, E_1)_{\varrho,q;J}$ if there exists a sequence $(u_n)_{n\in\mathbf{Z}}$, in $E_0 \cap E_1$, which satisfies

(1) $$x = \sum_{n\in\mathbf{Z}} u_n \quad \text{(convergence in } E_0+E_1),$$

(2) $$(u_n/\varrho(2^n))_{n\in\mathbf{Z}} \in l^q(E_0),$$

(3) $$(u_n 2^n/\varrho(2^n))_{n\in\mathbf{Z}} \in l^q(E_1).$$

We equip the space $(E_0, E_1)_{\varrho,q;J}$ with the norm

(4) $$\|x\|_{\varrho,q;J} = \inf_{x=\sum u_n} \max\big[\|(u_n/\varrho(2^n))_n\|_{l^q(E_0)}, \|(u_n 2^n/\varrho(2^n))_n\|_{l^q(E_1)}\big].$$

For $x \in (E_0, E_1)_{\varrho,q;J}$ and $(u_n)_{n\in\mathbf{Z}}$ satisfying 2.1(1) we have

(5) $$\sum_{n\in\mathbf{Z}} \|u_n\|_{E_0+E_1} < \infty.$$

Also, we say that $x \in E_0+E_1$ belongs to $(E_0, E_1)_{\varrho,\infty;K}$ if

$$\|x\|_{\varrho,\infty;K} = \sup_{t>0} \varrho(t)^{-1} K(t, x) < \infty, \tag{6}$$

where K is Peetre's K-functional.

Then, for all $\varrho \in \mathscr{P}^{+-}$ and $1 \leqslant p < \infty$, we have

$$(E_0, E_1)_{\varrho,1;J} \hookrightarrow \langle E_0, E_1 \rangle_{\varrho,p} \hookrightarrow (E_0, E_1)_{\varrho,\infty;K}. \tag{7}$$

3. Interpolation of $L^p(E)$ spaces. We shall show that L^p-spaces commute with the interpolation space $\langle E_0, E_1\rangle_{\varrho,p}$. First we need some lemmas.

3.1. LEMMA (Carlson). *Let E be a Banach lattice, $\varrho \in \mathscr{P}^{+-}$ and $1 \leqslant p < \infty$. Then if $(u_j)_{1\leqslant j\leqslant n}$ is a finite sequence in E, then*

$$\Big\|\sum_{j=1}^{n} |u_j|\Big\|_E \leqslant CR\left(\left\|\left(\sum_{j=1}^{n}\left(\frac{|u_j|}{\varrho(2^j)}\right)^p\right)^{1/p}\right\|_E, \left\|\left(\sum_{j=1}^{n}\left(\frac{2^j|u_j|}{\varrho(2^j)}\right)^p\right)^{1/p}\right\|_E\right), \tag{1}$$

where $R(s, t) = s\varrho(t/s)$.

Proof. Assume first $1 < p < \infty$. Let (u_j) be a sequence in the Banach lattice E. Following Lindenstrauss and Tzafriri [13], we have

$$\Big(\sum_{j=1}^{n} |u_j|^p\Big)^{1/p} = \text{l.u.b.}\Big\{\sum_{j=1}^{n} \alpha_j u_j \,\Big|\, (\alpha_j) \subset \mathbf{R} \text{ and } \sum_{j=1}^{n} |\alpha_j|^{p'} \leqslant 1\Big\},$$

where the l.u.b. is taken with respect to the order of the lattice E. Therefore, the following Hölder inequality holds:

$$\sum_{j=1}^{n} |\alpha_j u_j| \leqslant \Big(\sum_{j=1}^{n} |\alpha_j|^{p'}\Big)^{1/p'} \Big(\sum_{j=1}^{n} |u_j|^p\Big)^{1/p},$$

where the α_j are scalars and the u_j are elements of the lattice E.

Now, set

$$A_k = \left\|\left(\sum_{j=1}^{n}\left(\frac{2^{kj}}{\varrho(2^j)}|u_j|\right)^p\right)^{1/p}\right\|_E, \qquad k = 0, 1.$$

Then, upon taking $A_0 = U$ and $A_1 = V$ the proof follows exactly as in Gustavsson–Peetre [11].

Next, if $p = 1$, since ϱ is pseudoconcave, we have for $x \in \mathbf{R}_+$

$$\Big\|\sum_{j=1}^{n} u_j\Big\|_E \leqslant \Big\|\sum_{2^j\leqslant x} u_j\Big\|_E + \Big\|\sum_{2^j>x} u_j\Big\|_E \leqslant \varrho(x)\left\|\sum_{j=1}^{n}\frac{|u_j|}{\varrho(2^j)}\right\|_E + \frac{\varrho(x)}{x}\left\|\sum_{j=1}^{n}\frac{2^j|u_j|}{\varrho(2^j)}\right\|_E$$

and the proof follows as in the case $1 < p < \infty$.

3.2. LEMMA (Khinchin–Maurey, see [11, p. 49]). *Let E be a q-concave Banach lattice for some $q < \infty$. Then there exists a constant C such that for every sequence $(u_j)_{1\leqslant j\leqslant n}$ in E we have*

$$C^{-1}\Big\|\Big(\sum_{j=1}^{n} |u_j|^2\Big)^{1/2}\Big\|_E \leqslant \int_0^1 \Big\|\sum_{j=1}^{n} r_j(t) u_j\Big\|_E dt \leqslant C\Big\|\Big(\sum_{j=1}^{n} |u_j|^2\Big)^{1/2}\Big\|_E. \tag{1}$$

3.3. THEOREM. *Let (E_0, E_1) be a Banach couple of q-concave Banach lattices for some $q < \infty$. Then for all p, $1 \leqslant p < \infty$, we have*

$$\langle L^p(E_0), L^p(E_1)\rangle_{\varrho,p} = L^p(\langle E_0, E_1\rangle_{\varrho,p}), \tag{1}$$

with equivalent norms.

Proof. Step 1. Let $f \in L^p(\langle E_0, E_1\rangle_{\varrho,p})$, and assume that f is a step function. Then there are pairwise disjoint measurable sets $B_1, \ldots, B_N$ such that $f = \sum_{j=1}^{N} a_j \chi_{B_j}$, where $a_j \in \langle E_0, E_1\rangle_{\varrho,p}$. Since $a_j \in \langle E_0, E_1\rangle_{\varrho,p}$, $j = 1, \ldots, N$, there are admissible sequences $(u_n^j)_{n\in\mathbf{Z}}$ in $E_0 \cap E_1$, $j = 1, \ldots, N$, such that

$$2\|a_j\|_{\langle E_0,E_1\rangle_{\varrho,p}} \geqslant \max_{k=0,1} \sup_J \left\|\sum_{n\in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n^i\right\|_{L^p([0,1],E_k)}, \tag{2}$$

for each $j = 1, \ldots, N$. Next, for each $n \in \mathbf{Z}$, set

$$u_n(x) = \begin{cases} u_n^j & \text{if } x \in B_j,\ j = 1, \ldots, N, \\ 0 & \text{if } x \notin \bigcup_j B_j. \end{cases}$$

Then u_n is an $E_0 \cap E_1$-valued step function. Moreover, $u_n \in L^p(E_0 \cap E_1) = L^p(E_0) \cap L^p(E_1)$, and $f = \sum_{n\in\mathbf{Z}} u_n$ with convergence in $L^p(E_0) + L^p(E_1)$. In fact, for all finite subsets $J_0 \subset \mathbf{Z}_-$ and $J_1 \subset \mathbf{Z}_+$, we have, for $k = 0, 1$,

$$\Big\|\sum_{n\in J_k} u_n\Big\|_{L^p(E_k)} \leqslant 2 \sum_{n\in J_k} \frac{\varrho(2^n)}{2^{kn}} \|f\|_{L^p(\langle E_0,E_1\rangle_{\varrho,p})}.$$

Consequently, $(u_n)_{n\leqslant 0}$ is $L^p(E_0)$-summable and $(u_n)_{n>1}$ is $L^p(E_1)$-summable. Then setting $f_0 = \sum_{n=-\infty}^{0} u_n$ and $f_1 = \sum_{n=1}^{\infty} u_n$ we have $f_0 + f_1 = f$ and the desired convergence follows.

The $L^p([0, 1], L^p(E_k))$-summability of $(\tilde{r}_n(\cdot)2^{kn}u_n/\varrho(2^n))_{n\in\mathbf{Z}}$ follows from the $L^p([0, 1], E_k)$-summability of $(\tilde{r}_n(\cdot)2^{kn}u_n^j/\varrho(2^n))_{n\in\mathbf{Z}}$, $j = 1, \ldots, N$. In fact, given $\varepsilon > 0$ there are finite subsets $J_\varepsilon^j \subset \mathbf{Z}$, $j = 1, \ldots, N$, such that for all finite subsets $J' \subset \mathbf{Z}$ with $J' \cap J_\varepsilon^j = \varnothing$

$$\left\|\sum_{n\in J'} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n^j\right\|_{L^p([0,1],E_k)}^p < \varepsilon^p \Big/ \sum_{j=1}^{N} \mu(B_j), \qquad j = 1, \ldots, N.$$

Hence, setting $J_\varepsilon = \bigcup_{j=1}^{N} J_\varepsilon^j$, for all finite subsets $J' \subset \mathbf{Z}$ with $J' \cap J_\varepsilon = \varnothing$ we also have

$$\left\|\sum_{n\in J'} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)} u_n\right\|_{L^p([0,1],L^p(E_k))} = \left(\sum_{j=1}^{N} \int_{B_j}\int_0^1 \left\|\sum_{n\in J'} \tilde{r}_n(t)\frac{2^{kn}}{\varrho(2^n)} u_n^j\right\|_{E_k}^p dt\, d\mu\right)^{1/p} < \varepsilon.$$

Thus $(u_n)_{n\in\mathbf{Z}}$ is an admissible sequence for f. Consequently, from (2) we get

$$\|f\|_{\langle L^p(E_0),L^p(E_1)\rangle_{\varrho,p}}^p \leqslant \max_{k=0,1} \sup_J \int_0^1\int_X \left\|\sum_{n\in J} \tilde{r}_n(t)\frac{2^{kn}}{\varrho(2^n)} u_n(x)\right\|_{E_k}^p d\mu\, dt$$

$$\leqslant \int_X \Big(\max_{k=0,1} \sup_J \Big\| \sum_{n\in J} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} u_n(x) \Big\|_{L^p([0,1],E_k)} \Big)^p d\mu$$

$$\leqslant 2^p \int_X \| f(x)\|^p_{\langle E_0,E_1\rangle_{\varrho,p}} d\mu.$$

Finally, by density we conclude that

(3) $$L^p(\langle E_0, E_1\rangle_{\varrho,p}) \hookrightarrow \langle L^p(E_0), L^p(E_1)\rangle_{\varrho,p}.$$

Step 2. Let $f\in\langle L^p(E_0), L^p(E_1)\rangle_{\varrho,p}$, and let $(u_n)_{n\in\mathbf{Z}}$ be an admissible sequence for f such that

(4) $$\max_{k=0,1} \Big\| \sum_{n\in J} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} u_n \Big\|_{L^p([0,1],L^p(E_k))} \leqslant 2\|f\|_{\langle L^p(E_0),L^p(E_1)\rangle_{\varrho,p}},$$

for all finite subsets $J\subset\mathbf{Z}$. For each J we set $v_n=\chi_J u_n$ and we see that

$$\sum_{n\in J} u_n(x) = \sum_{n=-\infty}^{\infty} v_n(x) \quad (\text{in } E_0+E_1),$$

$(\tilde{r}_n(\cdot)2^{kn}v_n(x)/\varrho(2^n))$ is $L^p([0,1],E_k)$-summable and, since $v_n=0$ for $n\notin J$,

$$\Big\| \sum_{n\in I} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} v_n(x) \Big\|_{L^p([0,1],E_k)} < \infty,$$

for all finite subsets $I\subset\mathbf{Z}$. Hence, by Lemma 3.2,

$$\begin{aligned}\Big\|\sum_{n\in J} u_n(x)\Big\|_{\langle E_0,E_1\rangle_{\varrho,p}} &\leqslant \max_{k=0,1}\sup_I \Big\| \sum_{n\in I} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} v_n(x) \Big\|_{L^p([0,1],E_k)}\\ &\leqslant C \max_{k=0,1}\sup_{I\subset J} \Big\| \Big(\sum_{n\in I} \Big(\frac{2^{kn}}{\varrho(2^n)} |v_n(x)| \Big)^2 \Big)^{1/2} \Big\|_{E_k}\\ &\leqslant C \max_{k=0,1} \Big\| \Big(\sum_{n\in J} \Big(\frac{2^{kn}}{\varrho(2^n)} |v_n(x)| \Big)^2 \Big)^{1/2} \Big\|_{E_k}\\ &\leqslant C \max_{k=0,1} \Big\| \sum_{n\in J} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} u_n(x) \Big\|_{L^p([0,1],E_k)}.\end{aligned}$$

Therefore, by Fubini's theorem and (4), we get

(5) $$\begin{aligned}\int_X \Big\|\sum_{n\in J} u_n(x)\Big\|^p_{\langle E_0,E_1\rangle_{\varrho,p}} d\mu &\leqslant C \max_{k=0,1} \Big\| \sum_{n\in J} \tilde{r}_n(\cdot) \frac{2^{kn}}{\varrho(2^n)} u_n \Big\|^p_{L^p([0,1],L^p(E_k))}\\ &\leqslant C\|f\|^p_{\langle L^p(E_0),L^p(E_1)\rangle_{\varrho,p}}.\end{aligned}$$

Now, since $(\tilde{r}_n(\cdot)2^{kn}u_n/\varrho(2^n))_{n\in\mathbf{Z}}$ is $L^p([0,1],L^p(E_k))$-summable, $k=0,1$, the first inequality in (5) implies the $L^p(\langle E_0,E_1\rangle_{\varrho,p})$-summability of $(u_n)_n$. Moreover, Theorem 2.6, Step 1 and the identity $L^p(E_0+E_1)=L^p(E_0)+L^p(E_1)$ (see [9]) ensures that $f=\sum_{n=-\infty}^{\infty} u_n$ in $L^p(\langle E_0,E_1\rangle_{\varrho,p})$. Hence, after passing to the limit, the second inequality in (5) gives

$$\|f\|_{L^p(\langle E_0,E_1\rangle_{\varrho,p})} \leqslant C\|f\|_{\langle L^p(E_0),L^p(E_1)\rangle_{\varrho,p}},$$

as desired.

3.4. Remark. The problem of finding interpolation functors which commute with L^p-spaces was raised by J. Peetre. For the Lions–Peetre interpolation method, a commutation result like Theorem 3.3 holds without additional assumptions on the Banach couple (E_0, E_1). A similar result for another interpolation functor and with different hypotheses on the Banach lattices E_0 and E_1 was announced, with no indication of proof, in [3].

4. Characterization of Orlicz spaces. In this section we show that reflexive Orlicz spaces can be characterized as interpolation spaces between L^p-spaces. This is a converse of Gustavsson–Peetre's Theorem [11, Theorem 7.3].

By an *Orlicz function* we mean a convex and continuous function Φ: $[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$.

4.1. Theorem. *Let Φ be an Orlicz function and let $L^\Phi(E)$ be the corresponding E-valued Orlicz space on a σ-finite measure space, where E is a q-concave Banach lattice, with $q<\infty$. Suppose that Φ and the conjugate function Φ^* satisfy the Δ_2-condition. Then there exist p_0 and p_1, with $1<p_1<p_0<\infty$, such that*

(1) $$\langle L^{p_0}(E), L^{p_1}(E)\rangle_{\varrho,p} = L^\Phi(E), \quad \textit{where}$$

(2) $$\varrho(t) = \begin{cases} t^{p_1/(p_1-p_0)}\Phi^{-1}(t^{p_0p_1/(p_0-p_1)}), & t>0,\\ 0, & t=0,\end{cases}$$

and $1<p<\infty$.

Proof. Step 1. From the Δ_2-condition it can be seen that

(3) $$1<\bar{q}_\Phi\leqslant\bar{p}_\Phi<\infty,$$

where

$$\bar{q}_\Phi = \inf_{t>0} t\varphi(t)/\Phi(t), \qquad \bar{p}_\Phi = \sup_{t>0} t\varphi(t)/\Phi(t),$$

and $\varphi=D_+(\Phi)$. Moreover, if $0<s<1$ and $1<r<\bar{q}_\Phi$, or $s>1$ and $\bar{p}_\Phi<r<\infty$, we have

(4) $$s^{1/r}\Phi^{-1}(t) < \Phi^{-1}(st).$$

Step 2. Choose p_0 and p_1 such that

$$1<p_1<\bar{q}_\Phi\leqslant\bar{p}_\Phi<p_0<\infty,$$

with the (not essential) additional restrictions

$$1<p_1<\frac{\bar{q}_\Phi p_0}{p_0+\bar{q}_\Phi}<\bar{q}_\Phi\leqslant\bar{p}_\Phi<\frac{1}{2}p_0<\infty.$$

With such a choice of p_0 and p_1 fixed, the function ϱ given by 4.1(2) belongs to the class $\mathscr{P}^{+-}$. Indeed, setting $\varepsilon = p_1/(p_0 - p_1)$, it can be seen that $\varrho(t)/t^{\varepsilon}$ is increasing and $\varrho(t)/t^{1-\varepsilon}$ is decreasing. Consequently,

$$\bar{\varrho}(t) \leqslant \max(t^{\varepsilon}, t^{1-\varepsilon}), \qquad \bar{\varrho}(t) = o\big(\max(1, t)\big).$$

Step 3. Set $\Phi_0(t) = t^{p_0}$ and $\Phi_1(t) = t^{p_1}$. We have

$$\Phi^{-1} = \Phi_0^{-1}\varrho(\Phi_1^{-1}/\Phi_0^{-1}).$$

Following Gustavsson–Peetre [11, Theorem 7.1], let $h(z) = \big(\Phi(z)\big)^{1/p_1 - 1/p_0}$. Then

$$\Phi(z) = \Phi_0\big(z/\varrho(h(z))\big) = \Phi_1\big(zh(z)/\varrho(h(z))\big). \tag{5}$$

Thus, given $f \in L^{\Phi}(E)$ with $\|f\|_{L^{\Phi}(E)} \leqslant 1$, for each $n \in \mathbf{Z}$, we put

$$B_n = \{x \in X\colon \|f(x)\|_E \in h^{-1}([2^{n-1}, 2^n))\}, \qquad B_{-\infty} = \{x \in X\colon \|f(x)\|_E = 0\}.$$

The B_n are μ-measurable, disjoint and $X = \bigcup_n B_n$ a.e. We define

$$u_n(x) = \begin{cases} f(x) & \text{if } x \in B_n, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $(u_n)_n$ is an admissible sequence for f. Indeed, for all finite subsets $J_0 \subset \mathbf{Z}_-$ ($J_1 \subset \mathbf{Z}_+$, respectively), from (5) and recalling that ϱ is pseudoconcave, we have, for $k = 0, 1$,

$$\begin{aligned}
\int_X \Big(\Big\|\sum_{n \in J_k} u_n(x)\Big\|_E\Big)^{p_k} d\mu &\leqslant \sum_{n \in J_k} \int_{B_n} \left(\frac{2^{k(n-k)}}{\varrho(2^{n-k})}\|f(x)\|_E\right)^{p_k} d\mu \\
&\leqslant \sum_{n \in J_k} \int_{B_n} \left(\frac{\big(h(\|f(x)\|_E)\big)^k}{\varrho\big(h(\|f(x)\|_E)\big)}\|f(x)\|_E\right)^{p_k} d\mu \\
&= \sum_{n \in J_k} \int_{B_n} \Phi(\|f(x)\|_E)\, d\mu < \varepsilon
\end{aligned}$$

if $J_k \cap J_\varepsilon = \emptyset$ and J_ε is a finite subset of $\mathbf{Z}$ such that

$$\sum_{n \notin J_\varepsilon} \int_{B_n} \Phi(\|f(x)\|_E)\, d\mu < \varepsilon.$$

This can be done since

$$\int_X \Phi(\|f(x)\|_E)\, d\mu \leqslant 1.$$

Hence, $(u_n)_{n \leqslant 0}$ is $L^{p_0}(E)$-summable and $(u_n)_{n > 1}$ is $L^{p_1}(E)$-summable. Then setting $f_0 = \sum_{n=-\infty}^{0} u_n$ and $f_1 = \sum_{n=1}^{\infty} u_n$ we have $f_0 + f_1 = f$ and $f = \sum_{n=-\infty}^{\infty} u_n$ in $L^{p_0}(E) + L^{p_1}(E)$. Next, since

$$\begin{aligned}
\int_X \left(\left\|\sum_{n \in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)}u_n(x)\right\|_E\right)^{p_k} d\mu &\leqslant \sum_{n \in J} \int_{B_n} \left(\left\|\frac{2^{k(n-k)}}{\varrho(2^{k-n})}f(x)\right\|_E\right)^{p_k} d\mu \\
&\leqslant \sum_{n \in J} \int_{B_n} \Phi(\|f(x)\|_E)\, d\mu
\end{aligned}$$

it follows as before that $(\tilde{r}_n(\cdot)2^{kn}u_n/\varrho(2^n))_{n \in \mathbf{Z}}$ is $L^p([0, 1], L^{p_k}(E))$-summable, $k = 0, 1$. Finally, it can be analogously shown that

$$\max_{k=0,1} \sup_J \left\|\sum_{n \in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)}u_n\right\|_{L^p([0,1], L^{p_k}(E))} \leqslant 2.$$

This yields $L^{\Phi}(E) \hookrightarrow \langle L^{p_0}(E), L^{p_1}(E)\rangle_{\varrho,p}$.

Step 4. Let $f \in \langle L^{p_0}(E), L^{p_1}(E)\rangle_{\varrho,p}$ with $\|f\|_{\langle L^{p_0}(E), L^{p_1}(E)\rangle_{\varrho,p}} < 1$ and choose $(u_n)_{n \in \mathbf{Z}}$, an admissible sequence for f such that

$$\max_{k=0,1} \sup_J \left\|\sum_{n \in J} \tilde{r}_n(\cdot)\frac{2^{kn}}{\varrho(2^n)}u_n\right\|_{L^p([0,1], L^{p_k}(E))} \leqslant 1. \tag{6}$$

Since $(\tilde{r}_n(\cdot)2^{kn}u_n/\varrho(2^n))_{n \in \mathbf{Z}}$ is $L^p([0, 1], L^{p_k}(E))$-summable, given $\varepsilon > 0$, there exists a finite subset $J_\varepsilon \subset \mathbf{Z}$ such that for all finite subsets $J' \subset \mathbf{Z}$ with $J' \cap J_\varepsilon = \emptyset$, we have

$$\left\|\left(\sum_{n \in J'}\left(\frac{2^{kn}}{\varrho(2^n)}|u_n|\right)^2\right)^{1/2}\right\|_{L^{p_k}(E)} \leqslant \varepsilon^{1/p_k} \tag{7}$$

(after using the Khinchin–Maurey inequality!). Now, setting

$$A_k(x) = \left\|\left(\sum_{n \in J'}\left(\frac{2^{kn}}{\varrho(2^n)}|u_n(x)|\right)^2\right)^{1/2}\right\|_E$$

and $A(x) = \max_{k=0,1}(A_k(x)/2\varepsilon)^{p_k}$, from Lemma 3.1 and the fact that $\varrho(t)/t$ is nonincreasing we get

$$\begin{aligned}
\frac{1}{2\varepsilon}\Big\|\sum_{n \in J'} u_n(x)\Big\|_E &\leqslant C\frac{A_0(x)}{2\varepsilon}\varrho\left(\frac{A_1(x)}{A_0(x)}\right) \\
&\leqslant C\big(A(x)\big)^{1/p_0}\varrho\big((A(x))^{1/p_1 - 1/p_0}\big) = C\Phi^{-1}\big(A(x)\big).
\end{aligned}$$

Therefore, by (7)

$$\int_X \Phi\left(\frac{1}{2C\varepsilon}\Big\|\sum_{n \in J'} u_n(x)\Big\|_E\right) d\mu \leqslant \frac{1}{2}\int_X \left(\left(\frac{A_0(x)}{\varepsilon}\right)^{p_0} + \left(\frac{A_1(x)}{\varepsilon}\right)^{p_1}\right) d\mu \leqslant 1.$$

This implies that $\|\sum_{n \in J'} u_n\|_{L^{\Phi}(E)} \leqslant 2C\varepsilon$, i.e., the sequence $(u_n)_n$ satisfies the Cauchy condition in $L^{\Phi}(E)$. By the first embedding (Step 3) and Theorem 2.6 we have $f \in L^{\Phi}(E)$. Moreover, by a straightforward modification of the above calculation we have $\|f\|_{L^{\Phi}(E)} \leqslant C$, and the second embedding follows.

5. Besov–Orlicz spaces and Hardy–Sobolev–Orlicz spaces. The following Orlicz space version of Triebel's multiplier theorem (see [14]) holds.

5.1. THEOREM. *Let $K = (K_{i,j})_{i,j\in\mathbf{Z}}$ be a matrix, with $K_{i,j}$ in $L^1(\mathbf{R}^n)\cap\mathscr{S}'(\mathbf{R}^n)$. Suppose that $\mathscr{F}K_{i,j}$ is a regular distribution having classical derivatives in $\mathbf{R}^n\setminus\{0\}$ up to order $[n/2]+1$, $i,j\in\mathbf{Z}$. Further, assume that there exists a number B such that for $R>0$ and all multi-indices α with $|\alpha|\leqslant 1+[n/2]$,*

$$\int_{R/2\leqslant|t|\leqslant 2R}\sum_{i,j\in\mathbf{Z}}|D^\alpha(\mathscr{F}K_{i,j})(t)|^2\,dt\leqslant B^2R^{n-2|\alpha|}. \tag{1}$$

Then the operator $\mathscr{K}$,

$$(\mathscr{K}f)(x)=\int_{\mathbf{R}^n}K(x-y)f(y)\,dy, \tag{2}$$

where $f=(f_j)_{j\in\mathbf{Z}}$ is a quasi-null sequence of $C_c^\infty(\mathbf{R}^n)$-functions, has a bounded extension from $L^\Phi(\mathbf{R}^n, l^2)$ into itself, for all Orlicz functions Φ which satisfy, together with their conjugate functions Φ^*, the Δ_2-condition. Moreover, if additionally $K_{i,j}=0$ if $i\neq j$, then $\mathscr{K}$ has also a bounded extension from $L^\Phi(\mathbf{R}^n, l^r)$ into itself, for all r with $1<r<\infty$.

Proof. The assertions hold for $\Phi(t)=t^p$, $1<p<\infty$. Since l^r is a q-concave Banach lattice the result follows from Theorem 4.1.

5.2. Let $(\varphi_n)_n$ be a sequence in the Schwartz class $\mathscr{S}(\mathbf{R}^d)$ such that

$$\operatorname{supp}\mathscr{F}\varphi_n\subset\{2^{n-1}\leqslant|t|\leqslant 2^{n+1}\},\quad n=1,2,\ldots, \tag{1}$$

$$\operatorname{supp}\mathscr{F}\varphi_0\subset\{|t|\leqslant 2\}; \tag{2}$$

$$|\mathscr{F}\varphi_n(t)|\geqslant C_\varepsilon>0\quad\text{if } (2-\varepsilon)^{-1}2^n\leqslant|t|\leqslant(2-\varepsilon)2^n,\ n=1,2,\ldots, \tag{3}$$

$$|\mathscr{F}\varphi_0(t)|\geqslant C_\varepsilon>0\quad\text{if } |t|\leqslant 2-\varepsilon; \tag{4}$$

$$|D^\beta\mathscr{F}\varphi_n(t)|\leqslant C_\beta 2^{-|\beta|n}\quad\text{for all } \beta\in\mathbf{N}^d \text{ and } n\in\mathbf{N}; \tag{5}$$

$$\sum_{n=0}^\infty\mathscr{F}\varphi_n(t)=1. \tag{6}$$

The sequence (φ_n) is called a *system of test functions.*

5.3. DEFINITION. Suppose $s\in\mathbf{R}$ and let (φ_n) be a system of test functions. We define, for $1<q<\infty$,

$$H^s_{\Phi,q}=H^s_{\Phi,q}(\mathbf{R}^d)=\{f\in\mathscr{S}'(\mathbf{R}^d)\,|\,(\varphi_n*f)_n\in L^\Phi(l^q_s)\}; \tag{1}$$

and, for $1\leqslant q\leqslant\infty$, we set

$$B^s_{\Phi,q}=B^s_{\Phi,q}(\mathbf{R}^d)=\{f\in\mathscr{S}'(\mathbf{R}^d)\,|\,(\varphi_n*f)_n\in l^q_s(L^\Phi)\}. \tag{2}$$

We equip the spaces $H^s_{\Phi,q}$ and $B^s_{\Phi,q}$ with the norms

$$\|f\|_{H^s_{\Phi,q}}=\|(\varphi_n*f)_n\|_{L^\Phi(l^q_s)}=\big\|\|(\varphi_n*f)_n\|_{l^q_s}\big\|_{L^\Phi}, \tag{3}$$

$$\|f\|_{B^s_{\Phi,q}}=\|(\varphi_n*f)_n\|_{l^q_s(L^\Phi)}=\big\|(\|\varphi_n*f\|_{L^\Phi})_n\big\|_{l^q_s}. \tag{4}$$

5.4. It follows from Theorem 5.1 that the spaces $H^s_{\Phi,q}$ and $B^s_{\Phi,q}$ do not depend on the particular system of test functions used to define them.

As in the usual Hardy–Sobolev and Besov spaces we have

5.5. THEOREM. *Let $s\in\mathbf{R}$ and let Φ be an Orlicz function as before. Then:*

A. $H^s_{\Phi,q}$ is a retract of $L^\Phi(l^q_s)$ for $1<q<\infty$.
B. $B^s_{\Phi,q}$ is a retract of $l^q_s(L^\Phi)$ for $1\leqslant q\leqslant\infty$.

5.6. COROLLARY. *The spaces $H^s_{\Phi,q}$ and $B^s_{\Phi,q}$ are complete.*

5.7. For $s\in\mathbf{R}$ the Bessel (potential) operator J^s is defined on $f\in\mathscr{S}'(\mathbf{R}^d)$ by

$$J^sf=\mathscr{F}^{-1}(1+|\cdot|^2)^{s/2}\mathscr{F}f.$$

Here, $\mathscr{F}^{-1}$ is the inverse of the Fourier transform $\mathscr{F}$ on $\mathscr{S}'(\mathbf{R}^d)$. As in the usual cases the Bessel operator J^s is an isomorphism from $H^r_{\Phi,q}$ ($B^r_{\Phi,q}$, respectively) onto $H^{r-s}_{\Phi,q}$ ($B^{r-s}_{\Phi,q}$, respectively), $1<q<\infty$.

Next, we recall the definition of the Sobolev–Orlicz spaces and we show that they are the same as the spaces $H^s_{\Phi,2}$.

5.8. DEFINITION. Assume $s\in\mathbf{R}$ and let Φ be an Orlicz function. The *Sobolev–Orlicz space* $H^s_\Phi=H^s_\Phi(\mathbf{R}^d)$ is the linear space of all $f\in\mathscr{S}'(\mathbf{R}^d)$ such that $J^sf\in L^\Phi(\mathbf{R}^d)$.

The space H^s_Φ is equipped with the norm

$$\|f\|_{H^s_\Phi}=\|J^sf\|_{L^\Phi}.$$

A relationship between the Hardy–Sobolev–Orlicz spaces and the Sobolev–Orlicz spaces is a consequence of the following theorem of Littlewood–Paley type.

5.9. THEOREM. *Let $(\varphi_n)_n$ be a system of test functions as before. Assume that L^Φ is an Orlicz space where the Orlicz function Φ and its conjugate function satisfy the Δ_2-condition. Then there exists a constant $C>0$ such that for all $f\in L^\Phi$,*

$$C^{-1}\|f\|_{L^\Phi}\leqslant\Big\|\Big[\sum_{n=0}^\infty|\varphi_n*f|^2\Big]^{1/2}\Big\|_{L^\Phi}\leqslant C\|f\|_{L^\Phi}. \tag{1}$$

5.10. COROLLARY. *For all $s\in\mathbf{R}$ and every Orlicz function Φ which satisfies, together with its conjugate function Φ^*, the Δ_2-condition, we have*

$$H^s_{\Phi,2}=H^s_\Phi. \tag{2}$$

Proof. It follows from 5.7 that J^s is an isomorphism from $H^s_{\Phi,2}$ onto $H^0_{\Phi,2}$ and it is obvious that it is an isomorphism from H^s_Φ onto $H^0_\Phi=L^\Phi$. But the Littlewood–Paley theorem says that $H^0_{\Phi,2}=L^\Phi$.

To close this section we show that the Hardy–Sobolev–Orlicz spaces $H^s_{\Phi,q}$ and the Besov–Orlicz spaces $B^s_{\Phi,q}$ can be characterized as interpolation spaces between usual Sobolev and Besov spaces, respectively.

5.11. THEOREM. *Let Φ be an Orlicz function which satisfies, together with its conjugate function Φ^*, the Δ_2-condition. Then there exist p_0 and p_1, with $1 < p_0, p_1 < \infty$, such that*

$$(1) \qquad \langle H^s_{p_0,q}, H^s_{p_1,q}\rangle_{\varrho,p} = H^s_{\Phi,q},$$

$$(2) \qquad \langle B^s_{p_0,q}, B^s_{p_1,q}\rangle_{\varrho,p} = B^s_{\Phi,q},$$

where ϱ is given by 4.1(2).

Proof. This follows from Theorems 5.5 and 4.1.

6. The U.M.D. property. As a consequence of Theorems 3.3 and 4.1 it can be seen that reflexive Orlicz spaces are U.M.D. spaces.

We recall that a Banach space E is U.M.D. or has the U.M.D. property (see Burkholder [5]) if and only if the Hilbert transform has a bounded extension from $L^p(\mathbf{R}, E)$ into $L^p(\mathbf{R}, E)$. Consequently, by the foregoing, we have

6.1. THEOREM. *Let (E_0, E_1) be a Banach couple of U.M.D. spaces. Then the interpolation space $\langle E_0, E_1\rangle_{\varrho,p}$ is also a U.M.D. space.*

6.2. THEOREM. *Let L^Φ be an Orlicz space such that the Orlicz function Φ and the conjugate function Φ^* satisfy the Δ_2-condition. Then L^Φ has the U.M.D. property.*

Proof. The spaces L^q, $1 < q < \infty$, are U.M.D. Hence, for $1 < q_0, q_1 < \infty$, we have

$$H\colon L^p(L^{q_i}) \to L^p(L^{q_i}), \quad i = 1, 2,$$

where H stands for the Hilbert transform. Consequently,

$$H\colon \langle L^p(L^{q_0}), L^p(L^{q_1})\rangle_{\varrho,p} \to \langle L^p(L^{q_0}), L^p(L^{q_1})\rangle_{\varrho,p}.$$

Now, by Theorems 3.3 and 4.1 it follows that $H\colon L^p(L^\Phi) \to L^p(L^\Phi)$, and so L^Φ, with the above assumptions, is a U.M.D. space.

6.3. COROLLARY. *Let Φ and Ψ be Orlicz functions satisfying the Δ_2-condition together with their conjugate functions Φ^* and Ψ^*. Then $L^\Psi(L^\Phi)$ is also a U.M.D. space.*

Proof. The Calderón space $X(E)$ is U.M.D. if and only if X and E are U.M.D. spaces.

Besides $H^s_{\Phi,q}$ and $B^s_{\Phi,q}$ we can also consider the spaces $H^s_{\Phi,\Psi}$ and $B^s_{\Phi,\Psi}$.

6.4. COROLLARY. *Let Φ and Ψ be as in Corollary 6.3. Then the Besov–Orlicz spaces $B^s_{\Phi,\Psi}$ and the Hardy–Sobolev–Orlicz spaces $H^s_{\Phi,\Psi}$ (i.e. Besov and Hardy–Sobolev spaces modeled on Orlicz spaces) are also U.M.D. spaces.*

Proof. It follows from Theorem 4.1 and Triebel's multiplier theorem ([14]) that the spaces $H^s_{\Phi,\Psi}$ are retracts of $L^\Phi(l^\Psi_s)$ spaces. On the other hand, from Cobos–Fernandez [7, Prop. 6.2] we know that retracts of U.M.D. spaces are U.M.D. spaces. Consequently, the Hardy–Sobolev–Orlicz spaces are U.M.D. spaces. An analogous reasoning shows that the Besov–Orlicz spaces are also U.M.D. spaces.

6.5. COROLLARY. *Let Φ be an Orlicz function satisfying the Δ_2-condition together with its conjugate function Φ^*. Then the Schatten–Orlicz class S_Φ is also a U.M.D. space.*

Proof. This follows from a result of Arazy [1].

References

[1] J. Arazy, *Some remarks on interpolation theorems and the boundedness of the triangular projection in unitary matrix spaces*, Integral Equations Operator Theory 1 (1978), 453–495.

[2] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. 21 (1983), 163–168.

[3] A. V. Bukhvalov, *Interpolation of operators in spaces of vector-valued functions with applications to singular integral operators*, Soviet Math. Dokl. 30 (1984), 401–405.

[4] —, *Continuity of operators in spaces of vector functions with applications to the theory of bases*, J. Soviet Math. 44 (1989), 749-762.

[5] D. L. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, in: Proc. Conf. Harmonic Analysis in Honor of A. Zygmund, Chicago 1981, Wadsworth, Belmont 1983, 270–286.

[6] F. Cobos, *Some spaces in which martingale difference sequences are unconditional*, Bull. Polish Acad. Sci. 34 (1986), 695–703.

[7] F. Cobos and D. L. Fernandez, *Hardy–Sobolev spaces and Besov spaces with a function parameter*, in: Proc. U.S.-Swedish Seminar on Function Spaces, Lund 1986, Lecture Notes in Math. 1302, Springer 1987, 158–170.

[8] D. L. Fernandez, *Interpolation of 2^n Banach spaces and the Calderón spaces $X(E)$*, Proc. London Math. Soc. 56 (1988), 143–162.

[9] D. L. Fernandez and J. B. Garcia, *Interpolation of Orlicz-valued function spaces and U.M.D. property*, in: 26° Seminário Brasileiro de Análise, Rio de Janeiro, Nov. 1987, Trabalhos Apresentados, 269–281.

[10] J. Gustavsson, *On interpolation of weighted L^p-spaces and Ovchinnikov's theorem*, Studia Math. 72 (3) (1982), 237–251.

[11] J. Gustavsson and J. Peetre, *Interpolation of Orlicz spaces*, Studia Math. 60 (1977), 33–59.

[12] H. König, personal communication, 1987.

[13] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II. Function Spaces*, Springer, Berlin 1979.

[14] H. Triebel, *Spaces of distributions of Besov type on Euclidean n-space*, Ark. Mat. 11 (1973), 65–74.

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Added in proof (April 1991). I. Condition 2.2(2) implies convergence of the series in 2.2(1) and the summability assumption implies condition 2.2(2). However, a large part of the theory holds assuming only 2.2(1)–(2). The summability assumption is required for the density result (Theorem 2.10) and to commutate the interpolation functor and L^p (Theorem 3.3).

II. Theorem 2.6 can be proved directly (without appeal to Lemmas 2.4 and 2.5) by means of decomposition and ideas similar to the ones already used in Theorem 2.3.

III. The interpolation result 2.7 can be improved to obtain the constant $2M_0\,\bar{\varrho}(M_1/M_0)$. One has to follow Gustavsson–Peetre's ideas and recall the symmetry properties of the Rademacher functions.

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On two classes of Banach spaces with uniform normal structure

by

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Abstract. We give two classes of Banach spaces X that have uniform normal structure. The first class is closed under duality, and contains the uniformly convex spaces as well as the uniformly smooth spaces. The second class is defined by $J(X) < 3/2$, where $J(X) = \sup\{\|x+y\| \wedge \|x-y\|:$ $\|x\| = \|y\| = 1\}$. Both classes of spaces are uniformly nonsquare, their properties are being studied.

§ 1. Introduction. A Banach space X is said to have *normal structure* [2, 8] if for each bounded closed convex subset K in X that contains more than one point, there exists a point $x \in K$ such that

$$\sup\{\|x-y\|:\ y \in K\} < \operatorname{diam} K.$$

X is said to have *uniform normal structure* if there exists $0 < c < 1$ such that for any subset K as above, there exists $x \in K$ such that

$$\sup\{\|x-y\|:\ y \in K\} < c \operatorname{diam} K.$$

It is well known that uniform convexity in every direction implies normal structure [8, 28], whereas uniform convexity and uniform smoothness imply uniform normal structure [8, 27]. Our main purpose in this paper is to give two new classes of Banach spaces with uniform normal structure and study their relevant properties.

Let $S(X) = \{x \in X:\ \|x\| = 1\}$ be the unit sphere of X. For $x \in X$, let F_x denote the set of norm 1 supporting functionals f of $S(X)$ at x. In [16] Lau introduced the following notion to study the Chebyshev subset of X:

DEFINITION 1.1. A Banach space X is called a *U-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(1.1)\qquad \forall x, y \in S(X),\quad \|(x+y)/2\| > 1-\delta \Rightarrow \langle f, y\rangle > 1-\varepsilon,\ \forall f \in F_x.$$

Some of the properties of U-spaces in [16] are summarized in the following theorem.

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