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Generators of perfect σ-algebras of $\mathbf{Z}^d$-actions

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Abstract. Let Φ be a $\mathbf{Z}^d$-action, $d \geqslant 2$, with finite entropy $h(\Phi)$, on a Lebesgue space $(X, \mathscr{B}, \mu)$ and let Γ_Φ be the set of all countable measurable partitions P of X with finite entropy such that the mean entropy $h(P, \Phi)$ equals $h(\Phi)$. It is shown that if Φ is strongly ergodic then the set of all finite partitions of X which generate perfect σ-algebras of Φ is dense in Γ_Φ. If $h(\Phi) > 0$ then it is also a boundary set in Γ_Φ.

1. Introduction and notations. Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space and $\mathscr{Z}$ the set of all countable measurable partitions of X with finite entropy. We consider in $\mathscr{Z}$ the Rokhlin metric

$$\varrho(P, Q) = H(P\,|\,Q) + H(Q\,|\,P), \qquad P, Q \in \mathscr{Z}.$$

Let $\mathbf{Z}^d$ denote the group of d-dimensional integers and $<$ the lexicographical ordering in $\mathbf{Z}^d$ for $d \geqslant 2$ and the natural ordering for $d = 1$. Let $e^i \in \mathbf{Z}^d$ be the ith unit coordinate vector. We put

$$\mathbf{Z}^d_n = \{g = (m_1, \ldots, m_d) \in \mathbf{Z}^d;\ m_1 = \ldots = m_n = 0\}, \qquad 1 \leqslant n \leqslant d,$$
$$\mathbf{Z}^d_- = \{g \in \mathbf{Z}^d;\ g < (0, \ldots, 0)\}.$$

Let Φ be a $\mathbf{Z}^d$-*action* on $(X, \mathscr{B}, \mu)$, i.e. Φ is an isomorphism of the group $\mathbf{Z}^d$ into the group of all measure-preserving automorphisms of $(X, \mathscr{B}, \mu)$.

The restriction of Φ to $\mathbf{Z}^d_n$ is denoted by Φ_n. We denote by $T_1, \ldots, T_d$ the generators of the group $\Phi(\mathbf{Z}^d)$ which are the images by Φ of the vectors $e^1, \ldots, e^d$ respectively. We call them the *standard automorphisms* determined by Φ.

A $\mathbf{Z}^d$-action Φ is said to be *aperiodic* if

$$\mu(\{x \in X;\ \Phi^g x = x\}) = 0 \quad \text{for every } g \in \mathbf{Z}^d \setminus \{(0, \ldots, 0)\}.$$

Φ is said to be *ergodic* if for any Φ^g-invariant set $A \in \mathscr{B}$ and $g \in \mathbf{Z}^d$ either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We say that Φ is *strongly ergodic* if the automorphism T_d is ergodic. It is clear that every strongly ergodic action is ergodic.

Let $\mathscr{A}_i$, $i \in I$, be a family of measurable subsets of X. The smallest σ-algebra containing all $\mathscr{A}_i$, $i \in I$, is denoted by $\bigvee_{i \in I} \mathscr{A}_i$. For a given set $A \subset \mathbf{Z}^d$ and

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Consider now the Heisenberg group H^n and the sublaplacian $\mathscr{L}$ on H^n. Let $\varphi_N^\lambda(z)$ stand for the function $L_N^{n-1}(2|\lambda|\,|z|^2)\,e^{-|\lambda|\,|z|^2}$ where $z \in \mathbf{C}^n$ and L_N^{n-1} are the Laguerre polynomials of type $n-1$. Define

$$\psi_N^\lambda(z, t) = e^{-i\lambda t}\varphi_N^\lambda(z). \tag{1.7}$$

Then it can be checked that $f * \psi_N^\lambda(z, t)$ is an eigenfunction of the sublaplacian with eigenvalue $(2N+n)|\lambda|$. If we define the kernel $G_s(z, t)$ by

$$G_s(z, t) = c_n s^n \sum_{N=0}^{\infty} \nu^{n+1}\cos(\nu s t)\,\varphi_N^\nu(s^{1/2} z) \tag{1.8}$$

where $\nu = (2N+n)^{-1}$ then $P_s f = f * G_s$ will be an eigenfunction of $\mathscr{L}$ with eigenvalue s. The spectral decomposition of $\mathscr{L}$ can be written as

$$\mathscr{L}f = \int_0^\infty s P_s f\, ds. \tag{1.9}$$

This is the analogue of (1.3) and is known as the Strichartz formula for the spectral decomposition of $\mathscr{L}$ (see [6] and [7]).

Motivated by the estimate (1.5) D. Müller [5] investigated the mapping properties of the operator P_s. To state his results let us introduce the spaces $L^{(p,r)}(H^n) = L^p(\mathbf{C}^n, L^r(\mathbf{R}))$. Let $\|f\|_{(p,r)}$ stand for the norm of f in $L^{(p,r)}(H^n)$. Then Müller proved the following theorem.

THEOREM (D. Müller). *Let* $1 \leqslant p < 2$. *Then for all* f *in* $L^{(p,1)}(H^n)$

$$\|P_s f\|_{(p',\infty)} \leqslant c s^{n(2/p-1)}\|f\|_{(p,1)}.$$

Moreover, the result fails when $p > p_n = 4n/(2n-1)$.

The aim of this paper is to give two extensions of the restriction theorem of Müller. To motivate the first extension let us recall the following theorem of Zygmund for the Fourier transform [9].

THEOREM (A. Zygmund). *If* $f \in L^p(\mathbf{R}^2)$, $1 \leqslant p < 4/3$, $q = p'/3$ *then for* $s > 0$

$$\Big(\int_{|x|=s} |\hat{f}(x)|^q\, d\sigma\Big)^{1/q} \leqslant C s^{1/p'}\|f\|_p. \tag{1.10}$$

We would like to see if a similar result is true for the restriction operators P_s. In other words, does there exist an inequality of the form

$$\|P_s f\|_{(q,\infty)} \leqslant C_s\|f\|_{(p,1)} \tag{1.11}$$

where q is smaller than p'? When f is *zonal*, i.e., f is of the form $f(z, t) = f(|z|, t)$ we can prove the following theorem.

THEOREM 1. *Let* $(2n-1)/(2n+1) \leqslant \gamma \leqslant (2n+1)/(2n-1)$, $1 \leqslant p < 1+\gamma$ *and* $q = \gamma p'$. *Then for* $f \in L^{(p,1)}(H^n)$, f *zonal,*

$$\|P_s f\|_{(q,\infty)} \leqslant C_s\|f\|_{(p,1)}. \tag{1.12}$$

It would be interesting to see if this theorem remains true for all functions. The proof we are going to give works only for zonal functions.

To motivate the other extension let us write the Euclidean Plancherel theorem in the following way:

$$\|f\|_2^2 = \int_0^\infty r^{n-1}\, dr\Big(\int_{S^{n-1}} |\hat{f}(ru)|^2\, d\sigma\Big). \tag{1.13}$$

The Stein–Tomas restriction theorem is a result about the inner integral. It states that

$$\int_{S^{n-1}} |\hat{f}(ru)|^2\, d\sigma \leqslant C r^{-2n/p'}\|f\|_p^2. \tag{1.14}$$

On the Heisenberg group we have a Fourier transform. To define it, let us recall that all the infinite-dimensional irreducible unitary representations of H^n are parametrized by $\lambda \in \mathbf{R}$, $\lambda \neq 0$, and are given by

$$\pi_\lambda(z, t)\,\varphi(\xi) = e^{i\lambda t} e^{2i\lambda(2\xi - x)\cdot y}\varphi(\xi - x) \tag{1.15}$$

where $z = x+iy$ and $\varphi \in L^2(\mathbf{R}^n)$. The Fourier transform of a function f is defined by

$$\hat{f}(\lambda) = \int_{H^n} f(z, t)\,\pi_\lambda(z, t)\, dz\, dt. \tag{1.16}$$

For each $\lambda \neq 0$, $\hat{f}(\lambda)$ is a bounded operator on $L^2(\mathbf{R}^n)$. For this Fourier transform we have the Plancherel formula

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{\infty} |\lambda|^n\, \|\hat{f}(\lambda)\|_{\mathrm{HS}}^2\, d\lambda. \tag{1.17}$$

We want to rewrite this formula in a form which resembles (1.13). To do that let us introduce the following notations.

The one-dimensional normalized Hermite functions are defined by the formula

$$h_j(s) = (2^j\sqrt{\pi}\, j!)^{-1/2} e^{-s^2/2} H_j(s) \tag{1.18}$$

where $H_j(s) = (-1)^j e^{s^2}(d/ds)^j(e^{-s^2})$. For each multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ we define the n-dimensional Hermite functions $\Phi_\alpha(x)$ by

$$\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j).$$

Let P_N^n be the projection of $L^2(\mathbf{R}^n)$ onto the Nth eigenspace spanned by $\{\Phi_\alpha\colon |\alpha| = N\}$. Let $\Phi_\alpha^\lambda(x) = (2|\lambda|^{1/2})^{n/2}\Phi_\alpha(2|\lambda|^{1/2}x)$ be the scaled Hermite functions and let $P_N^n(\lambda)$ be the corresponding projections. Let V_N^λ be the span of $\{\Phi_\alpha^\lambda\colon |\alpha| = N\}$. Then it has been observed in [2] by Geller that V_N^λ are the natural analogues of the spheres rS^{n-1} in $\mathbf{R}^n$ and V_N^λ can be thought of as a sphere in $L^2(\mathbf{R}^n)$ with radius $(2N+n)|\lambda|$.

Let $\mathcal{O}(V_N^\lambda)$ be the space of all bounded linear operators from V_N^λ into $L^2(\mathbf{R}^n)$. On this space we can define an inner product by setting

$$(R, S)_N = (2|\lambda|)^{n-1} \sum_{|\alpha|=N} (R\Phi_\alpha^\lambda, S\Phi_\alpha^\lambda). \tag{1.19}$$

This space $\mathcal{O}(V_N^\lambda)$ is the natural analogue of $L^2(rS^{n-1})$. Observe that for operators T on $L^2(\mathbf{R}^n)$

$$(T, T)_N = (2|\lambda|)^{n-1} \|TP_N^n(\lambda)\|_{\mathrm{HS}}^2. \tag{1.20}$$

Having made these observations we can write the Plancherel formula (1.17) in the form

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} |\lambda|^n \|\hat{f}(\lambda) P_N(\lambda)\|_{\mathrm{HS}}^2 \, d\lambda.$$

By making a change of variables we have

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int_0^\infty r^n \sum_{N=0}^{\infty} \nu^{n+1} (\|\hat{f}(\nu r) P_N^n(\nu r)\|_{\mathrm{HS}}^2 + \|\hat{f}(-\nu r) P_N^n(\nu r)\|_{\mathrm{HS}}^2) \, dr \tag{1.21}$$

where we have set $\nu = \nu(N) = (2N+n)^{-1}$. If we define

$$(R(f, r))^2 = \sum_{N=0}^{\infty} \nu^{n+1} (\|\hat{f}(\nu r) P_N^n(\nu r)\|_{\mathrm{HS}}^2 + \|\hat{f}(-\nu r) P_N^n(\nu r)\|_{\mathrm{HS}}^2) \tag{1.22}$$

then (1.21) becomes

$$\|f\|_2^2 = \frac{2^{n-1}}{\pi^{n+1}} \int_0^\infty r^n (R(f, r))^2 \, dr \tag{1.23}$$

and (1.23) is the analogue of (1.13). So we would like to obtain an analogue of the Stein–Tomas restriction theorem for the "restrictions" $R(f, r)$. The following is the restriction theorem we can prove for $R(f, r)$.

THEOREM 2. *Assume that* $1 \leqslant p < 2$ *and* $f \in L^{(p,1)}(H^n)$. *Then*

$$R(f, r) \leqslant C r^{-n/p'} \|f\|_{(p,1)}. \tag{1.24}$$

Müller's theorem is the analogue of the estimate (1.5) whereas Theorem 2 is the direct analogue of the Stein–Tomas restriction theorem (1.6). In the case of $\mathbf{R}^n$, (1.6) is derived from the estimate (1.5). Similarly in the present case we can derive Theorem 2 from Theorem 1 (with $q = p'$) for zonal functions. In fact, in [5] Müller derived Theorem 2 from his Theorem for polyradial functions (Corollary 3.6 of [5]) though it was not stated in the present form. We remark that Theorem 2 is valid in the full range $1 \leqslant p < 2$ whereas the Stein–Tomas restriction theorem is proved only in the smaller range $1 \leqslant p \leqslant 2(n+1)/(n+3)$.

For notations and results concerning the Fourier transform on the Heisenberg group we refer to [1], [2] and [5]. Finally, the author wishes to thank E. M. Stein for bringing this problem to his attention and the referee for many useful comments which helped improving the exposition.

2. Proof of Theorem 1. The restriction operators P_s are convolution operators with kernels G_s where

$$G_s(z, t) = C_n s^n \sum_{N=0}^{\infty} \nu^{n+1} \cos(\nu st) \varphi_N^\nu(s^{1/2} z). \tag{2.1}$$

By means of dilation we can assume that $s = 1$. It is therefore enough to study the operator $Pf = f * G$ where

$$G(z, t) = \sum_{N=0}^{\infty} \nu^{n+1} e^{-i\nu t} \varphi_N^\nu(z). \tag{2.2}$$

Let us recall the definition of the twisted convolution of two functions f and g on $\mathbf{C}^n$. The *λ-twisted convolution* is defined by

$$f *_\lambda g(z) = \int_{\mathbf{C}^n} f(z-w) g(w) e^{2i\lambda \mathrm{Im}(z\cdot\bar{w})} \, dw. \tag{2.3}$$

When $\lambda = -1/4$ we simply call it the *twisted convolution*. If $\psi_N^\nu(z, t) = e^{-i\nu t} \varphi_N^\nu(z)$, then a simple calculation shows that

$$f * \psi_N^\nu(z, t) = e^{-i\nu t} f^\nu *_\nu \varphi_N^\nu(z) \tag{2.4}$$

where f^ν is the partial inverse Fourier transform:

$$f^\nu(z) = \int_{-\infty}^{\infty} f(z, t) e^{i\nu t} \, dt.$$

For simplicity of notation assume that $f(z, t) = h(t) g(z)$. Then

$$Pf(z, t) = \sum_{N=0}^{\infty} \nu^{n+1} e^{-i\nu t} \hat{h}(-\nu) g *_\nu \varphi_N^\nu(z). \tag{2.5}$$

Since $|\hat{h}(\nu)| \leqslant \|h\|_1$ we have the estimate

$$\|Pf\|_{(q,\infty)} \leqslant \|h\|_1 \sum_{N=0}^{\infty} \nu^{n+1} \|g *_\nu \varphi_N^\nu(z)\|_q.$$

Now by making a change of variables it is easy to see that

$$\overline{g *_\nu \varphi_N^\nu(z)} = (2\sqrt{\nu})^{-2n} (g_\nu^* \times \varphi_N)(2\sqrt{\nu} z)$$

where $\varphi_N(z) = L_N^{n-1}(\frac{1}{2}|z|^2) e^{-|z|^2/4}$, $g_\nu^*(z) = \bar{g}(z/(2\sqrt{\nu}))$ and $\times$ stands for the twisted convolution:

$$g_\nu^* \times \varphi_N(z) = \int_{\mathbf{C}^n} g_\nu^*(z-w) \varphi_N(w) e^{-(i/2)\mathrm{Im}(z\cdot\bar{w})} \, dw.$$

With this observation we have obtained

$$\|Pf\|_{(q,\infty)} \leqslant C\|h\|_1 \sum_{N=0}^{\infty} \nu^{1-n/q} \|g_\nu^* \times \varphi_N\|_q. \tag{2.6}$$

Up to this point we have not used the fact that g is a radial function.

When g is radial the twisted convolution $g_\nu^* \times \varphi_N$ is given by a simple formula. To obtain it we need to recall several properties of the Weyl transform W (see [4] for all the properties we use). When g is radial the Weyl transform $W(g)$ reduces to the Laguerre transform:

$$W(g) = \sum_{N=0}^{\infty} R_N(g) P_N^n.$$

Here $R_N(g)$ is defined by

$$R_N(g) = \frac{N!}{(N+n-1)!} \int_{\mathbf{C}^n} g(z)\varphi_N(z)\,dz.$$

It is clear that $W(\varphi_N) = P_N^n$. In view of the formula $W(f \times g) = W(f)\,W(g)$ one obtains

$$g_\nu^* \times \varphi_N = R_N(g_\nu^*)\,\varphi_N.$$

Since $N!/(N+n-1)! = O(N^{-n+1})$, Hölder's inequality gives the estimate

$$\|g_\nu^* \times \varphi_N\|_q \leqslant C\nu^{n-1}\|g_\nu^*\|_p \|\varphi_N\|_{p'} \|\varphi_N\|_q = C\nu^{n-1+n/p}\|g\|_p \|\varphi_N\|_{p'} \|\varphi_N\|_q. \tag{2.7}$$

Thus for zonal functions we have proved the estimate

$$\|Pf\|_{(q,\infty)} \leqslant C\|f\|_{(p,1)} \sum_{N=0}^{\infty} (2N+n)^{-n+n/q-n/p} \|\varphi_N\|_{p'} \|\varphi_N\|_q. \tag{2.8}$$

To prove Theorem 1 it is therefore necessary to find estimates for the norms of φ_N. The required bounds are given in the following lemma.

Lemma 2.1.

(i) $\|\varphi_N\|_q \leqslant CN^{n/q-1/2}$ *if* $q < \dfrac{4n}{2n-1}$,

(ii) $\|\varphi_N\|_q \leqslant CN^{(n-1)/2}N^{-1/4}(\log N)^{1/q}$ *if* $q = \dfrac{4n}{2n-1}$,

(iii) $\|\varphi_N\|_q \leqslant CN^{n-1}N^{-n/q}$ *if* $q > \dfrac{4n}{2n-1}$.

These estimates will follow from some well known estimates for the normalized Laguerre functions. If

$$\mathscr{L}_N^\alpha(r) = \left(\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(r)\,e^{-r/2}r^{\alpha/2}$$

are the normalized Laguerre functions then the following bounds have been proved in [3] (see p. 24).

Lemma 2.2 (C. Markett).

$$\begin{aligned} \|\mathscr{L}_N^{\alpha+\beta}(r)\,r^{-\beta/2}\|_q &\leqslant CN^{1/q-1/2-\beta/2} && \textit{if } q \leqslant 4,\ \beta < 2/q-1/2,\\ &\leqslant CN^{1/q-1/2-\beta/2}(\log N)^{1/q} && \textit{if } q \leqslant 4,\ \beta = 2/q-1/2,\\ &\leqslant CN^{\beta/2-1/q} && \textit{if } q \leqslant 4,\ \beta > 2/q-1/2\\ & && \textit{or } q \geqslant 4,\ \beta > 4/(3q)-1/3. \end{aligned}$$

It is easy to see how Lemma 2.1 follows from Lemma 2.2. A simple calculation shows that

$$\|\varphi_N\|_q \leqslant CN^{(n-1)/2}\|\mathscr{L}_N^{n-1}(r)\,r^{-\beta/2}\|_q$$

where $\beta = 2(n-1)(1/2-1/q)$. From this the various estimates of Lemma 2.1 follow from Lemma 2.2.

Now we can complete the proof of Theorem 1. In view of (2.8) it is enough to prove the convergence of the series

$$\sum_{N=0}^{\infty} (2N+n)^{-n+n/q-n/p} \|\varphi_N\|_{p'} \|\varphi_N\|_q.$$

First assume that $1 \leqslant p < 4n/(2n+1)$ so that $p' > 4n/(2n-1)$ and $q > 4n/(2n+1)$. There are three cases to consider. When $q > 4n/(2n-1)$ the series reduces to

$$\sum_{N=0}^{\infty} (2N+n)^{-n-n/p+n/q+n-1-n/p'+n-1-n/q} = \sum_{N=0}^{\infty} (2N+n)^{-2} \leqslant C.$$

When $q < 4n/(2n-1)$ the series reduces to

$$\sum_{N=0}^{\infty} (2N+n)^{-1-(n+1/2)+2n/q} \leqslant C$$

since $q > 4n/(2n+1)$. Similarly when $q = 4n/(2n-1)$ the series converges. This settles the case when $1 \leqslant p < 4n/(2n+1)$.

Next assume that $4n/(2n+1) < p < 1+\gamma$. Since $q = \gamma p'$ we have $p < 1+\gamma < q$. Again there are three cases to consider. Here $p' < 4n/(2n-1)$ and so when $q > 4n/(2n-1)$ the series becomes

$$\sum_{N=0}^{\infty} (2N+n)^{-1+(n-1/2)-2n/p} \leqslant C$$

since $p < 4n/(2n-1)$. When $q < 4n/(2n-1)$ we need to check if

$$\sum_{N=0}^{\infty} (2N+n)^{-1+2n(1/q-1/p)} \leqslant C.$$

Since $p < 1+\gamma < q$, we have $1/q-1/p < 0$ and the above series converges. When $q = 4n/(2n-1)$ again a similar calculation shows that the series is convergent. The case $p = 4n/(2n+1)$ can be treated in a similar way. This completes the proof of Theorem 1.

3. Proof of Theorem 2. In order to prove Theorem 2 we need to recall some more facts about the Fourier transform on the Heisenberg group. First we recall that when f is zonal then $\hat{f}(\lambda)$ is given by the simple formula

$$\hat{f}(\lambda) = \sum_{N=0}^{\infty} R_N(\lambda, f)\, P_N^n(\lambda) \tag{3.1}$$

where

$$R_N(\lambda, f) = \omega_{2n-1} \frac{N!}{(N+n-1)!} \int_0^{\infty} \tilde{f}(r, \lambda)\, L_N^{n-1}(2|\lambda| r^2)\, e^{-|\lambda| r^2} r^{2n-1}\, dr. \tag{3.2}$$

Here ω_{2n-1} is the surface measure of the unit sphere in $\mathbf{R}^{2n}$ and $\tilde{f}(r, \lambda)$ is the Fourier transform of f in the second variable:

$$\tilde{f}(r, \lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(r, t)\, dt. \tag{3.3}$$

The inversion formula becomes

$$f(r, t) = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{\infty} e^{-i\lambda t} \Big(\sum_{N=0}^{\infty} R_N(\lambda, f)\, L_N^{n-1}(2|\lambda| r^2)\, e^{-|\lambda| r^2} \Big) |\lambda|^n\, d\lambda. \tag{3.4}$$

Therefore, if we set

$$G_N(z, t) = \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{\infty} e^{-i\lambda t} L_N^{n-1}(2|\lambda| r^2)\, e^{-|\lambda| r^2} |\lambda|^n\, d\lambda \tag{3.5}$$

where $r = |z|$ then it is clear that $\hat{G}_N(\lambda) = P_N^n(\lambda)$. Thus

$$(f * G_N)^{\wedge}(\lambda) = \hat{f}(\lambda)\, \hat{G}_N(\lambda) = \hat{f}(\lambda)\, P_N^n(\lambda). \tag{3.6}$$

To calculate the Hilbert–Schmidt norm of $(f * G_N)^{\wedge}(\lambda)$ we use the fact that it is an integral operator. In fact, its kernel can be easily calculated. Recall that the representations π_λ are given by

$$\pi_\lambda(z, t)\, \varphi(\xi) = e^{i\lambda t} e^{2i\lambda(2\xi - x)\cdot y} \varphi(\xi - x)$$

where $z = x+iy$. For a function g on H^n, $\hat{g}(\lambda)$ is an integral operator whose kernel can be easily calculated to be

$$K_g^\lambda(x, y) = \mathscr{F}_{2,3}\, g(x-y,\, 2\lambda(x+y),\, \lambda) \tag{3.7}$$

where $\mathscr{F}_{2,3}\, g$ is the Fourier transform of g with respect to y and t (where $z = x+iy$).

Therefore, the Hilbert–Schmidt norm of $\hat{g}(\lambda)$ is given by

$$\|\hat{g}(\lambda)\|_{\mathrm{HS}}^2 = \int |\mathscr{F}_{2,3}\, g(x-y,\, 2\lambda(x+y),\, \lambda)|^2\, dx\, dy. \tag{3.8}$$

An easy change of variables gives that

$$\|\hat{g}(\lambda)\|_{\mathrm{HS}}^2 = C|\lambda|^{-n} \int |\mathscr{F}_{2,3}\, g(x, y, \lambda)|^2\, dx\, dy. \tag{3.9}$$

Applying the Euclidean Plancherel theorem we obtain

$$\|\hat{g}(\lambda)\|_{\mathrm{HS}}^2 = C|\lambda|^{-n} \int |\mathscr{F}_3\, g(x, y, \lambda)|^2\, dx\, dy. \tag{3.10}$$

Thus to estimate $\|\hat{f}(\lambda)\, P_N^n(\lambda)\|_{\mathrm{HS}}^2$ we need a bound for the right hand side of (3.10) when $g = f * G_N$.

Assume that $\lambda > 0$. The case when $\lambda < 0$ is similar. If we let $g^\lambda(x, y) = \mathscr{F}_3\, g(x, y, \lambda)$ then an easy calculation shows that

$$g^\lambda(x, y) = (f * G_N)^\lambda(x, y) = f^\lambda *_\lambda G_N^\lambda(x, y) \tag{3.11}$$

where

$$G_N^\lambda(z) = \frac{2^{n-1}}{\pi^{n+1}} L_N^{n-1}(2|\lambda| r^2)\, e^{-|\lambda| r^2} |\lambda|^n.$$

Writing out the definition of the twisted convolution and making a change of variables we obtain

$$f^\lambda *_\lambda G_N^\lambda\Big(\frac{z}{2\sqrt{|\lambda|}}\Big) = C_n \int_{\mathbf{C}^n} e^{(i/2)\operatorname{Im}(z\cdot\bar{v})} f_\lambda^\lambda(z-v)\, \varphi_N(v)\, dv \tag{3.12}$$

where $f_\lambda^\lambda(z) = f^\lambda(z/(2\sqrt{|\lambda|}))$. Thus we have proved that

$$f^\lambda *_\lambda G_N^\lambda\Big(\frac{z}{2\sqrt{|\lambda|}}\Big) = c_n f_\lambda^\lambda \times \varphi_N(z) \tag{3.13}$$

where $f \times g$ stands for the usual twisted convolution.

Thus formula (3.10) gives

$$\|\hat{f}(\lambda)\, P_N^n(\lambda)\|_{\mathrm{HS}}^2 = c_n |\lambda|^{-2n} \|f_\lambda^\lambda \times \varphi_N\|_2^2. \tag{3.14}$$

If we can show that

$$\|f \times \varphi_N\|_2^2 \leqslant C N^{(n-1)(1-2/p')} \|f\|_p^2 \tag{3.15}$$

for functions f on $\mathbf{C}^n$ and for $1 \leqslant p \leqslant 2$ then we are done. To see this, from (3.14) we obtain

$$\|\hat{f}(\lambda)\, P_N^n(\lambda)\|_{\mathrm{HS}}^2 \leqslant C|\lambda|^{-2n} N^{(n-1)(1-2/p')} \|f_\lambda^\lambda\|_p^2,$$

which gives $\|\hat{f}(\lambda)\, P_N(\lambda)\|_{\mathrm{HS}}^2 \leqslant C|\lambda|^{-2n/p'} N^{(n-1)(1-2/p')} \|f^\lambda\|_p^2$. Since

$$\int_{\mathbf{C}^n} |f^\lambda(z)|^p\, dz = \int |\mathscr{F}_3 f(z, \lambda)|^p\, dz \leqslant \int \Big(\int |f(z, t)|\, dt\Big)^p dz = \|f\|_{(p,1)}^p$$

we have proved that

(3.16) $$\|\hat{f}(\lambda)P_N^n(\lambda)\|_{HS}^2 \leqslant c|\lambda|^{-2n/p'} N^{(n-1)(1-2/p')} \|f\|_{(p,1)}^2.$$

A similar estimate can be proved when $\lambda < 0$. From (1.15) we then obtain

(3.17) $$(R(f, r))^2 \leqslant C \sum_{N=0}^{\infty} \nu^{n+1}(\nu r)^{-2n/p'} \nu^{-(n-1)(1-2/p')} \|f\|_{(p,1)}^2.$$

Recalling that $\nu = (2N+n)^{-1}$ we have

$$(R(f, r))^2 \leqslant cr^{-2n/p'} \|f\|_{(p,1)}^2 \Big(\sum_{N=0}^{\infty} (2N+n)^{-2/p}\Big).$$

Since the series converges for $1 \leqslant p < 2$, this proves that

(3.18) $$R(f, r) \leqslant Cr^{-n/p'} \|f\|_{(p,1)}.$$

Therefore, it remains to prove the estimate (3.15). But (3.15) follows immediately from the estimates

(3.19) $$\|f \times \varphi_N\|_2 \leqslant CN^{(n-1)/2} \|f\|_1,$$

(3.20) $$\|f \times \varphi_N\|_2 \leqslant C\|f\|_2$$

by interpolation. (3.19) is a consequence of Young's inequality and the fact that $\|\varphi_N\|_2 \leqslant CN^{(n-1)/2}$. (3.20) follows from the fact that $W(f \times \varphi_N) = W(f)P_N^n$ where W is the Weyl transform, and from the Plancherel theorem for the Weyl transform:

$$\|f\|_2^2 = (2\pi)^{-n} \|W(f)\|_{HS}^2.$$

For the facts about the Weyl transform we refer to the paper [4] of Mauceri. Hence the inequality (3.15) is proved.

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