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Continuity of a homomorphism on commutative subalgebras is not sufficient for continuity

by

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Abstract. Examples of topological algebras are given to show that the structure of all commutative subalgebras does not determine the structure of the given algebra.

1. In the list of problems [2] formulated by Professor W. Żelazko concerning topological algebras there is the following one (Problem 4): Suppose that there are two topologies τ_1 and τ_2 making an algebra X a topological algebra. Suppose that for each commutative subalgebra $Y \subseteq X$ we have $\tau_1 | Y = \tau_2 | Y$, where $\tau | Y$ is the restriction of the topology τ to Y. Does it follow that $\tau_1 = \tau_2$? A negative answer to this question provides a negative answer to the next problem on the list: Suppose that a homomorphism h of topological algebras X into Y is continuous when restricted to each commutative subalgebra of X. Does it follow that h is continuous?

The following theorem holds.

THEOREM. *There is an algebra* $\mathbf{X}$ *and two different topologies* τ_1 *and* τ_2 *such that* (X, τ_1) *and* (X, τ_2) *are topological algebras and* τ_1, τ_2 *coincide on commutative subalgebras.*

Proof. Recall the example given in [1]. Let S be the free semigroup (with unit) of words over the alphabet A. Assume that A is infinite. For $B \subseteq A$, let $S(B)$ denote the free semigroup with unit over the alphabet B; in this notation $S = S(A)$. Let $\mathbf{R}^S$ be the algebra of real functions defined on S, with the convolution $x * y$ as multiplication. Let X be the set of those $x \in \mathbf{R}^S$ for which there is a finite $B \subseteq A$ such that $x(s) = 0$ for $s \in S \setminus S(B)$, and $\sum\{|x(s)|: s \in S\} < \infty$. For $x \in X$ we define two norms: $\|x\| = \sum\{|x(s)|: s \in S\}$ (the usual l^1-norm) and $\|x\|_1 = \sum\{2^{n(s)}|x(s)|: s \in S\}$ (the weighted l^1-norm) where $n(s)$ is the cardinality of the set of different letters appearing in s. It is easy to check that both are multiplicative norms, so both $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ are topological algebras.

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The following lemma is proved in [1].

LEMMA. *Let S be a semigroup of words over the alphabet A. Then if* $x, y \in l^1(S)\setminus\{0\}$, $x(\emptyset) = y(\emptyset) = 0$ *and* $x*y = y*x$, *then* $\mathrm{lett}(\mathrm{supp}\,x) = \mathrm{lett}(\mathrm{supp}\,y)$, *where* $\mathrm{supp}\,x$ *denotes the support of x and* $\mathrm{lett}(E)$ *denotes the set of letters appearing in words from E.*

By the Lemma for any commutative subalgebra Y of X there is a finite $B \subset A$ such that if $x \in Y$ then $\mathrm{supp}\,x \subseteq S(B)$, so $\|x\| \leqslant \|x\|_1 \leqslant 2^{\mathrm{card}\,B}\|x\|$. This demonstrates that on commutative subalgebras both topologies coincide, i.e. $\|\cdot\|$ is equivalent to $\|\cdot\|_1$. On the other hand, the unit ball of $(X, \|\cdot\|_1)$ does not contain any ball of $(X, \|\cdot\|)$. To see this let $\{a_1, a_2, \ldots\} \subseteq A$ and consider $x_n = n^{-1}\delta_{a_1a_2\ldots a_n}$, where δ_s denotes the function equal to 1 at s and 0 elsewhere. We have $\|x_n\| = n^{-1}$, while $\|x_n\|_1 = 2^n n^{-1}$.

2. The example $(X, \|\cdot\|)$ given above provides also a negative solution to another problem (Problem 16) from [2]:

Suppose that X is a topological algebra and all its commutative subalgebras are Banach ones. Does it follow that X is a Banach algebra?

In fact, for any commutative subalgebra Y of X there is a finite subset B of A such that if $x \in Y$ then $\mathrm{supp}\,x \subseteq S(B)$. So $\{x \in X\colon \mathrm{supp}\,x \in S(B)\}$ is a Banach algebra in $\|\cdot\|$-norm. So any commutative closed subalgebra is a Banach algebra but obviously $(X, \|\cdot\|)$ is not a Banach algebra (recall that the set A is infinite).

Moreover, our example $(X, \|\cdot\|)$ is locally convex, which solves negatively Problem 15 of [2].

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Equivalence et orthogonalité des mesures aléatoires engendrées par martingales positives homogènes

par

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Abstract. A positive homogeneous indexed martingale yields an operator which transforms a measure into a random measure. We compare the images of a measure by two such operators and demonstrate that either they are mutually singular or one is absolutely continuous with respect to the other. We also obtain a sufficient condition for one to be L^p-integrable with respect to the other. We give applications of such results to: Riesz products, random coverings, Mandelbrot's martingales, exponentiation of Gaussian processes etc.

Introduction. La source du présent travail est le fameux théorème dichotomique de Kakutani [10]: Soient μ_n et ν_n deux probabilités définies sur un espace mesurable $(\Omega_n, \mathscr{B}_n)$ $(n \geqslant 1)$. Supposons que ν_n soit absolument continue par rapport à μ_n. Alors la mesure produit $\otimes\nu_n$ est soit absolument continue par rapport à $\otimes\mu_n$, soit singulière par rapport à $\otimes\mu_n$, et le premier cas a lieu si et seulement si

$$\prod \int (d\nu_n/d\mu_n)^{1/2}\, d\mu_n > 0$$

où $d\nu_n/d\mu_n$ désigne la dérivée de Radon–Nikodym.

Autour de ce théorème beaucoup de travaux ont été réalisés dont une préoccupation majeure est de se dégager de la forte indépendance intervenant dans les hypothèses du théorème. Cette indépendance est que les fonctions $d\nu_n/d\mu_n$, considérées comme des variables définies sur $\otimes\Omega_n$, sont indépendantes par rapport à $\otimes\nu_n$ ainsi qu'à $\otimes\mu_n$.

Cela est possible comme l'indiquent les produits de Riesz qu'ont étudiés Brown–Moran [2], Peyrière [13, 14], Kilmer–Saeki [12], et les autres produits infinis qui ont été étudiés par Brown–Moran [1], Kanter [11], Ritter [15].

Les mesures dont nous nous préoccupons sont des mesures aléatoires qui se fabriquent de la façon que Kahane a décrite dans [7], ce que l'on appelle multiplication aléatoire. Ce cadre est vaste. On peut y encadrer tous les problèmes suivants: produits de Riesz aléatoires, recouvrements aléatoires, martingales de Mandelbrot, exponentiations de processus gaussiens etc.

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