Choose $\varepsilon = 1/(2\min(\varepsilon_1, \varepsilon_2))$. Then

$$(e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2})(x) = \int_{-\infty}^{\infty} e^{-\varepsilon_1\tilde{\psi}_1(y)-\varepsilon_2\tilde{\psi}_2(x-y)}\,dy$$

$$\leqslant e^{-\varepsilon\tilde{\psi}(x)} \int_{-\infty}^{\infty} e^{-\varepsilon_1\tilde{\psi}_1(y)/2}\, e^{-\varepsilon_2\tilde{\psi}_2(x-y)/2}\,dy$$

$$\leqslant Ce^{-\varepsilon\tilde{\psi}(x)} \int_{-\infty}^{\infty} e^{-\varepsilon_1\tilde{\psi}_1(y)/2}\,dy$$

$$\leqslant C\gamma(\tilde{\psi}_1)e^{-\varepsilon\tilde{\psi}(x)} \leqslant C\gamma(\psi_1)^{-1}e^{-\varepsilon\tilde{\psi}(x)}$$

by Lemmas 2 and 3. This completes the proof. ∎

Proof of the Theorem. Let $\psi \in \Gamma$ and suppose that $\psi(1) = 1$. If the number of terms in ψ is 1, then the theorem was proved in Lemma 4. If the number of terms in ψ is greater than 1, then we split ψ as $\psi_1 + \psi_2$ where $\psi_1, \psi_2 \in \Gamma$ and the number of terms of ψ_j is less than the number of terms in ψ. By the induction hypothesis, we know that there are positive constants C_j and ε_j $(j = 1, 2)$ such that

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi_j(x)}\,dx\Big| \leqslant C_j\gamma(\tilde{\psi}_j)e^{-\varepsilon_j\tilde{\psi}_j(t)}, \quad j = 1, 2.$$

Without loss of generality, we may assume that $\gamma(\psi_1) \geqslant 1/2$. Then

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx\Big| = \Big|\int_{-\infty}^{\infty} e^{ixt-\psi_1(x)-\psi_2(x)}\,dx\Big| \leqslant C\gamma(\psi_1)\gamma(\psi_2)(e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2})(t)$$

$$\leqslant C\gamma(\psi_2)(e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2})(t) \leqslant Ce^{-\varepsilon\tilde{\psi}(t)}$$

by Lemma 5. If $\psi(1) \neq 1$, we again make a change of variables $x \to \gamma(\psi)x$. This completes the proof. ∎

A final remark. It would be very interesting to see whether our theorem holds for general smooth convex functions.

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Received February 8, 1990 (2652)

STUDIA MATHEMATICA 98 (3) (1991)

Point derivations and prime ideals in $R(X)$

by

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Abstract. Let X be a compact plane set. Then $R(X)$ is the uniform algebra of all continuous functions on X which may be uniformly approximated on X by rational functions with poles off X. We give an example of a compact plane set X such that $R(X)$ is normal, and $R(X)$ contains a prime ideal whose closure is not prime. In order to construct this example, we give an example of a compact plane set X with $0 \in X$ for which $R(X)$ has a non-zero, continuous point derivation at 0, but such that the polynomial Z^2 may be uniformly approximated on X by functions in $R(X)$ which vanish on a neighbourhood of 0.

1. Introduction. Many counterexamples to conjectures in the theory of uniform algebras have been obtained by studying $R(X)$ for suitable compact plane sets X. In this paper, we shall work with one particular kind of compact plane set, the Swiss cheese.

In [M], McKissick was able to produce the first known example of a non-trivial, normal uniform algebra by constructing a suitable Swiss cheese. Wermer has found a Swiss cheese X for which $R(X)$ is non-trivial, but has no non-zero, continuous point derivations at any point of X (see [WE]). In [WA2], Wang gives an example of a Swiss cheese X for which $R(X)$ is strongly regular at a non-peak point. O'Farrell, in [OF], has given an example of a Swiss cheese X for which $R(X)$ is normal, but has a non-zero, continuous, infinite-order point derivation at a point of X.

We shall use methods adapted from those of these papers to produce various examples of Swiss cheeses, including an example of X such that $0 \in X$ and $R(X)$ has a non-zero, continuous point derivation at 0, but such that the polynomial Z^2 may be uniformly approximated on X by functions in $R(X)$ which vanish on a neighbourhood of 0. In Section 5, we shall give an example of a Swiss cheese X and a prime ideal P in $R(X)$, such that $\bar{P}$ is not prime.

Other results on point derivations of various orders have been obtained using the tools of analytic capacity. See, for example, [H].

1985 *Mathematics Subject Classification*: Primary 46J10.
This paper is based on part of the author's Doctoral Thesis, [F]. I would like to thank Garth Dales for his help and guidance. Financial support for my research was provided by the Science and Engineering Research Council.

2. Preliminary results, definitions and notation. Throughout this paper, all algebras considered are over the complex field **C**.

The following result about prime ideals follows immediately from Theorem 2.2 of [C, Chapter 9].

2.1. PROPOSITION. *Let I_1, I_2 be ideals in a commutative algebra A, with $I_1 \subseteq I_2$ and $I_2 \neq A$. Then the following statements are equivalent.*

(i) *There exists a prime ideal P with $I_1 \subseteq P \subseteq I_2$.*

(ii) *No element of I_1 may be expressed as a finite product of elements of $A \backslash I_2$.* ■

2.2. DEFINITION. Let A be a commutative algebra, and let φ be a character on A. A *point derivation at φ* is a linear functional d on A satisfying

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

A *point derivation of order n at φ* is a sequence $(d_k)_{k=0}^n$ of linear functionals on A, with $d_0 = \varphi$, and satisfying

$$d_k(ab) = \sum_{j=0}^{k} d_j(a)d_{k-j}(b) \quad (a, b \in A,\ k \in \{0, 1, 2, \ldots, n\}).$$

An *infinite-order point derivation at φ* is a sequence $(d_k)_{k=0}^\infty$ such that, for each $n \in \mathbf{N}$, $(d_k)_{k=0}^n$ is a point derivation of order n at φ.

A point derivation of a given order is *degenerate* if $d_1 = 0$.

2.3. DEFINITION. Let A be a normed algebra, and let (d_n) be a point derivation of some order on A. Then (d_n) is *continuous* if each linear functional d_n in the sequence is continuous.

Let S be a set, and let f be a bounded, complex-valued function on S. For each set E contained in S, the *uniform norm of f on E*, denoted by $|f|_E$, is defined by $|f|_E = \sup\{|f(x)|: x \in E\}$.

In our terminology, a *compact space* is a compact, Hausdorff topological space.

Let X be a compact space. Then $C(X)$ denotes the uniform algebra of all continuous functions from X into **C**. Let A be a uniform algebra on X. Then A is *trivial* if $A = C(X)$.

Let A be a uniform algebra on a compact space X. For each $x \in X$ the *evaluation character* ε_x is defined by $\varepsilon_x(f) = f(x)$ $(f \in A)$. We define ideals M_x, J_x by

$$M_x = \operatorname{Ker} \varepsilon_x, \quad J_x = \{f \in A: f^{-1}(\{0\}) \text{ is a neighbourhood of } x\}.$$

The uniform algebra A is *strongly regular at x* if J_x is dense in M_x.

2.4. DEFINITION. Let X be a compact subset of **C**. Then $R_0(X)$ is the set of restrictions to X of rational functions with poles off X, and $R(X)$ is the closure of $R_0(X)$ in $C(X)$.

For any compact plane set X, the character space of $R(X)$ is just X itself. For this, and other standard results about $R(X)$ see, for example, [BR].

Notation. Let $a \in \mathbf{C}$, and let $r > 0$. We denote by $\Delta(a, r)$ the set $\{z \in \mathbf{C}: |z-a| < r\}$, and by $\bar{\Delta}(a, r)$ the set $\{z \in \mathbf{C}: |z-a| \leqslant r\}$.

Let Δ be any disc in the plane. Then $r(\Delta)$ denotes the radius of Δ, and $s(\Delta)$ denotes the distance from 0 to Δ.

Remark. In this paper we shall work with the particular point 0, but we could have chosen to work with any other point in the open unit disc $\Delta(0, 1)$.

Notation. Throughout, Lebesgue measure on the plane is denoted by m. Let X be a compact plane set. Then Z denotes the coordinate functional on X. If X is an infinite set, then, for each $n \in \mathbf{N}$, we set

$$\delta_n(f) = \frac{1}{n!} f^{(n)}(0) \quad (f \in R_0(X)).$$

Clearly the sequence $(\delta_n)_{n=0}^\infty$ is an infinite-order point derivation on $R_0(X)$ at 0.

2.5. PROPOSITION [WE, p. 28]. *Let X be an infinite compact set with $0 \in X$.*

(i) *Let d be a point derivation on $R(X)$ at 0. Then the restriction of d to $R_0(X)$ is a multiple of δ_1.*

(ii) *There is a non-zero, continuous point derivation on $R(X)$ at 0 if and only if the linear functional δ_1 is continuous on $R_0(X)$.* ■

The following result is elementary.

2.6. LEMMA. *Let X be an infinite, compact subset of* **C** *such that $0 \in X$. Then any second-order point derivation at 0 on $R_0(X)$ has the form $(\delta_0, \alpha\delta_1, \alpha^2\delta_2 + \beta\delta_1)$ for some $\alpha, \beta \in$* **C**. ■

The compact plane sets X for which $R(X)$ is trivial can be characterized in terms of point derivations.

2.7. PROPOSITION [BR, p. 178]. *Let X be a compact plane set. Then $R(X) = C(X)$ if and only if there are no non-zero point derivations on $R(X)$.* ■

It was conjectured by Browder that this result would be false if "point derivation" were replaced by "continuous point derivation". This conjecture was proved by Wermer in [WE], where he gave the example, mentioned in the introduction, of a compact plane set X for which $R(X) \neq C(X)$, and $R(X)$ has no non-zero, continuous point derivations.

2.8. DEFINITION. Let X be a compact plane set. Then $R(X)$ is *regular* if, for each closed set F contained in X and each $x \in X \backslash F$, there exists $f \in R(X)$ with

$f(x)=1$ and $f(F) \subseteq \{0\}$; $R(X)$ is *normal* if, for each pair E, F of disjoint closed sets contained in X, there exists $f \in R(X)$ with $f(E) \subseteq \{1\}$ and $f(F) \subseteq \{0\}$.

In fact, whenever $R(X)$ is regular it is also normal (see [S, 27.2]).

We now introduce a useful family of compact plane sets, the Swiss cheeses.

For the rest of this section, $\mathscr{D}$ will denote a countable family of open discs in the plane, $\mathscr{D} = \{\Delta_1, \Delta_2, \ldots\}$.

2.9. DEFINITION. The *Swiss cheese* obtained from the family $\mathscr{D}$ of discs is the compact plane set

$$X = \bar{\Delta}(0, 1) \setminus \bigcup \{\Delta: \Delta \in \mathscr{D}\}.$$

We shall only be interested in Swiss cheeses X for which $R(X)$ is non-trivial.

2.10. LEMMA [BD, pp. 28–29]. *Let $n \in \mathbf{N}$, and set*

$$X_n = \bar{\Delta}(0, 1) \setminus \bigcup \{\Delta_k: 1 \leqslant k \leqslant n\}.$$

Then the boundary of X_n provides a closed contour Γ_n satisfying:

(i) *Γ_n has winding number 0 round each point of $\mathbf{C} \setminus X_n$;*

(ii) *Γ_n has winding number 1 round each point of the interior of X_n.* ∎

The following lemma is a slight generalization of a result of Browder (see [WE], p. 34).

2.11. LEMMA. *Let X be the Swiss cheese obtained from the family $\mathscr{D}$. If $\mathscr{D}$ satisfies*

$$\sum_{\Delta \in \mathscr{D}} r(\Delta)/s(\Delta)^2 < \infty, \tag{1}$$

then there exists a non-zero, continuous point derivation on $R(X)$ at 0.

Proof. We shall use the notation of 2.10. Clearly $0 \in \operatorname{Int}(X_n)$ $(n \in \mathbf{N})$. Let f be a rational function with no poles in X. There exists $N_0 \in \mathbf{N}$ such that f has no poles in X_n $(n \geqslant N_0)$. By 2.10 and Cauchy's formula for the derivative,

$$f'(0) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(z)\,dz}{z^2} \quad (n \geqslant N_0).$$

Since Γ_n is the boundary of X_n,

$$\left| \int_{\Gamma_n} \frac{f(z)\,dz}{z^2} \right| \leqslant \left(\sum_{j=1}^{n} \frac{2\pi r(\Delta_j)}{s(\Delta_j)^2} + 2\pi \right) |f|_{X_n},$$

and so $|f'(0)| \leqslant C|f|_{X_n}$ $(n \geqslant N_0)$, where $C = 1 + \sum_{\Delta \in \mathscr{D}} r(\Delta)/s(\Delta)^2$. Taking the limit as $n \to \infty$, we obtain $|f'(0)| \leqslant C|f|_X$. The result now follows from 2.5(ii). ∎

3. Point derivations of order one or two.

In this section, we shall consider point derivations of order less than or equal to 2 on $R(X)$. First we recall the definition of an approximate limit.

3.1. DEFINITION. Let E be a Lebesgue measurable subset of $\mathbf{C}$. Then E has *full area density* at a point $z \in \mathbf{C}$ if

$$\lim_{r \to 0+} \frac{m(E \cap \Delta(z, r))}{\pi r^2} = 1.$$

3.2. DEFINITION. Let S be a subset of $\mathbf{C}$, let f be a function from S into $\mathbf{C}$, and let $z_0 \in \mathbf{C}$. Then f has an *approximate limit* as $z \to z_0$ if there exists a Lebesgue set E contained in S satisfying:

(i) E has full area density at z_0;

(ii) $f(z)$ is convergent, as $z \to z_0$ through values in E.

The limit in (ii) is independent of the choice of E, and we shall denote it by $\operatorname{app\,lim}_{z \to z_0} f(z)$.

The following lemma is a special case of [BR, 3.3.9].

3.3. PROPOSITION. *Let X be a compact plane set, and let $z \in X$. If z is a non-peak point for $R(X)$, then X has full area density at z.* ∎

The following result is the case $p = 1$ of [WA1, 3.6], taking $x = 0$.

3.4. PROPOSITION. *Let X be a compact subset of $\mathbf{C}$ containing 0, and suppose that d_1 is a continuous point derivation on $R(X)$ at 0, extending δ_1. Then for each $f \in R(X)$, $(f(z) - f(0))/z$ has an approximate limit as $z \to 0$, and*

$$\operatorname*{app\,lim}_{z \to 0} \frac{f(z) - f(0)}{z} = d_1(f).$$

3.5. COROLLARY. *With the same assumptions as in 3.4, and denoting the evaluation character at 0 by d_0, the following further conclusions may be drawn.*

(i) *There is a linear functional d_2 on $R(X)$ extending δ_2, such that (d_0, d_1, d_2) is a second-order point derivation on $R(X)$ at 0.*

(ii) *Let I be the closed ideal $M_0 \cap \operatorname{Ker} d_1$. Then I is not prime in $R(X)$, but $J_0 \subseteq I$, and there exists a prime ideal P with $J_0 \subseteq P \subseteq I$.*

Proof. (i) Let $f, g \in M_0$. Then, by 3.4, $\operatorname{app\,lim}_{z \to 0} f(z)g(z)/z^2$ exists, and is equal to $d_1(f)d_1(g)$. It now follows that there exists a linear functional d_2 on $R(X)$, such that $d_2(1) = 0$, $d_2(Z) = 0$, and $d_2(h) = \operatorname{app\,lim}_{z \to 0} h(z)/z^2$ $(h \in M_0^2)$. It is clear that (d_0, d_1, d_2) is a second-order point derivation on $R(X)$ at 0. By 2.6, $d_2|R_0(X)$ is of the form $\delta_2 + \beta\delta_1$, for some $\beta \in \mathbf{C}$. Since $d_2(Z) = 0$, we deduce that $\beta = 0$. Thus d_2 has the required properties.

(ii) Set $I = M_0 \cap \operatorname{Ker} d_1$. It is clear that I is a closed ideal in $R(X)$, and that I is not prime, because $Z^2 \in I$, but $Z \notin I$. The fact that $J_0 \subseteq I$ is immediate from 3.4. By 2.1, to show that the prime ideal P exists it is sufficient to show that no element of J_0 may be expressed as a finite product of elements of $R(X) \setminus I$.

It is easy to show that if $f \in R(X)$, then $f \notin I$ if and only if there exists $C > 0$, and a Lebesgue set E contained in X with full area density at 0, such that $|f(z)|$

$\geqslant C|z|$ $(z \in E)$. The result now follows easily, since the intersection of finitely many sets of full area density at 0 also has full area density at 0. ∎

The following result can be obtained from the calculations in [OF], or from first principles. The proof is similar to that of McKissick's theorem (see [M]). The extra property is obtained by eliminating some of the redundancy in the usual choice of the discs to be deleted.

3.6. PROPOSITION. *There is a countable collection $\mathscr{D}$ of open discs in the plane satisfying* (1), *and such that if X is the Swiss cheese obtained from the family $\mathscr{D}$, then $R(X)$ is regular, and $R(X)$ has a non-zero, continuous point derivation at* 0. ∎

The following result is known. We provide a short proof.

3.7. THEOREM. *There is a countable collection $\mathscr{D}$ of open discs in the plane with the following properties.*

(i) *The family $\mathscr{D}$ satisfies condition* (1).

(ii) *Set $X = \bar{\Delta}(0, 1)\setminus\bigcup\{\Delta\colon \Delta \in \mathscr{D}\}$. Then $0 \in X$, and δ_2 is discontinuous on $R_0(X)$.*

(iii) *There is no non-degenerate, continuous, second-order point derivation at 0 on $R_0(X)$.*

(iv) *There is a non-degenerate, discontinuous, second-order point derivation at 0 on $R(X)$.*

Proof. Set $a_n = 2^{-n}$, $r_n = 2^{-(2n+1)}/n^2$, and $\Delta_n = \Delta(a_n, r_n)$ $(n \in \mathbf{N})$. Set $\mathscr{D} = \{\Delta_n\colon n \in \mathbf{N}\}$. We shall show that $\mathscr{D}$ satisfies (i) to (iv).

(i) This is clear.

(ii) For each $n \in \mathbf{N}$, set

$$f_n = r_n/(a_n - Z).$$

Then $f_n \in R_0(X)$, and $|f_n|_X \leqslant 1$ $(n \in \mathbf{N})$. But

$$\delta_2(f_n) = 2r_n/a_n^3 = 2^n/n^2 \qquad (n \in \mathbf{N}),$$

and so $|\delta_2(f_n)| \to \infty$ as $n \to \infty$. Thus δ_2 is discontinuous on $R_0(X)$.

(iii) By 2.11, δ_1 is continuous on $R_0(X)$. By 2.6, any second-order point derivation at 0 on $R_0(X)$ is of the form $(\delta_0, \alpha\delta_1, \alpha^2\delta_2 + \beta\delta_1)$, for some $\alpha, \beta \in \mathbf{C}$. Since δ_1 is continuous, and δ_2 is discontinuous, such a point derivation must either be discontinuous or degenerate.

(iv) By 3.5, there is a second-order point derivation (d_0, d_1, d_2) on $R(X)$ at 0, such that $d_j|R_0(X) = \delta_j$ $(j = 1, 2)$. Since δ_2 is discontinuous, so is d_2. ∎

4. Functions vanishing in a neighbourhood of 0. In [WA2] Wang gave an example of a Swiss cheese X with $0 \in X$ such that 0 is not a peak point for $R(X)$, but the coordinate functional Z is in the closure of the ideal J_0, and hence $R(X)$ is strongly regular at 0. In this section we shall give an example of a Swiss cheese X with $0 \in X$ such that $R(X)$ has a non-zero, continuous point derivation at 0, but $Z^5 \in \bar{J}_0$. The power five will be replaced by two in the next section, where we combine all the results of Sections 2 to 4.

We shall need some technical results. We quote the following result of Körner.

4.1. LEMMA [K, 2.3]. *There exists $N_0 \geqslant 2$ such that, for each $n \geqslant N_0$, there is a finite collection $A(n)$ of pairwise disjoint open discs, and a rational function g_n with the following properties:*

(i) $\sum_{\Delta \in A(n)} r(\Delta) = 1/n^2$;

(ii) *the poles of g_n lie in* $\bigcup\{\Delta\colon \Delta \in A(n)\}$;

(iii) $|g_n(z)| \leqslant (n+1)^{-4}$ $(|z| \geqslant 1 - 2^{-(2n+1)})$;

(iv) $|1 - g_n(z)| \leqslant (n+1)^{-4}$ $(|z| \leqslant 1 - 2^{-(2n-1)})$;

(v) $|g_n(z)| \leqslant 2n^2$ $(z \in \mathbf{C}\setminus\bigcup\{\Delta\colon \Delta \in A(n)\})$;

(vi) *the rational function g_n has no zeros in* $\mathbf{C}$;

(vii) $\bigcup\{\Delta\colon \Delta \in A(n)\} \subseteq \{z \in \mathbf{C}\colon 1 - 2^{-(2n-1)} \leqslant |z| \leqslant 1 - 2^{-(2n+1)}\}$;

(viii) $g_n(0) = 1$ (this is not stated in Körner's lemma). ∎

Using the notation of Lemma 4.1, for $m \geqslant N_0$ set

$$f_{m,n} = \prod_{r=m}^{n} g_r \qquad (n \geqslant m),$$

and let $(\Delta_{m,k})_{k=1}^{\infty}$ be an enumeration of the discs in $\bigcup_{r=m}^{\infty} A(r)$. Set

$$r_{m,k} = r(\Delta_{m,k}) \qquad (k \in \mathbf{N}),$$

$$V_m = \bigcup_{k=1}^{\infty} \Delta_{m,k}.$$

We shall need the following two technical lemmas. The proof of the first is the simple induction mentioned during the proof of [K, 2.4]. The second is essentially [K, 2.4], slightly rewritten.

4.2. LEMMA. *With notation as above, let $m \geqslant N_0$, and let $z \in \mathbf{C}\setminus V_m$. Then, for each $n \geqslant m$, the following three conditions are satisfied:*

$(\mathrm{i})_n$ *if* $|z| \leqslant 1 - 2^{-(2n-1)}$, *then*

$$|f_{m,n}(z)| \leqslant 2(m+1)^4 \prod_{j=m}^{n} (1 + (j+1)^{-4});$$

$(\mathrm{ii})_n$ *if* $1 - 2^{-(2n-1)} \leqslant |z| \leqslant 1 - 2^{-(2n+1)}$, *then* $|f_{m,n}(z)| \leqslant 2(m+1)^4 n^{-2}$;

$(\mathrm{iii})_n$ *if* $|z| \geqslant 1 - 2^{-(2n+1)}$, *then* $|f_{m,n}(z)| \leqslant (m+1)^4 (n+1)^{-4}$. ∎

4.3. LEMMA. *With notation as above, let $m \geqslant N_0$. Then the following hold.*

(i) $\sum_{k=1}^{\infty} r_{m,k} \leqslant 1/(m-1)$.

(ii) *The poles of $f_{m,n}$ lie in V_m* $(n \in \mathbf{N})$.

(iii) *The sequence* $(f_{m,n})_{n=m}^{\infty}$ *converges uniformly on* $\mathbf{C}\setminus V_m$ *to a function* F_m. *Furthermore,* F_m *is nowhere zero on* $\Delta(0, 1)\setminus V_m$, *while* F_m *is identically zero on* $\mathbf{C}\setminus\Delta(0, 1)$.

(iv) $V_m \subseteq \{z\in\mathbf{C}\colon 1-2^{-(2m-1)} < |z| < 1\}$.

(v) *The discs* $(\Delta_{m,k})_{k=1}^{\infty}$ *are pairwise disjoint.*

(vi) *Set* $C = 2\prod_{i=1}^{\infty}(1+(1+r)^{-4})$. *Then* C *is independent of* m, *and*

$$|f_{m,n}|_{\mathbf{C}\setminus V_m} \leqslant C(m+1)^4 \quad (n \geqslant m).$$

(vii) $f_{m,n}(0) = 1$ $(n \geqslant m)$.

Proof. (i) This follows from 4.1(i), and from the fact that

$$\int_{m-1}^{\infty}\frac{dt}{t^2} = \frac{1}{m-1}.$$

For (ii) to (vi), see [K, 2.4].

(vii) This is trivial. ∎

The following elementary result concerning the images of open discs under certain transformations is seen by direct calculation.

4.4. Lemma. *Let* a *be a non-zero complex number, suppose that* $0 < r < |a|$, *and let* $\lambda\in\mathbf{C}\setminus\{0\}$. *Set* $\Delta = \Delta(a, r)$. *Then the set* $\{\lambda/z\colon z\in\Delta\}$ *is an open disc* Δ', *with centre at* a', *and with radius* r', *where*

$$a' = \frac{\lambda\bar{a}}{|a|^2-r^2}, \qquad r' = \frac{|\lambda| r}{|a|^2-r^2}.$$

If also $r \leqslant |a|/3$, *then*

$$\frac{r(\Delta')}{s(\Delta')^2} \leqslant \frac{9r}{4|\lambda|}. \ ∎$$

For the rest of this section, our notation will be as in Lemmas 4.1 to 4.3.

4.5. Lemma. *The functions* F_m *satisfy the following three conditions.*

(i) *Let* $\varrho\in(0, 1)$. *Then for all sufficiently large* m, $\bar{\Delta}(0, \varrho)\cap V_m = \emptyset$, *and, as* $m\to\infty$, F_m *converges to* 1 *uniformly on* $\bar{\Delta}(0, \varrho)$.

(ii) *There is a constant* $C > 0$ *with*

$$|F_m|_{\mathbf{C}\setminus V_m} \leqslant C(m+1)^4 \quad (m \geqslant N_0).$$

(iii) $F_m(0) = 1$ $(m \geqslant N_0)$.

Proof. Conditions (ii) and (iii) are immediate from 4.3.

(i) There exists $N \geqslant N_0$ such that $\varrho < 1-2^{-(2N-1)}$. For $m \geqslant N$, we have $\varrho < 1-2^{-(2m-1)}$, and so $\bar{\Delta}(0, \varrho) \subseteq \mathbf{C}\setminus V_m$.

By 4.1(iv), for $n \geqslant N$ and $z\in\bar{\Delta}(0, \varrho)$, we have $|1-g_n(z)| \leqslant (n+1)^{-4}$. Thus by

[R, 15.3 and 15.4],

$$|F_m - 1|_{\bar{\Delta}(0,\varrho)} \leqslant \exp\Big(\sum_{n=m}^{\infty}(n+1)^{-4}\Big)-1 \quad (m \geqslant N).$$

The result now follows. ∎

We are now able to prove the result mentioned earlier concerning Z^5.

4.6. Lemma. *There is a countable collection* $\mathscr{D}$ *of open discs in the plane with the following properties.*

(i) *The family* $\mathscr{D}$ *satisfies condition* (1).

(ii) *Set* $X = \bar{\Delta}(0, 1)\setminus\bigcup\{\Delta\colon \Delta\in\mathscr{D}\}$. *Then, in* $R(X)$, $Z^5\in\bar{J}_0$.

Proof. With the above notation, we shall denote by $r_{m,k}$, and $a_{m,k}$, respectively, the radius and the centre of the open disc $\Delta_{m,k}$ $(m \geqslant N_0,\ k\in\mathbf{N})$. It is clear that $r_{m,k} \leqslant |a_{m,k}|/3$, and this will be important in later calculations.

For each $n \geqslant N_0$, set

$$M_n = n|F_{2^n}|_{\mathbf{C}\setminus V_{2^n}},$$

and define a rational function L_n by

$$L_n(z) = \tfrac{1}{2}M_n^{-1/5}z^{-1}.$$

Note that $L_n\circ L_n$ is the identity on $\mathbf{C}\setminus\{0\}$. Set

$$\Delta'_{n,k} = L_n(\Delta_{2^n,k}) \quad (k\in\mathbf{N}).$$

By Lemma 4.4, each $\Delta'_{n,k}$ is an open disc. We set

$$\mathscr{D} = \{\Delta'_{n,k}\colon n \geqslant N_0,\ k\in\mathbf{N}\}.$$

We shall show that this $\mathscr{D}$ satisfies the conditions of the lemma. By Lemma 4.4 again,

$$r(\Delta'_{n,k})/s(\Delta'_{n,k})^2 \leqslant \tfrac{18}{4}M_n^{1/5}r_{2^n,k} \quad (n \geqslant N_0,\ k\in\mathbf{N}).$$

By 4.5(ii), there is a constant C such that, for each $m \geqslant N_0$, $|F_m|_{\mathbf{C}\setminus V_m} \leqslant C(m+1)^4$. Thus

$$M_n \leqslant Cn(2^n+1)^4 \quad (n \geqslant N_0).$$

From Lemma 4.3, we have

$$\sum_{k=1}^{\infty} r_{2^n,k} \leqslant 1/(2^n-1),$$

and so

$$\sum_{n=N_0}^{\infty}\sum_{k=1}^{\infty}\frac{r(\Delta'_{n,k})}{s(\Delta'_{n,k})^2} \leqslant \sum_{n=N_0}^{\infty}\frac{18}{4(2^n-1)}(Cn(2^n+1)^4)^{1/5} < \infty.$$

This proves part (i).

To prove that (ii) is true, set

$$G_n(z) = F_{2^n}(L_n(z)) \quad (n \geqslant N_0,\ z \in \mathbf{C}\backslash L_n(V_{2^n});\ z \neq 0).$$

Clearly each G_n is continuous on $\mathbf{C}\backslash(L_n(V_{2^n}) \cup \{0\})$, and is zero on $L_n(\mathbf{C}\backslash\bar{\Delta}(0, 1))$, which is a punctured disc with centre at 0. Thus G_n extends to a continuous function on $\mathbf{C}\backslash L_n(V_{2^n})$ which is zero on a neighbourhood of 0. It is clear that G_n is the uniform limit on $\mathbf{C}\backslash L_n(V_{2^n})$ as $k \to \infty$ of the rational functions $f_{2^n,k} \circ L_n$, which have no poles in X. Thus $G_n|X \in J_0$ in $R(X)$ $(n \in \mathbf{N})$.

We now show that

$$|Z^5 - Z^5 G_n|_X \to 0 \quad \text{as } n \to \infty.$$

For any $n \geqslant N_0$, $z \in X$, consider the following two cases.

(a) If $|z| \geqslant M_n^{-1/5}$, then $|L_n(z)| \leqslant 1/2$, and so

$$|z^5|\,|G_n(z) - 1| \leqslant |F_{2^n} - 1|_{\bar{\Delta}(0,1/2)}.$$

(b) If $|z| \leqslant M_n^{-1/5}$, then $|z|^5 \leqslant (n|F_{2^n}|_{\mathbf{C}\backslash V_{2^n}})^{-1} \leqslant (n|G_n|_X)^{-1}$. Since $|F_{2^n}|_{\mathbf{C}\backslash V_{2^n}} \geqslant F_{2^n}(0) = 1$, we also have $|z|^5 \leqslant n^{-1}$, and so

$$|z|^5 |G_n(z) - 1| \leqslant 2/n.$$

Combining these estimates gives us the following:

$$|Z^5 - Z^5 G_n|_X \leqslant \max\{2/n,\ |F_{2^n} - 1|_{\bar{\Delta}(0,1/2)}\} \quad (n \geqslant N_0).$$

Thus, by 4.5(i) with $\varrho = 1/2$, $|Z^5 - Z^5 G_n|_X \to 0$ as $n \to \infty$. Since $Z^5 G_n | X \in J_0$ $(n \in \mathbf{N})$, the result is proved. ∎

5. A prime ideal in $R(X)$ whose closure is not prime. The following is a naive conjecture: The closure of a prime ideal in a commutative Banach algebra is also prime. We now give an example to show that this conjecture fails even for regular uniform algebras. This example will use the results of Sections 3 and 4. A much easier example is available for regular, commutative Banach algebras: the Banach algebra $C^1[0, 1]$, consisting of those $f \in C([0, 1])$ which are continuously differentiable on the unit interval, contains many prime ideals whose closures are not prime.

5.1. THEOREM. *There is a countable collection $\mathscr{D}$ of open discs in the plane with the following properties.*

(i) *The family $\mathscr{D}$ satisfies condition* (1).

(ii) *Set $X = \bar{\Delta}(0, 1)\backslash \bigcup\{\Delta\colon \Delta \in \mathscr{D}\}$. Then $R(X)$ is regular.*

(iii) *The linear functional δ_1 is continuous on $R_0(X)$, but the linear functional δ_2 is discontinuous there. The functional δ_1 extends to a continuous point derivation d on $R(X)$ at* 0.

(iv) *There are no non-degenerate, continuous, second-order point derivations on $R(X)$ at* 0.

(v) *There is a non-degenerate, discontinuous, second-order point derivation on $R(X)$ at* 0.

(vi) *In $R(X)$,* (a) *$Z^2 \in \bar{J}_0$, and* (b) *$\bar{J}_0 = M_0 \cap \operatorname{Ker} d$.*

(vii) *The ideal $\bar{J}_0$ is not prime, but there exists a prime ideal P with $J_0 \subseteq P \subseteq \bar{J}_0$.*

(viii) *The regular uniform algebra $R(X)$ contains a prime ideal P such that $\bar{P}$ is not prime in $R(X)$.*

Proof. Let $\mathscr{D}$ be the union of the three countable collections obtained in 3.6, 3.7 and 4.6. We shall now show that for this $\mathscr{D}$, properties (i) to (viii) hold.

By the choice of $\mathscr{D}$, and 3.6, 3.7, it is clear that properties (i) to (iii) hold for the Swiss cheese X. As in 3.7, properties (iv) and (v) hold.

(vi) Since X is contained in the Swiss cheese of 4.6, it is clear that $Z^5 \in \bar{J}_0$. Thus $\overline{Z^5 R(X)} \subseteq \bar{J}_0$.

To prove (a), it is enough to show that $Z^2 \in \overline{Z^3 R(X)}$. Since $\operatorname{Ker} \delta_2$ is dense in $R_0(X)$, there is a sequence (r_n) of elements of $\operatorname{Ker} \delta_2$ such that $|r_n - Z^2|_X \to 0$ as $n \to \infty$. Set

$$R_n = r_n - r_n(0)1 - r_n'(0)Z \quad (n \in \mathbf{N}).$$

Then each R_n is an element of $M_0 \cap \operatorname{Ker} \delta_1 \cap \operatorname{Ker} \delta_2$, and $R_n \to Z^2$ as $n \to \infty$. Clearly, each R_n is an element of $Z^3 R_0(X)$, and so $Z^2 \in \overline{Z^3 R(X)}$, proving (a).

It is easy to see that

$$M_0 \cap \operatorname{Ker} d = \overline{Z^2 R(X)}.$$

Thus (b) now follows from (a).

(vii) Since $\bar{J}_0 = M_0 \cap \operatorname{Ker} d$, this is immediate from 3.5(ii).

(viii) This is immediate from (vii).

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Received February 21, 1990
Revised version September 18, 1990 (2657)

STUDIA MATHEMATICA 98 (3) (1991)

Continuity of a homomorphism on commutative subalgebras is not sufficient for continuity

by

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Abstract. Examples of topological algebras are given to show that the structure of all commutative subalgebras does not determine the structure of the given algebra.

1. In the list of problems [2] formulated by Professor W. Żelazko concerning topological algebras there is the following one (Problem 4): Suppose that there are two topologies τ_1 and τ_2 making an algebra X a topological algebra. Suppose that for each commutative subalgebra $Y \subseteq X$ we have $\tau_1 | Y = \tau_2 | Y$, where $\tau | Y$ is the restriction of the topology τ to Y. Does it follow that $\tau_1 = \tau_2$? A negative answer to this question provides a negative answer to the next problem on the list: Suppose that a homomorphism h of topological algebras X into Y is continuous when restricted to each commutative subalgebra of X. Does it follow that h is continuous?

The following theorem holds.

THEOREM. *There is an algebra* $\mathbf{X}$ *and two different topologies* τ_1 *and* τ_2 *such that* (X, τ_1) *and* (X, τ_2) *are topological algebras and* τ_1, τ_2 *coincide on commutative subalgebras.*

Proof. Recall the example given in [1]. Let S be the free semigroup (with unit) of words over the alphabet A. Assume that A is infinite. For $B \subseteq A$, let $S(B)$ denote the free semigroup with unit over the alphabet B; in this notation $S = S(A)$. Let $\mathbf{R}^S$ be the algebra of real functions defined on S, with the convolution $x * y$ as multiplication. Let X be the set of those $x \in \mathbf{R}^S$ for which there is a finite $B \subseteq A$ such that $x(s) = 0$ for $s \in S \setminus S(B)$, and $\sum\{|x(s)|: s \in S\} < \infty$. For $x \in X$ we define two norms: $\|x\| = \sum\{|x(s)|: s \in S\}$ (the usual l^1-norm) and $\|x\|_1 = \sum\{2^{n(s)}|x(s)|: s \in S\}$ (the weighted l^1-norm) where $n(s)$ is the cardinality of the set of different letters appearing in s. It is easy to check that both are multiplicative norms, so both $(X, \|\cdot\|)$ and $(X, \|\cdot\|_1)$ are topological algebras.

1985 *Mathematics Subject Classification*: 46H10, 43A20.