PROPOSITION 2. *A measure of the form $\sigma * \sigma$ can never be a semi-Kronecker measure.*

Proof. This measure has positive Fourier coefficients; hence it is impossible to approximate the constant -1 by characters.

COROLLARY 1. (W, S), *which is spectrally isomorphic to a rank one system, is not of rank one.*

COROLLARY 2. *Rank is not a spectral invariant. Also, WCT is not a spectral property, and it is still not a spectral property when we restrict ourselves to the class of systems which are spectrally isomorphic to rank one systems.*

Remarks. The rank of (W, S) is not known.

For any measure σ on the circle, we can define the systems (Y, T) and (W, S) in the same way. These give spectrally isomorphic systems; in several other cases we can prove that they are not metrically isomorphic, for example when σ is singular and $\sigma * \sigma$ is absolutely continuous ([7]) or when σ is concentrated on a semi-Kronecker set (using Thouvenot's [8] theory of Gaussian–Kronecker factors). Is this true for every singular σ?

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On the Fourier transform of $e^{-\psi(x)}$

by

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Abstract. We prove that the Fourier transform of $e^{-\psi(x)}$ where $\psi(x)$ is a convex polynomial with positive coefficients can be estimated by $e^{-\tilde{\psi}(x)}$ where $\tilde{\psi}(x)$ is the Legendre transform of $\psi(x)$.

1. Introduction. In this paper we investigate the behavior of the Fourier transform of the function $e^{-\psi(x)}$ where ψ is a convex polynomial on $\mathbf{R}$. Since $e^{-\psi(x)}$ belongs to the Schwartz class, we know that the Fourier transform of $e^{-\psi(x)}$ decays faster than the reciprocal of any polynomial. But, since the decay of $e^{-\psi(x)}$ is exponential, we should be able to say more about its Fourier transform. In fact, we prove that the behavior of the Fourier transform of $e^{-\psi(x)}$ is controlled by $e^{-\tilde{\psi}(x)}$ where $\tilde{\psi}$ is the Legendre transform of ψ when ψ belongs to a certain class of functions. The Legendre transform of a convex function $\psi(x)$ such that $\psi(0) = \psi'(0) = 0$ is defined by

$$\tilde{\psi}(x) = \sup_{p\in\mathbf{R}} (xp - \psi(p)). \tag{1}$$

For a geometric meaning of the Legendre transform, see [1]. A precise statement of our result is as follows.

THEOREM. *Let $\psi(x) = \sum_{j=1}^{m} a_j x^{2j}$ be an even convex polynomial. Assume that $a_j \geqslant 0$ for all j. Then there are positive constants C and ε depending only on m such that*

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx\Big| \leqslant C\psi^{-1}(1)e^{-\varepsilon\tilde{\psi}(t)} \tag{2}$$

where $\psi^{-1}(1)$ is the positive number u such that $\psi(u) = 1$.

If $\psi(x) = x^2$, then $\tilde{\psi}(x) = x^2$, and hence the theorem holds for e^{-x^2} since the Fourier transform of e^{-x^2} is $\sqrt{\pi}e^{-x^2/(16\pi^2)}$ [2].

2. Proofs. Let Γ be the class of all nonzero even convex polynomials $\psi(x) = \sum_{j=1}^{m} a_j x^{2j}$ where $a_j \geqslant 0$ (not all zero). We prove the theorem by induction on the number of terms in the polynomial in Γ. We begin with some preliminary observations on convex functions and their Legendre transforms.

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LEMMA 1. *If ψ is a convex function such that $\psi(0) = \psi'(0) = 0$, then*

$$\tilde{\psi}(x) = \int_0^x (\psi')^{-1}(t)\,dt. \tag{3}$$

In other words, $(\tilde{\psi})'$ is the inverse function of ψ'.

Proof. For any $y \in \mathbf{R}$, we have

$$\psi((\psi')^{-1}(y)) = \int_0^{(\psi')^{-1}(y)} \psi'(t)\,dt = y(\psi')^{-1}(y) - \int_0^y (\psi')^{-1}(s)\,ds$$

by a change of variable $s = \psi'(t)$. Therefore, since $xp - \psi(p)$ attains its maximum at $p = (\psi')^{-1}(x)$,

$$\tilde{\psi}(x) = \sup(xp - \psi(p)) = x(\psi')^{-1}(x) - \psi((\psi')^{-1}(x)) = \int_0^x (\psi')^{-1}(t)\,dt.$$

This completes the proof. ∎

DEFINITION. Let ψ be a convex function. We denote by $\gamma(\psi)$ the positive number such that $\psi(\gamma(\psi)) = 1$.

LEMMA 2. *Let $\psi \in \Gamma$. Then there are positive constants c and C depending only on the degree of ψ such that*

$$c \leqslant \gamma(\psi)\gamma(\tilde{\psi}) \leqslant C.$$

Proof. Let $\psi(x) = \sum_{j=1}^m a_j x^{2j}$ with $a_j \geqslant 0$. Suppose that $\gamma(\psi) = 1$. Then $\sum_{j=1}^m a_j = 1$ and hence there are positive constants a, b, c, and d depending only on m such that $a < 1 < b$ and $c < \psi'(a) < 1 < \psi'(b) < d$. Since $\int_0^c (\tilde{\psi})'(x)\,dx < 1$ and since $\int_0^{2d} (\tilde{\psi})'(x)\,dx \geqslant \int_d^{2d} (\tilde{\psi})'(x)\,dx \geqslant 1$, we have $c < \gamma(\tilde{\psi}) < 2d$.

If $\gamma(\psi) \neq 1$, then let $\varphi(x) = \psi(\gamma(\psi)x)$. Then $\gamma(\varphi) = 1$ and hence $c < \gamma(\tilde{\varphi}) < 2d$. Since $\tilde{\varphi}(x) = \tilde{\psi}(\gamma(\psi)^{-1}x)$, $\gamma(\tilde{\varphi}) = \gamma(\psi)\gamma(\tilde{\psi})$. So Lemma 2 follows with $C = 2d$. ∎

LEMMA 3. *Let $\psi \in \Gamma$. Then there exists a constant $C > 0$ depending only on the degree of ψ such that*

$$\int_{-\infty}^{\infty} e^{-\psi(x)}\,dx \leqslant C\gamma(\psi) \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\tilde{\psi}(x)}\,dx \leqslant C\gamma(\tilde{\psi}).$$

Proof. Let $\psi(x) = \sum_{j=1}^m a_j x^{2j}$ be in the class Γ. Suppose that $\gamma(\psi) = 1$. Let $\alpha = \psi'(1) = \sum_{j=1}^m 2ja_j$. Then $\alpha x \leqslant \psi'(x) \leqslant \alpha x^{2m-1}$ for $x \geqslant 1$ and $\alpha x^{2m-1} \leqslant \psi'(x) \leqslant \alpha x$ for $x \leqslant 1$. Hence $\alpha^{-1}x^{1/(2m-1)} \leqslant (\tilde{\psi})'(x) \leqslant \alpha^{-1}x$ for $x \geqslant 1$ and $\alpha^{-1}x \leqslant (\tilde{\psi})'(x) \leqslant \alpha^{-1}x^{1/(2m-1)}$ for $x \leqslant 1$. This implies Lemma 3 when $\gamma(\psi) = 1$. If $\gamma(\psi) \neq 1$, then we use the same change of variables as in the proof of Lemma 2. This completes the proof. ∎

LEMMA 4. *Let $\psi(x) = ax^{2m}$ where a is a constant. Then there are positive constants C and ε such that*

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx\Big| \leqslant Ca^{-1/(2m)}e^{-\varepsilon\tilde{\psi}(t)}.$$

Proof. We first assume that $a = 1$. Observe that there are positive constants α and β depending only on m such that $\operatorname{Re}(1+iu)^{2m} \geqslant \alpha - \beta u^{2m}$ for all $u \in \mathbf{R}$. And $\operatorname{Re}(x+iy)^{2m} \geqslant \alpha x^{2m} - \beta y^{2m}$ for all $x, y \in \mathbf{R}$. With this β, we put $y = (\tilde{\psi})'(\beta^{-1}t)$. Then $t = \beta\psi'(y)$ and we have

$$\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx = \int_{-\infty}^{\infty} e^{i\beta\psi'(y)x-\psi(x)}\,dx = e^{-\beta\psi'(y)y}\int_{-\infty}^{\infty} e^{i\beta\psi'(y)x-\psi(x+iy)}\,dx.$$

Here we made a contour change $x \to x+iy$. Therefore,

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx\Big| \leqslant e^{-\beta\psi'(y)y}\Big|\int_{-\infty}^{\infty} e^{i\beta\psi'(y)x-\psi(x+iy)}\,dx\Big| \leqslant e^{-\beta\psi'(y)y}\Big|\int_{-\infty}^{\infty} e^{-\operatorname{Re}\psi(x+iy)}\,dx\Big|$$

$$\leqslant e^{-\beta\psi'(y)y}\Big|\int_{-\infty}^{\infty} e^{-\alpha\psi(x)+\beta\psi(y)}\,dx\Big| \leqslant e^{-\beta\psi'(y)y+\beta\psi(y)}\Big|\int_{-\infty}^{\infty} e^{-\alpha\psi(x)}\,dx\Big|$$

$$\leqslant Ce^{-\beta\psi'((\tilde{\psi})'(\beta^{-1}t))(\tilde{\psi})'(\beta^{-1}t)+\beta\psi((\tilde{\psi})'(\beta^{-1}t))}.$$

But, by Lemma 1, the last exponent equals

$$-t(\tilde{\psi})'(\beta^{-1}t) + t(\tilde{\psi})'(\beta^{-1}t) - \beta\int_0^{\beta^{-1}t} (\tilde{\psi})'(y)\,dy = -\beta\tilde{\psi}(\beta^{-1}t),$$

and hence

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx\Big| \leqslant Ce^{-\beta\tilde{\psi}(\beta^{-1}t)} \leqslant Ce^{-\varepsilon\tilde{\psi}(t)}$$

for some constant ε depending only on m since $\tilde{\psi}(t) = (2m)^{-2m/(2m-1)}(2m-1) \times t^{2m/(2m-1)}$. If $a \neq 1$, we make a change of variables $x \to a^{-1/(2m)}x$ as in the proof of Lemma 2. This completes the proof. ∎

LEMMA 5. *Let ψ_1, ψ_2, and ψ be in the class Γ and let $\psi = \psi_1 + \psi_2$. Assume that $\gamma(\psi) = 1$. Then for any ε_1 and ε_2, there exist positive constants $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2)$ and $C = C(\varepsilon_1, \varepsilon_2, \deg(\psi))$ such that*

$$e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2} \leqslant C\min(1/\gamma(\psi_1), 1/\gamma(\psi_2))e^{-\varepsilon\tilde{\psi}}.$$

Proof. Without loss of generality, we may assume that $\gamma(\psi_1) \geqslant \gamma(\psi_2)$. Observe that

$$\tilde{\psi}(x) = \sup_p (xp - \psi(p)) = \sup_p (yp - \psi_1(p) + (x-y)p - \psi_2(p))$$

$$\leqslant \psi_1(y) + \psi_2(x-y).$$

Choose $\varepsilon = 1/(2\min(\varepsilon_1, \varepsilon_2))$. Then

$$(e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2})(x) = \int_{-\infty}^{\infty} e^{-\varepsilon_1\tilde{\psi}_1(y)-\varepsilon_2\tilde{\psi}_2(x-y)}\,dy$$
$$\leqslant e^{-\varepsilon\tilde{\psi}(x)}\int_{-\infty}^{\infty} e^{-\varepsilon_1\tilde{\psi}_1(y)/2}e^{-\varepsilon_2\tilde{\psi}_2(x-y)/2}\,dy$$
$$\leqslant Ce^{-\varepsilon\tilde{\psi}(x)}\int_{-\infty}^{\infty} e^{-\varepsilon_1\tilde{\psi}_1(y)/2}\,dy$$
$$\leqslant C\gamma(\tilde{\psi}_1)e^{-\varepsilon\tilde{\psi}(x)} \leqslant C\gamma(\psi_1)^{-1}e^{-\varepsilon\tilde{\psi}(x)}$$

by Lemmas 2 and 3. This completes the proof. ■

Proof of the Theorem. Let $\psi \in \Gamma$ and suppose that $\psi(1) = 1$. If the number of terms in ψ is 1, then the theorem was proved in Lemma 4. If the number of terms in ψ is greater than 1, then we split ψ as $\psi_1 + \psi_2$ where $\psi_1, \psi_2 \in \Gamma$ and the number of terms of ψ_j is less than the number of terms in ψ. By the induction hypothesis, we know that there are positive constants C_j and ε_j $(j = 1, 2)$ such that

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi_j(x)}\,dx\Big| \leqslant C_j\gamma(\tilde{\psi}_j)e^{-\varepsilon_j\tilde{\psi}_j(t)}, \quad j = 1, 2.$$

Without loss of generality, we may assume that $\gamma(\psi_1) \geqslant 1/2$. Then

$$\Big|\int_{-\infty}^{\infty} e^{ixt-\psi(x)}\,dx\Big| = \Big|\int_{-\infty}^{\infty} e^{ixt-\psi_1(x)-\psi_2(x)}\,dx\Big| \leqslant C\gamma(\psi_1)\gamma(\psi_2)(e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2})(t)$$
$$\leqslant C\gamma(\psi_2)(e^{-\varepsilon_1\tilde{\psi}_1} * e^{-\varepsilon_2\tilde{\psi}_2})(t) \leqslant Ce^{-\varepsilon\tilde{\psi}(t)}$$

by Lemma 5. If $\psi(1) \neq 1$, we again make a change of variables $x \to \gamma(\psi)x$. This completes the proof. ■

A final remark. It would be very interesting to see whether our theorem holds for general smooth convex functions.

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Point derivations and prime ideals in $R(X)$

by

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Abstract. Let X be a compact plane set. Then $R(X)$ is the uniform algebra of all continuous functions on X which may be uniformly approximated on X by rational functions with poles off X. We give an example of a compact plane set X such that $R(X)$ is normal, and $R(X)$ contains a prime ideal whose closure is not prime. In order to construct this example, we give an example of a compact plane set X with $0 \in X$ for which $R(X)$ has a non-zero, continuous point derivation at 0, but such that the polynomial Z^2 may be uniformly approximated on X by functions in $R(X)$ which vanish on a neighbourhood of 0.

1. Introduction. Many counterexamples to conjectures in the theory of uniform algebras have been obtained by studying $R(X)$ for suitable compact plane sets X. In this paper, we shall work with one particular kind of compact plane set, the Swiss cheese.

In [M], McKissick was able to produce the first known example of a non-trivial, normal uniform algebra by constructing a suitable Swiss cheese. Wermer has found a Swiss cheese X for which $R(X)$ is non-trivial, but has no non-zero, continuous point derivations at any point of X (see [WE]). In [WA2], Wang gives an example of a Swiss cheese X for which $R(X)$ is strongly regular at a non-peak point. O'Farrell, in [OF], has given an example of a Swiss cheese X for which $R(X)$ is normal, but has a non-zero, continuous, infinite-order point derivation at a point of X.

We shall use methods adapted from those of these papers to produce various examples of Swiss cheeses, including an example of X such that $0 \in X$ and $R(X)$ has a non-zero, continuous point derivation at 0, but such that the polynomial Z^2 may be uniformly approximated on X by functions in $R(X)$ which vanish on a neighbourhood of 0. In Section 5, we shall give an example of a Swiss cheese X and a prime ideal P in $R(X)$, such that $\bar{P}$ is not prime.

Other results on point derivations of various orders have been obtained using the tools of analytic capacity. See, for example, [H].

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