Next for any $f(x)$ we may assume that $\int |f(x)|^p v(x)\,dx < \infty$. Then Carleson and Jones' condition (1.7) and Hölder's inequality show that $\int |f(x)|(1+|x|)^{-n}\,dx < \infty$ and $T^*f(x)$ is well defined.

Let $f_M(x) = f(x)\chi_{\{|x|\leqslant M\}}(x)$. Then from (4.1) we obtain

$$\int (T^*f_M(x))^p w(x)\,dx \leqslant C\int |f(x)|^p v(x)\,dx,$$

where the bound C is independent of M. By taking M to tend to infinity Fatou's lemma shows the conclusion (1.6) of our Theorem. ■

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The structure of module derivations of $C^n[0, 1]$ into $L_p(0, 1)$

by

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Abstract. We completely determine the structure of all continuous and discontinuous module derivations $D\colon C^n[0,1]\to L_p(0,1)$, $n = 1, 2, \ldots$ and $1\leqslant p<\infty$, where $C^n[0, 1]$ is the Banach algebra of n times continuously differentiable functions on the unit interval $[0, 1]$ and $L_p(0, 1)$ is considered as a $C^n[0, 1]$-module with module multiplication defined by the $C^n[0, 1]$-operational calculus for the operator $M-nJ$ where $M\colon f(t)\mapsto tf(t)$ and $J\colon f(t)\mapsto\int_0^t f(s)\,ds$.

1. Preliminaries. Let $C^n[0, 1]$ denote the algebra of all complex-valued functions on $[0, 1]$ which have n continuous derivatives. It is well known that $C^n[0, 1]$ is a Banach algebra under the norm

$$\|f\| = \max_{t\in[0,1]} \sum_{k=0}^{n} |f^{(k)}(t)|/k!,$$

and that its structure space is $[0, 1]$. We will need a characterization of the squares of the closed primary ideals with finite codimension in $C^n[0, 1]$. We use the notation

$$M_{n,k}(t_0) = \{f\in C^n[0, 1] \mid f^{(j)}(t_0) = 0,\ j = 0, 1, \ldots, k\}.$$

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(t_0)$ which consists of functions vanishing at t_0. Throughout this paper we write $M_{n,k}$ for $M_{n,k}(0)$ and set $z(t) = t$, $0\leqslant t\leqslant 1$. We have

1.1. THEOREM. *Let n be a positive integer. Then*

(i) $M_{n,0}^2 = zM_{n,0} = \{f \mid f(0) = f'(0) = 0$ *and* $f^{(n+1)}(0)$ *exists*$\}$,

(ii) $M_{n,k}^2 = z^{k+1}M_{n,k}$, $1\leqslant k\leqslant n-1$,

(iii) $M_{n,n}^2 = z^n M_{n,n}$.

Part (i) is from [1, Example 3]. Part (ii) is due to Dales and McClure [3, Theorem 3.1.]. The proof of part (iii) can be found in [2].

The squares of the closed primary ideals $M_{n,k}(t_0)$ at other points t_0 in $[0, 1]$ are given by exactly similar formulas, where z is replaced by $z-t_0$.

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We say that $L_p(0,1)$ is a $C^n[0,1]$-module if there exists a continuous homomorphism ϱ_n of $C^n[0,1]$ into $B(L_p(0,1))$, the bounded operators on $L_p(0,1)$. The homomorphism ϱ_n is used to define module multiplication on $L_p(0,1)$. In this paper we consider the case when ϱ_n is the operational calculus for the operator $M-nJ$: $L_p(0,1)\to L_p(0,1)$, i.e. $\varrho_n(z)=M-nJ$. Here M: $f\mapsto zf$, where $(zf)(t)=tf(t)$ and $Jf=u*f$, where $u(t)=1$, $0\leqslant t\leqslant 1$, and $*$ denotes convolution. Kantorovitz [6] showed that ϱ_n is given by the formula

$$\varrho_n(f)x=\sum_{j=0}^{n}\binom{n}{j}(-1)^j J^j(f^{(j)}x),\qquad f\in C^n[0,1],\ x\in L_p(0,1).$$

A linear map D: $C^n[0,1]\to L_p(0,1)$ is a *derivation*, or a *module derivation* if D satisfies the identity

$$D(fg)=\varrho_n(f)D(g)+\varrho_n(g)D(f),\qquad f,g\in C^n[0,1].$$

We are mainly interested in the form taken by discontinuous derivations. To measure the discontinuity of a derivation D one introduces the *separating space* $S(D)$. This is a subspace of $L_p(0,1)$ defined by

$$S(D)=\{y\in L_p(0,1)\mid \text{there exists } \{f_k\}\subset C^n[0,1] \text{ with } f_k\to 0 \text{ and } D(f_k)\to y\}.$$

It is easily checked that $S(D)$ is a closed submodule of $L_p(0,1)$ and that the derivation D is continuous if and only if $S(D)=\{0\}$. The *continuity ideal* for a derivation D: $C^n[0,1]\to L_p(0,1)$ is defined as

$$I(D)=\{f\in C^n[0,1]\mid \varrho_n(f)S(D)=\{0\}\}.$$

The ideal $I(D)$ is closed in $C^n[0,1]$. It is proved in [1, Theorem 3.2] that

$$I(D)=\{f\in C^n[0,1]\mid D_f \text{ is continuous}\},$$

where $D_f(\cdot)=\varrho_n(f)D(\cdot)$.

The hull F of $I(D)$ is called the *singularity set* for D. If D is a derivation from $C^n[0,1]$ with singularity set F, then F is finite, and $I(D)\supseteq\bigcap_{t\in F}M_{n,n-1}(t)$. Moreover, we can decompose D into a finite sum of derivations whose singularity sets consist of exactly one point [2, Theorems 1.2 and 3.2]. Throughout this paper we shall assume that a discontinuous derivation has the point zero for its singularity set.

We also need the notion of the differential subspace of a $C^n[0,1]$-module, a concept first introduced by Kantorovitz, who named it "semisimplicity manifold" [4, 5]. Let $\mathcal{M}$ be a Banach space which is a $C^n[0,1]$-module with multiplication defined by a continuous homomorphism ϱ: $C^n[0,1]\to B(\mathcal{M})$. The *differential subspace* is the set W of all vectors m in $\mathcal{M}$ such that the map $p\to\varrho(p')m$ is continuous on P, where P is the dense subalgebra of polynomials in z. We quote the following result from [2].

1.2. THEOREM. *Let $\mathcal{M}$ be a $C^n[0,1]$-module. A vector m lies in the differential subspace W if and only if the map $\varrho\mapsto\varrho(p)m$ is continuous for the $C^{n-1}[0,1]$-norm on P. For $m\in W$, define*

$$|||m|||=\sup\{\|\varrho(p)m\|\mid \|p\|_{n-1}=1\}.$$

Then

(1) $\|m\|\leqslant|||m|||$, $m\in W$,

(2) *W is a Banach space with respect to the norm $|||\cdot|||$,*

(3) *W is a $C^{n-1}[0,1]$-module. There exists a unique continuous homomorphism γ: $C^{n-1}[0,1]\to W$ such that*

$$\gamma(p)m=\varrho(p)m,\qquad m\in W,\ p\in P.$$

Since $D(f)=\gamma(f')D(z)$, $f\in C^n[0,1]$, for every continuous derivation D: $C^n[0,1]\to\mathcal{M}$ [2, Theorem 4.5], a computation of γ, given $\mathcal{M}$ and ϱ: $C^n[0,1]\to B(\mathcal{M})$, will give us an explicit structure of continuous derivations of $C^n[0,1]$ into $\mathcal{M}$.

A non-zero derivation D: $C^n[0,1]\to\mathcal{M}$ is called *singular* if D vanishes on P (equivalently if $D(z)=0$). A singular derivation is necessarily discontinuous. We say that a discontinuous derivation D is *decomposable* if D can be expressed in the form $D=E+F$, where E is continuous and F is singular. Such a splitting is unique. It was shown in [2] that a discontinuous derivation D: $C^n[0,1]\to\mathcal{M}$ is decomposable if and only if $D(z)\in W$. If D is decomposable and $D=E+F$, then its singular part F vanishes also on $I(D)^2$. An *indecomposable derivation* is a discontinuous derivation which is not decomposable. In 1978 Curtis [2] computed an explicit example of an indecomposable derivation from $C^1[0,1]$ into $L_p(0,1)$ which is discontinuous on every dense subalgebra of $C^1[0,1]$. In this example module multiplication on $L_p(0,1)$ was given by the operational calculus for the operator $M-J$. In 1988 a description of all derivations from $C^1[0,1]$ into $L_p(0,1)$, with the same module structure, was given by myself in [7]. This paper is continuation of work of Bade and Curtis [1, 2] and myself [7].

2. The differential subspace of $L_p(0,1)$. We fix p with $1\leqslant p<\infty$ and consider $L_p(0,1)$ as $C^n[0,1]$-modules ($n=1,2,3,\ldots$) with module action ϱ_n: $C^n[0,1]\to B(L_p(0,1))$ defined by

$$(*)\qquad \varrho_n(f)x=\sum_{j=0}^{n}\binom{n}{j}(-1)^j J^j(f^{(j)}x),\qquad f\in C^n[0,1],\ x\in L_p(0,1).$$

We shall characterize $W(\varrho_n)$, the differential subspaces of $L_p(0,1)$. In the following if x is a function of bounded variation, we write $x(ds)$ and $v(x)(ds)$ for the measures corresponding to x and its total variation.

2.1. THEOREM. *Let $L_p(0,1)$, $1\leqslant p<\infty$, be given the $C^n[0,1]$-module operations defined by $(*)$. An element x of $L_p(0,1)$ belongs to $W(\varrho_n)$ if and only if*

(1) *x is of bounded variation on each interval $[0,t]$, $0<t<1$, and*

(2) $\int_0^1\big(J^{n-1}(v(x)([0,t]))\big)^p\,dt<\infty.$

Proof. The case $n=1$ was proved by Bade and Curtis in [2]. We use a similar argument to prove this theorem for $n \geqslant 2$.

Let x satisfy (1) and (2). We can suppose that x is right continuous. For all polynomials h in P we have

$$(\varrho_n(h)x)(t) = \sum_{j=0}^{n-2} \binom{n}{j}(-1)^j\big(J^j(h^{(j)}x)\big)(t)+(-1)^{n-1}(n-1)\big(J^{n-1}(h^{(n-1)}x)\big)(t)$$
$$+(-1)^{n-1}J^{n-1}\big(h^{(n-1)}(t)x(t)-\int_0^t h^{(n)}(s)x(s)\,ds\big).$$

Integrating the last term by parts yields

$$(\varrho_n(h)x)(t) = \sum_{j=0}^{n-2} \binom{n}{j}(-1)^j\big(J^j(h^{(j)}x)\big)(t)+(-1)^{n-1}(n-1)\big(J^{n-1}(h^{(n-1)}x)\big)(t)$$
$$+(-1)^{n-1}h^{(n-1)}(0)x(0)\frac{t^{n-1}}{(n-1)!}+(-1)^{n-1}J^{n-1}\Big(\int_0^t h^{(n-1)}(s)x(ds)\Big).$$

Letting $G(t) = J^{n-1}\big(v(x)([0,\,t])\big)$, it follows from Minkowski's inequality for integrals that

$$\|\varrho_n(h)x\|_p \leqslant \sum_{j=0}^{n-2} \binom{n}{j}\,\|(h^{(j)}\|_\infty\,\|x\|_p+(n-1)\,\|h^{(n-1)}\|_\infty\|x\|_p$$
$$+\|h^{(n-1)}\|_\infty|x(0)|+\|h^{(n-1)}\|_\infty\|G\|_p,$$

hence

$$\|\varrho_n(h)x\|_p \leqslant n!\,\|h\|_{n-1}\big(|x(0)|+2^n\|x\|_p+\|G\|_p\big).$$

By [2, Theorem 4.3], $x \in W(\varrho_n)$.

Now let $x \in W(\varrho_n)$ and $y \in L_q(0,\,1)$ where $1/p+1/q=1$ if $p>1$, $q=\infty$ if $p=1$. We write

$$\int_0^1 y(t)\big(\varrho_n(f)x\big)(t)\,dt = \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\int_0^1 y(t)\big(J^j(f^{(j)}x)\big)(t)\,dt$$
$$+(-1)^n\int_0^1 y(t)\big(J^n(f^{(n)}x)\big)(t)\,dt.$$

The sum on the right-hand side only involves the first $n-1$ derivatives of f so that the map $f \mapsto \sum_{j=0}^{n-1}(-1)^j\binom{n}{j}\int_0^1 y(t)\big(J^j(f^{(j)}x)\big)(t)\,dt$ is continuous for the C^{n-1}-norm on $C^n[0,\,1]$. And since $x \in W(\varrho_n)$, it follows from Theorem 1.2 that the map $f \mapsto \int_0^1 y(t)\big(\varrho_n(f)x\big)(t)\,dt$ is also continuous for the C^{n-1}-norm on $C^n[0,\,1]$, thus we have

$$\Big|\int_0^1 y(t)\int_0^t\int_0^{r_1}\ldots\int_0^{r_{n-1}} f^{(n)}(s)x(s)\,ds\,dr_{n-1}\ldots dr_1\,dt\Big| \leqslant K\,\|y\|_q\,\|f\|_{n-1},$$

for $y \in L_q(0,\,1)$ and $f \in C^n[0,\,1]$. Note that in the above inequality the term $\|x\|_p$ is absorbed into the constant K. Repeated application of Fubini's theorem yields

$$\int_0^1 y(t)\int_0^t\int_0^{r_1}\ldots\int_0^{r_{n-1}} f^{(n)}(s)x(s)\,ds\,dr_{n-1}\ldots dr_1\,dt$$
$$=\int_0^1 f^{(n)}(s)x(s)\int_s^1\int_{r_{n-1}}^1\ldots\int_{r_1}^1 y(t)\,dt\,dr_1\ldots dr_{n-1}\,ds.$$

If we let $Y(t) = \int_t^1\int_{r_{n-1}}^1\ldots\int_{r_1}^1 y(s)\,ds\,dr_1\ldots dr_{n-1}$, then

$$\Big|\int_0^1 Y(t)x(t)f^{(n)}(t)\,dt\Big| \leqslant K\,\|y\|_q\,\|f\|_{n-1}.$$

Thus for each $y \in L_q(0,\,1)$, there exist constants $b_0, \ldots, b_{n-2}$ and a finite Borel measure v_y on $[0,\,1]$ such that

$$\int_0^1 Y(t)x(t)f^{(n)}(t)\,dt = \sum_{j=0}^{n-2} b_j\frac{f^{(j)}(0)}{j!}+\int_0^1 f^{(n-1)}(t)\,v_y(dt),$$

for all $f \in C^n[0,\,1]$. If we let f run over the set $\{1, z, \ldots, z^{n-2}\}$ we get $b_0 = b_1 = \ldots = b_{n-2} = 0$ so that

$$(**)\qquad \int_0^1 Y(t)x(t)f^{(n)}(t)\,dt = \int_0^1 f^{(n-1)}(t)\,v_y(dt), \qquad f \in C^n[0,\,1].$$

Let S_y and U_y be the distributions on $\mathbf{R}$ whose value on a test function φ is

$$S_y(\varphi) = \int_0^1 \varphi(t)Y(t)x(t)\,dt \quad\text{and}\quad U_y(\varphi) = \int_0^1 \varphi(t)\,v_y(dt).$$

Then from (**) we have $S_y^{(n)}(\varphi) = -U_y^{(n-1)}(\varphi)$, so that

$$S_y'(\varphi) = (-1)^n U_y(\varphi)+\int_{-\infty}^{\infty}\varphi(t)\Big(\sum_{j=0}^{n-2} c_j t^j\Big)\,dt,$$

where $c_0, \ldots, c_{n-2}$ are constants and $S_y^{(k)}$ and $U_y^{(k)}$ denote the kth derivatives of S_y and U_y, respectively, in the sense of distributions. It follows that S_y is a distribution which has a measure for its derivative. Therefore its function $Y(t)x(t)$ is of bounded variation on each bounded interval of $\mathbf{R}$ [8, p. 54]. Let $y \equiv n!$ on $[0,\,1]$. Then $Y(t) = (1-t)^n$, so that $x(t)(1-t)^n$ is of bounded variation on $[0,\,1]$. Thus $x(t)$ is of bounded variation on $[0,\,c]$ for $0<c<1$.

We next prove that $G \in L_q(0,\,1)$, where $G(t) = J^{n-1}\big(v(x)([0,\,t])\big)$. Define J^*: $g(t) \mapsto \int_t^1 g(s)\,ds$. Then $Y(t) = \big((J^*)^n y\big)(t)$. We write $v_x(t)$ for $v(x)([0,\,t])$. Let $y \in L_q(0,\,1)$. Then for all g in $C[0,\,1]$,

$$\int_0^1 g(t)(xY)(dt) = \int_0^1 g(t)x(t)\big((J^*)^{n-1}y\big)(t)\,dt+\int_0^1 g(t)\,Y(t)\,x(dt).$$

Thus $Y(t)x(dt)$ is a finite measure on $[0, 1]$ whose total variation on $[0, c]$ is $\int_0^c |Y(t)|v(x)(dt)$. Restricting ourselves to $y \geqslant 0$ and applying Fubini's theorem, we have

$$\int_0^c ((J^*)^n y)(t)v(x)(dt) = \sum_{j=0}^{n-1} (J^j v_x)(c)\cdot((J^*)^{n-j}y)(c) + \int_0^c y(t)(J^{n-1}v_x)(t)dt.$$

The left side is bounded as $c \to 1$. All the terms on the right side are positive. It follows that

$$\int_0^c y(t)(J^{n-1}v_x)(t)dt < \infty, \qquad y \in L_q(0, 1),\ y \geqslant 0.$$

We conclude that $G \in L_p(0, 1)$.

2.2. Remark. *Let D: $C^n[0, 1] \to L_p(0, 1)$ be a discontinuous derivation with singularity set $F = \{0\}$. Then $D(f)$ is of bounded variation on each interval $[a, c]$ for $0 < a < c < 1$.*

Proof. Since $z^n \in I(D)$, $\varrho_n(z^n)D(\cdot)$ is a continuous derivation so that $\varrho_n(z^n)D(f) \in W(\varrho_n)$ for all $f \in C^n[0, 1]$. Hence

$$t^n D(f)(t) = -\sum_{j=1}^{n} \binom{n}{j}(-1)^j J^j\big((z^n)^{(j)}D(f)\big)(t) + y(t),$$

where $y \in W(\varrho_n)$. All the terms on the right-hand side are of bounded variation on $[0, c]$ for $0 < c < 1$. Thus $D(f)$ is of bounded variation on $[a, c]$ for $0 < a < c < 1$.

The spaces $W(\varrho_n)$ equipped with the norm described in Theorem 1.2 are $C^{n-1}[0, 1]$-modules whose module action will be denoted by γ_n: $C^{n-1}[0, 1] \to B(W(\varrho_n))$. We need the following two technical lemmas for the computation of γ_n. The explicit formula that we obtain for γ_n is essential in the description of continuous and discontinuous derivations of $C^n[0, 1]$ into $L_p(0, 1)$.

2.3. Lemma. *Let x be of bounded variation on $[0, t]$ where $0 < t < 1$, and let $g \in C[0, 1]$. For $k = 1, 2, 3, \ldots$, we have*

$$J^k\Big(\int_0^t g(s)x(ds)\Big) = \frac{1}{k!}\int_0^t (t-s)^k g(s)x(ds).$$

Proof. If $\psi(s) = \int_0^s g(u)x(du)$, then ψ is of bounded variation on $[0, t]$ and

$$J^k(\psi(t)) = \frac{1}{(k-1)!}\int_0^t (t-s)^{k-1}\psi(s)ds$$
$$= \frac{1}{(k-1)!}\left(\frac{-(t-s)^k}{k}\psi(s)\Big|_0^t - \int_0^t \frac{-(t-s)^k}{k}\psi(ds)\right) = \frac{1}{k!}\int_0^t (t-s)^k g(s)x(ds).$$

2.4. Lemma. *Let $g \in C^m[0, 1]$, $m = 0, 1, 2, \ldots$ For $i = 0, 1, \ldots, m$*

$$(z^i g)^{(m)} = m! \sum_{k=0}^{i} \binom{i}{k} z^k \frac{g^{(m-i+k)}}{(m-i+k)!}.$$

Proof. The case when $m = 0$ or $i = 0$ is trivial. Let $i = 1$. An induction on m shows that

$$(zg)^{(m)} = mg^{(m-1)} + zg^{(m)}, \qquad m = 1, 2, \ldots$$

Suppose that the lemma is true for i; by writing

$$(z^{i+1}g)^{(m)} = (z(z^i g))^{(m)} = m(z^i g)^{(m-1)} + z(z^i g)^{(m)},$$

a short computation shows that

$$(z^{i+1}g)^{(m)} = m! \sum_{k=0}^{i+1} \binom{i+1}{k} z^k \frac{g^{(m-(i+1)+k)}}{(m-(i+1)+k)!}.$$

2.5. Theorem. *Let $L_p(0, 1)$, $1 \leqslant p < \infty$, be given the $C^n[0, 1]$-module operation defined by*

$$(\varrho_n(f)x)(t) = \sum_{j=0}^{n} \binom{n}{j}(-1)^j J^j(f^{(j)}x)(t).$$

Let γ_n be the corresponding continuous homomorphism of $C^{n-1}[0, 1]$ into $B(W(\varrho_n))$. Then for all $f \in C^{n-1}[0, 1]$ and $x \in W(\varrho_n)$,

$$(\gamma_n(f)x)(t) = x(0)\sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j \delta_j(f)t^j$$
$$+ \frac{1}{(n-1)!}\sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j t^j \int_0^t (z^{n-1-j}f)^{(n-1)}(s)x(ds),$$

where $\delta_j(f) = f^{(j)}(0)/j!$ and $x(0) = \lim_{t\to 0^+} x(t)$ since we may assume that x is right continuous.

Proof. Let $p \in P$ and $x \in W(\varrho_n)$. Then

$$(\gamma_n(p)x)(t) = (\varrho_n(p)x)(t) = \sum_{j=0}^{n} \binom{n}{j}(-1)^j J^j(p^{(j)}x)(t).$$

Using the fact that $\binom{n}{j+1} = \binom{n-1}{j} + \binom{n-1}{j+1}$ we write

$$(\gamma_n(p)x)(t) = \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j J^j\Big(p^{(j)}(t)x(t) - \int_0^t p^{(j+1)}(s)x(s)ds\Big).$$

Integrating the last integral by parts yields

$$(\gamma_n(p)x)(t) = \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j p^{(j)}(0)x(0)\frac{t^j}{j!} + \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j J^j\Big(\int_0^t p^{(j)}(s)x(ds)\Big).$$

Letting $\mathrm{SUM} = x(0)\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\delta_j(p)t^j$ and applying Lemma 2.3, we obtain

$$(\gamma_n(p)x)(t) = \mathrm{SUM} + \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\int_0^t (t-s)^j\frac{p^{(j)}(s)}{j!}x(ds).$$

Writing $(t-s)^j = \sum_{k=0}^{j}\binom{j}{k}t^{j-k}(-1)^k s^k$ and changing the order of summation yields

$$(\gamma_n(p)x)(t) = \mathrm{SUM} + \sum_{k=0}^{n-1}\sum_{j=k}^{n-1}\binom{n-1}{j}(-1)^j\binom{j}{k}(-1)^k t^{j-k}\int_0^t s^k\frac{p^{(j)}(s)}{j!}x(ds).$$

We shift the index of summation, replacing j by $j+k$, and observe that $\binom{n-1}{j+k}\binom{j+k}{k} = \binom{n-1}{j}\binom{n-j-1}{k}$. Then

$$(\gamma_n(p)x)(t) = \mathrm{SUM} + \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}\binom{n-1}{j}(-1)^j\binom{n-j-1}{k}t^j\int_0^t s^k\frac{p^{(j+k)}(s)}{(j+k)!}x(ds).$$

Switching the order of summation again, we find that

$$(\gamma_n(p)x)(t) = \mathrm{SUM} + \sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j t^j\int_0^t \sum_{k=0}^{n-1-j}\binom{n-1-j}{k}s^k\frac{p^{(j+k)}(s)}{(j+k)!}x(ds).$$

By Lemma 2.4,

$$(\gamma_n(p)x)(t) = \mathrm{SUM} + \frac{1}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j t^j\int_0^t (z^{n-1-j}p)^{(n-1)}(s)x(ds).$$

Since γ_n is continuous for the C^{n-1}-norm and the right-hand side only involves the first $n-1$ derivatives of p, the same formula holds for all f in $C^{n-1}[0, 1]$.

Immediately following from this theorem we obtain a formula for continuous derivations of $C^n[0, 1]$ into $L_p(0, 1)$, $1 \leqslant p < \infty$.

2.6. THEOREM. *Let D: $C^n[0, 1]\to L_p(0, 1)$, $1\leqslant p<\infty$, be a continuous derivation. Then $D(z)\in W(\varrho_n)$, and for all $f\in C^n[0, 1]$,*

$$D(f) = \gamma_n(f')D(z) = D(z)(0)\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\delta_j(f')t^j$$
$$+\frac{1}{(n-1)!}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j t^j\int_0^t (z^{n-1-j}f')^{(n-1)}(s)D(z)(ds).$$

3. Discontinuous derivations of $C^n[0, 1]$ into $L_p(0, 1)$. We next turn to the characterization of singular derivations. Let D: $C^n[0, 1]\to L_p(0, 1)$, $1\leqslant p<\infty$, be a singular derivation. By [2, Theorem 3.2] we may assume that its continuity ideal $I(D)$ equals $M_{n,k-1}$ for some $1\leqslant k\leqslant n$. By [7, Corollary 2.6] the range of D is contained in the kernel of $\varrho_n(z^k)$. We prove that $\ker\varrho_n(z^k)$ is a finite-dimensional cyclic submodule of $L_p(0, 1)$.

3.1. LEMMA. *For $k = 1, 2, \ldots, n$, $\ker\varrho_n(z^k) = \operatorname{span}\{z^{n-k}, \ldots, z^{n-1}\}$.*

Proof. From [6, Lemma 12], $\ker\varrho_n(z)$ is the one-dimensional subspace spanned by z^{n-1}. Suppose that the lemma holds for $k-1$. Let $x\in\ker\varrho_n(z^k)$. Since $\varrho_n(z^{k-1})(\varrho_n(z)x) = 0$, there exist constants $a_{n-(k-1)}, \ldots, a_{n-1}$ such that

$$\varrho_n(z)x = a_{n-k+1}z^{n-k+1}+\ldots+a_{n-1}z^{n-1}.$$

Since

$$\varrho_n(z)z^j = \frac{j+1-n}{j+1}z^{j+1},\quad j = 0, 1, 2, \ldots,$$

we can write

$$\varrho_n(z)x = \varrho_n(z)(b_{n-k}z^{n-k}+\ldots+b_{n-2}z^{n-2})$$

for some constants $b_{n-k}, \ldots, b_{n-2}$. It follows from the form of $\ker\varrho_n(z)$ that $x\in\operatorname{span}\{z^{n-k}, \ldots, z^{n-1}\}$.

The structure of singular cyclic derivations into finite-dimensional modules was determined in [2, Theorem 5.3]. Using this result we obtain

3.2. THEOREM. *Let D: $C^n[0, 1]\to L_p(0, 1)$, $1\leqslant p<\infty$, be a singular derivation with continuity ideal $I(D) = M_{n,k-1}$, where $1\leqslant k\leqslant n$. Then there exists a discontinuous linear functional θ on $C^n[0, 1]$ which vanishes on polynomials and on the principal ideal $z^kC^n[0, 1]$ such that*

$$D(f) = \sum_{i=0}^{k-1}\theta(z^{k-1-i}f)\varrho_n(z^i)z^{n-k},\qquad f\in C^n[0, 1].$$

We now turn to the structure of an arbitrary discontinuous derivation D of $C^n[0, 1]$ into $L_p(0, 1)$. We show that D is the sum of a continuous linear map, a singular derivation, and a discontinuous part consisting of finitely many discontinuous linear functionals on $C^n[0, 1]$. We need the following lemmas.

3.3. LEMMA. *Let D: $C^n[0, 1]\to L_p(0, 1)$, $1\leqslant p<\infty$, be a discontinuous derivation with continuity ideal $I(D) = M_{n,k-1}$, where $1\leqslant k\leqslant n$. Let $\mu(t) = (\varrho_n(z^k)D(z))(t)$. Then $\lim_{t\to 0^+}\mu(t) = 0$.*

Proof. Since $z^k\in M_{n,k-1} = I(D)$, $\varrho_n(z^k)D(\cdot)$ is continuous so that $\mu = \varrho_n(z^k)D(z)$ is in $W(\varrho_n)$. By Theorem 2.1, μ is of bounded variation on $[0, c]$ for $0<c<1$, thus we may assume that μ is right continuous at 0, so $\lim_{t\to 0^+}\mu(t)$ exists. Suppose that $\lim_{t\to 0^+}\mu(t)\neq 0$. Then $\lim_{t\to 0^+}|t^kD(z)(t)|\neq 0$ since

$$\mu(t) = t^kD(z)(t)+\sum_{j=1}^{n}\binom{n}{j}(-1)^jJ^j\big((s^k)^{(j)}D(z)(s)\big),$$

and each term of the sum has limit zero at 0. This means that there exist $\varepsilon>0$ and $\delta>0$ such that $|t^kD(z)(t)|>\varepsilon$ for $0<t<\delta$. But then $|D(z)(t)|>\varepsilon/t^k$ on $(0, \delta)$ and this is a contradiction to $D(z)\in L_p(0, 1)$.

3.4. LEMMA. *Let D: $C^n[0, 1]\to L_p(0, 1)$, $1\leqslant p<\infty$, be a discontinuous derivation with singularity set $F = \{0\}$. Then we can write $D = D_1+D_2$, where*

D_1 is a continuous derivation and $D_2(z)$ is of bounded variation on $[a, 1]$ for all $0 < a < 1$.

Proof. By Remark 2.2, $D(z)$ is of bounded variation on $[a, c]$, $0 < a < c < 1$. Again by [2, Theorem 3.2], $I(D) = M_{n,k-1}$ for some $1 \leqslant k \leqslant n$, so that $\varrho_n(z^k)D(\cdot)$ is continuous. Let $\mu = \varrho_n(z^k)D(z)$ as before; then μ is in $W(\varrho_n)$. Now, fix b in $(0, 1)$ and let $y = \chi_{[b,1]}D(z)$, where $\chi_{[b,1]}$ denotes the characteristic function of $[b, 1]$. Then y is of bounded variation on $[0, c]$ for all $0 < c < 1$. From the characterization of $W(\varrho_n)$ in Theorem 2.1, $z^k D(z)$ is in $W(\varrho_n)$ since μ and all the terms in the above sum are. Since y vanishes near zero, it follows that $J^{n-1}(v(y)([0, s])) \in L_p(0, 1)$. Thus $y \in W(\varrho_n)$. By [2, Proposition 4.2] there exists a continuous derivation D_1: $C^n[0, 1] \to L_p(0, 1)$ such that $D_1(z) = y$. Then $D_2 = D - D_1$ is a derivation with $D_2(z) = \chi_{[0,b]}D(z)$ a function of bounded variation on $[a, 1]$ for $0 < a < 1$.

Given a discontinuous derivation D, decompose D as in Lemma 3.4. Since D_1 is continuous, it can be described by Theorem 2.6. So it is left to consider D_2, a discontinuous derivation with the property that $D_2(z)$ is of bounded variation on $[a, 1]$ for $0 < a < 1$. First we consider discontinuous derivations from $C^n[0, 1]$ to $L_1(0, 1)$.

3.5. THEOREM. *Let D: $C^n[0, 1] \to L_1(0, 1)$ be a discontinuos derivation with $D(z)$ a function of bounded variation on $[a, 1]$ for all $0 < a < 1$. Suppose that the continuity ideal is $I(D) = M_{n,k-1}$ for some $1 \leqslant k \leqslant n$. Then there exist discontinuous linear functionals $\alpha_1, \ldots, \alpha_k$ on $C^n[0, 1]$ such that*

$$D(f) = T(f) + \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j}(-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n},$$

where T is a continuous linear map from $C^n[0, 1]$ which is completely determined by $D(z)$ and ϱ_n.

Proof. Let $\mu = \varrho_n(z^k)D(z)$. Then μ is in $W(\varrho_n)$, $\mu(0) = 0$, and μ is of bounded variation on $[0, 1]$ since $D(z)$ is of bounded variation on $[a, 1]$, $0 < a < 1$. For $f \in I(D)^2$, we have, by [7, Theorem 2.3] and Theorem 2.5,

$$D(f)(t) = \gamma_n(f'/z^k)\mu(t) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^j t^j \int_0^t (z^{n-1-j-k} f')^{(n-1)}(s)\mu(ds).$$

If $0 \leqslant j \leqslant n-k-1$ then the map $f(t) \mapsto t^j \int_0^t (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds)$ defines a continuous linear map from $C^n[0, 1]$ to $L_1(0, 1)$ which vanishes on $1, z, \ldots, z^{j+k}$. For $n-k \leqslant j \leqslant n-1$ we write

$$t^j \int_0^t (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds) = t^j \int_0^1 (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds) - t^j \int_t^1 (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds).$$

The map $f(t) \mapsto S(t) = t^j \int_t^1 (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds)$ is a continuous linear map from $M_{n,k-1}$ to $L_1(0, 1)$ for $n-k \leqslant j \leqslant n-1$. To see this we write $f = \sum_{i=k}^n \delta_i(f)z^i + R_n(f)$, where $R_n(f) \in M_{n,n}$. Then

$$\|S\|_{L_1} \leqslant \int_0^1 t^j \int_t^1 \Big|\Big(z^{n-1-j-k} \sum_{i=k}^n i\delta_i(f) z^{i-1}\Big)^{(n-1)}(s)\Big| v(\mu)(ds)\,dt + \int_0^1 t^j \int_t^1 \big|(z^{n-1-j-k} R_n(f)')^{(n-1)}(s)\big| v(\mu)(ds)\,dt.$$

An application of Leibniz's rule, Fubini's theorem, together with the fact that the map $f \to f/z$ from $M_{n,n-1}$ to $M_{n-1,n-2}$ is continuous [7, Remark 2.2], shows that the sums on the right-hand side are bounded by $Kv(\mu)([0, 1])\|f\|_n$ for some $K > 0$. The integrals $\int_0^1 (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds)$ are defined on $z^{n-j}M_{n,k-1}$. We set

$$\beta_i(f) = \int_0^1 (z^{-1+i-k}f')^{(n-1)}(s)\mu(ds), \quad f \in z^{k-i+1} M_{n,k-1},\ i = 1, \ldots, k.$$

Using the Hahn–Banach theorem, we can extend $\beta_1, \ldots, \beta_k$ to linear functionals on $M_{n,n}$. For $f \in M_{n,n}$, let

$$\bar{D}(f)(t) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1-k} \binom{n-1}{j}(-1)^j t^j \int_0^t (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds) - \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j}(-1)^j t^j \int_t^1 (z^{n-1-j-k}f')^{(n-1)}(s)\mu(ds) + \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j}(-1)^j t^j \beta_{n-j}(f).$$

The integrals on the right-hand side define continuous linear maps from $C^n[0, 1]$ into $L_1(0, 1)$ so that the discontinuity of $\bar{D}$ arises from the linear functionals β_i. Since $\varrho_n(z^k)$ annihilates $z^{n-k}, \ldots, z^{n-1}$, the map $\varrho_n(z^k)\bar{D}(f)$ is continuous on $M_{n,n}$. Now z^k is in the continuity ideal of D so that $\varrho_n(z^k)D(f)$ is also continuous. Since D and $\bar{D}$ agree on $M_{n,k-1}^2$, which is dense in $M_{n,n}$, we conclude that $\varrho_n(z^k)D(f) = \varrho_n(z^k)\bar{D}(f)$ for all $f \in M_{n,n}$. Since $D(f) - \bar{D}(f) \in \ker \varrho_n(z^k)$, by Lemma 3.1 we can write

$$D(f) = \bar{D}(f) + \sum_{j=n-k}^{n-1} c_{n-j}(f) z^j, \quad f \in M_{n,n},$$

for some linear functionals $c_1, \ldots, c_k$. This establishes the desired result.

We obtain a similar result for discontinuous derivations from $C^n[0, 1]$ to $L_p(0, 1)$.

3.6. THEOREM. *Let D: $C^n[0, 1] \to L_p(0, 1)$, $1 \leqslant p < \infty$, be a discontinuous* *derivation with $D(z)$ a function of bounded variation on $[a, 1]$ for all $0 < a < 1$. Suppose that the continuity ideal is $I(D) = M_{n,k-1}$ for some $1 \leqslant k \leqslant n$. Then there exist discontinuous linear functionals $\alpha_1, \ldots, \alpha_k$ on $C^n[0, 1]$ such that*

$$D(f) = T(f) + \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j} (-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n},$$

where T is a continuous linear map from $C^n[0, 1]$ to $L_p(0, 1)$ which is completely determined by $D(z)$ and ϱ_n.

Proof. Since $L_p(0, 1) \subseteq L_1(0, 1)$ for $p \geqslant 1$, we can consider D as a derivation from $C^n[0, 1]$ into $L_1(0, 1)$. By Theorem 3.5 we can write

$$D(f) = T(f) + \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j} (-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n},$$

so that

$$T(f) = D(f) - \frac{1}{(n-1)!} \sum_{j=n-k}^{n-1} \binom{n-1}{j} (-1)^j \alpha_{n-j}(f) z^j, \quad f \in M_{n,n}.$$

Since all the terms on the left-hand side are in $L_p(0, 1)$, $T(f) \in L_p(0, 1)$ for all $f \in M_{n,n}$. Let $y \in L_p(0, 1)$ be in the separating space $S(T)$ of T. There exists $f_m \to 0$ in $C^n[0, 1]$ and $T(f_m) \to y$ in $L_p(0, 1)$. By Theorem 3.5, T is a continuous linear map from $C^n[0, 1]$ into $L_1(0, 1)$, so that $T(f_m) \to 0$ in $L_1(0, 1)$. Thus $y = 0$, and we conclude that T is continuous. This completes the proof.

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On the integrability and L^1-convergence of double trigonometric series

by

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Abstract. We study double cosine and sine series whose coefficients form a null sequence of bounded variation. In particular, we consider the special cases where the double sequence of coefficients is monotone decreasing, or convex, or quasiconvex. We are mainly concerned with the following problems: (i) the series in question converges pointwise, (ii) the sum of the series is integrable, (iii) the series is the Fourier series of its sum, (iv) the series converges in L^1-norm.

Among other things, we extend the classical theorems of Kolmogorov and Young from one-dimensional cosine and sine series to two-dimensional ones in an essentially more general setting. Our basic tools are Sidon type inequalities.

0. Introduction. The following theorems are well known for one-dimensional cosine and sine series.

THEOREM A (Kolmogorov [6] and see also [11, Vol. 1, pp. 183–184]). *If $\{a_j: j \geqslant 0\}$ is a quasiconvex null sequence, then the cosine series*

$$\tfrac{1}{2}a_0 + \sum_{j=1}^{\infty} a_j \cos jx \tag{0.1}$$

converges, except possibly at $x = 0$, to an integrable function $f(x)$, is the Fourier series of f, and the partial sums converge in $L^1(0, \pi)$-norm to f if and only if $a_j \ln j \to 0$ as $j \to \infty$.

THEOREM B (W. H. Young [10] and see also [11, Vol. 1, pp. 185–186]). *If $\{a_j: j \geqslant 1\}$ is a monotone decreasing null sequence, then the sine series*

$$\sum_{j=1}^{\infty} a_j \sin jx \tag{0.2}$$

converges to a function $g(x)$ at every x, and g is integrable if and only if $\sum (a_j - a_{j+1}) \ln j < \infty$. If this condition is satisfied, then (0.2) is the Fourier series of g, and the partial sums converge in $L^1(0, \pi)$-norm to g.

In this paper we will extend these results to two-dimensional trigonometric series (see Corollary 3 in Section 2 and Theorem 5 in Section 6) in an essentially

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