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A condition for a two-weight norm inequality for singular integral operators

by

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Abstract. We give a sufficient condition in order that a two-weight norm inequality holds for singular integral operators. In the case of equal weights our condition for the weight is equivalent to Muckenhoupt's A_p condition.

1. Introduction. Let $K(x)$ be a function defined outside the origin in the n-dimensional Euclidean space $\mathbf{R}^n$ which satisfies

$$|K(x)| \leqslant C_1 |x|^{-n} \quad \text{for } x \neq 0, \tag{1.1}$$

$$|K(x)-K(x-y)| \leqslant C_2 |y|^{\delta}/|x|^{n+\delta} \quad \text{for } |x| > 2|y|, \tag{1.2}$$

where C_1, C_2 and δ are positive constants.

We define a maximal integral operator T^* with kernel $K(x)$ as follows:

$$T^* f(x) = \sup_{\varepsilon>0} \Big| \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy \Big|$$

for any Lebesgue measurable function $f(x)$ on $\mathbf{R}^n$. If $T^*f(x)$ is not well defined, we set $T^*f(x) = \infty$. We also assume that T^* is of weak type (1, 1), that is,

$$\big|\{x \in \mathbf{R}^n \colon |T^*f(x)| > \lambda\}\big| \leqslant \frac{C_3}{\lambda} \int_{\mathbf{R}^n} |f(x)|\, dx \tag{1.3}$$

for any measurable function $f(x)$ and any $\lambda > 0$. Here $|E|$ denotes the Lebesgue measure of a Lebesgue measurable subset E of $\mathbf{R}^n$.

If $K(x) = c_n x_j/|x|^{n+1}$ where $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$, T^*f is the jth (maximal) Riesz transform of f, which satisfies (1.3). As is well known, Coifman and C. Fefferman [4] showed that if a nonnegative function $w(x)$ satisfies Muckenhoupt's A_p condition:

$$\sup_I |I|^{-1} \int_I w(x)\, dx \big(|I|^{-1} \int_I w(x)^{1-p'}\, dx\big)^{p-1} < \infty \tag{A_p}$$

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where $1 < p < \infty$, $(1-p)(1-p') = 1$ and I denotes a cube in $\mathbf{R}^n$ with sides parallel to the axes, then

$$\int_{\mathbf{R}^n} (T^*f(x))^p w(x)\,dx \leqslant C_4 \int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx \tag{1.4}$$

for all $f(x)$. Here the constant C_4 is independent of f.

A more general result for the inequality (1.4) was obtained by Sawyer [13], who improved the method of Coifman and C. Fefferman.

The A_p condition is also sufficient and necessary in order that the weighted norm inequality holds for the Hardy–Littlewood maximal function $M^*f(x)$:

$$M^*f(x) = \sup_{I \ni x} |I|^{-1} \int_I |f(x)|\,dx.$$

This was shown by Muckenhoupt [11]; later Sawyer [12] found that the following statements (S_p) and (1.5) are equivalent:

(S_p) There exists a positive constant C_5, depending only on w, v, $p \in (1, \infty)$ and n, such that

$$\int_I (M^*(\chi_I v^{1-p'}))^p w(x)\,dx \leqslant C_5 \int_I v^{1-p'}(x)\,dx < \infty$$

for every cube I, where $\chi_I(x)$ is the characteristic function of I.

There exists a positive constant C_6 such that

$$\int_{\mathbf{R}^n} (M^*f(x))^p w(x)\,dx \leqslant C_6 \int_{\mathbf{R}^n} |f(x)|^p v(x)\,dx \tag{1.5}$$

for every measurable function $f(x)$.

In this note we shall consider the inequality (1.4) from the point of view taken by Córdoba and C. Fefferman [5] and Jawerth and Torchinsky [10]; that is, the sharp maximal function of $T^*f(x)$ is majorized by $\{M^*(|f|^r)(x)\}^{1/r}$, $r > 1$, or $M^*f(x)$ pointwise.

Let $w(x)$ and $v(x)$ be nonnegative measurable functions defined on $\mathbf{R}^n$ and let $w(x)$ be not identically zero. We set for $w(x)$ and $v(x)$ the following condition:

(W_p) There exist constants $\alpha \in (0, 1)$, β and $C_0 \in (0, \infty)$ such that for every cube I and for all measurable subsets E and F of I with $E \cap F = \emptyset$ and $|F| \geqslant \alpha |I|$

$$\int_E w(x)\,dx \Big(|I|^{-1} \int_{c(n,\alpha)I} v^{1-p'}(x)\,dx\Big)^p \leqslant C_0 (|E|/|I|)^\beta \int_F v^{1-p'}(x)\,dx < \infty,$$

where $1 < p < \infty$ and $c(n, \alpha)I$ is the cube with the same center as I expanded $c(n, \alpha)$ times, and $c(n, \alpha)$ is a constant greater than 1 depending only on n and α which is increasing with respect to α.

We shall determine $c(n, \alpha)$ in Lemma 3.3. Our result is the following:

THEOREM. *Let $1 < p < \infty$. Suppose that a pair (w, v) of nonnegative functions satisfies the condition (W_p). Then there exists a positive constant C_7 depending only on C_1, C_2, δ, C_3, C_0, α, β, p and n such that*

$$\int_{\mathbf{R}^n} (T^*f(x))^p w(x)\,dx \leqslant C_7 \int_{\mathbf{R}^n} |f(x)|^p v(x)\,dx \tag{1.6}$$

for every measurable function $f(x)$.

Remark 1.1. If a pair (w, v) satisfies (W_p) then

$$\int_{\mathbf{R}^n} (1+|x|^n)^{-p'} v^{1-p'}(x)\,dx < \infty. \tag{1.7}$$

It is the condition found by Carleson and Jones [3] in order that there exists $w \neq 0$ satisfying (1.6).

Remark 1.2. The condition (W_p) also implies that

$$\int_I w(x)\,dx \Big(|I|^{-1} \int_{c(n,\alpha)I} v^{1-p'}(x)\,dx\Big)^p \leqslant C \int_F v^{1-p'}(x)\,dx < \infty, \tag{1.8}$$

where F is a measurable subset of a cube I and $|F| \geqslant (1+\alpha)|I|/2$. By [7, pp. 473–474, Theorem 2] and (1.8) we see that (W_p) implies (S_p).

Remark 1.3. Obviously (W_p) is not necessary for (1.6) in general; however, when $w = v$, (W_p) is equivalent to the A_p condition because $w(x)$ and $w^{1-p'}(x)$ satisfy the A_∞ condition.

We give another example of a pair (w, v) which satisfies (W_p):

EXAMPLE. Let $u(x)$ satisfy the A_∞ condition, that is,

$$\int_E u(x)\,dx \leqslant C(|E|/|I|)^\beta \int_I u(x)\,dx,$$

for a cube I and a measurable subset E of I. We fix a cube Q_0 and set $Q_\varepsilon = (1+\varepsilon)Q_0$ where $\varepsilon > 0$. We define $v(x)$ and $w(x)$ as follows:

$$v(x) = \big(u(x)(1-\chi_{Q_0}(x)) + g(x)\chi_{Q_0}(x)\big)^{1-p},$$

$$w(x) = u(x)(1-\chi_{Q_\varepsilon}(x))\Big(\sup_{I \ni x} |I|^{-1} \int_{c(n,\alpha)I} v^{1-p'}(x)\,dx\Big)^{-p},$$

where $g(x)$ is any nonnegative locally integrable function and we also assume that $\sup_{I \ni x} |I|^{-1} \int_{c(n,\alpha)I} v^{1-p'}(x)\,dx < \infty$, and α is determined by $u(x)$ and ε. Then the pair (w, v) satisfies (W_p) even though $w(x)$ and $v^{1-p'}(x)$ are not in A_∞.

Remark 1.4. If $(w, v) \in (W_p)$ and if $w_1(x) \leqslant w(x)$, then $(w_1, v) \in (W_p)$; and also if (w, v) satisfies a sufficient condition by Sawyer [13] for (1.6) and if $v_1(x) \geqslant v(x)$ then (w, v_1) satisfies the same condition. So we see that between (W_p) and the classes of pairs of weights determined by Sawyer's condition in [13] there is no inclusion relation simply as classes of weights, but we do not know yet any precise relation between them as sufficient conditions for (1.6).

We prove our Theorem in § 4 by using the sharp maximal functions defined with the use of median values over cubes which were introduced by Strömberg [14]. Our method is also found in [8].

In § 2 we state some fundamental lemmas on median values over cubes, and in § 3 we prove some lemmas which are main parts of our proof of the Theorem.

Throughout this note Q, I and J denote cubes $\prod_{i=1}^{n}[a_i, a_i+l)$ where $a_i \in \mathbf{R}$ and $l > 0$, CE denotes the complement of a subset E of $\mathbf{R}^n$, and cI denotes the cube with the same center as I which is expanded c times. And we will take $0\cdot\infty$ to be 0.

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2. Properties of the median. In this section we shall mention some fundamental properties of median values over cubes and prove them. The reader should also consult [14] and [10].

Let $f(x)$ be a real-valued measurable function on $\mathbf{R}^n$. If $f(x)$ is almost everywhere finitely valued, then for every finite cube Q in $\mathbf{R}^n$ we can define a finite number $m_f(Q)$ as follows:

$$m_f(Q) = \max\{M \in \mathbf{R}:\ |\{x \in Q:\ f(x) < M\}| \leqslant \tfrac{1}{2}|Q|\}.$$

The number $m_f(Q)$ is called the *median* of f over Q. We learned this definition from Carleson [2]. From the definition it follows that

$$(2.1)\qquad |\{x \in Q:\ f(x) \geqslant m_f(Q)\}| \geqslant \tfrac{1}{2}|Q|,$$

$$(2.2)\qquad |\{x \in Q:\ f(x) \leqslant m_f(Q)\}| \geqslant \tfrac{1}{2}|Q|.$$

LEMMA 2.1. *For a measurable function $f(x)$ on $\mathbf{R}^n$ and a cube Q*

$$(2.3)\qquad |m_f(Q)| \leqslant m_{|f|}(Q).$$

Proof. For $m_f(Q) > 0$ the conclusion is immediate from $\{x \in Q:\ |f(x)| < m_f(Q)\} \subset \{x \in Q:\ f(x) < m_f(Q)\}$. If $m_f(Q) \leqslant 0$, then $\{x \in Q:\ |f(x)| < |m_f(Q)|\} \subset \{x \in Q:\ f(x) > m_f(Q)\}$ and from (2.2) we have (2.3). ∎

LEMMA 2.2. *For every almost everywhere finitely valued measurable function $f(x)$*

$$(2.4)\qquad \lim_{|Q|\to 0,\, Q \ni x} m_f(Q) = f(x) \quad \textit{for a.e. } x \textit{ in } \mathbf{R}^n.$$

Proof. For every $k \geqslant 1$ we define

$$s_k(x) = \sum_{j=-\infty}^{\infty} \alpha_{k,j}\chi_{E_{k,j}}(x)$$

where $\alpha_{k,j} = (j-1)/2^k$ and $E_{k,j} = \{x:\ (j-1)/2^k \leqslant f(x) < j/2^k\}$. Then

$$(2.5)\qquad 0 \leqslant f(x) - s_k(x) \leqslant 2^{-k} \quad \text{for all } x \text{ in } \mathbf{R}^n,$$

$$(2.6)\qquad m_{s_k}(Q) \leqslant m_f(Q) \quad \text{for every } Q.$$

Let

$$F = \bigcap_{k=1}^{\infty} \{x \in \mathbf{R}^n:\ \text{there is } j \text{ with } x \in E_{k,j} \text{ and } \lim_{\substack{|Q|\to 0\\ Q \ni x}} |E_{k,j} \cap Q|/|Q| = 1\}.$$

Lebesgue's theorem on differentiating the integral implies that almost every x in $\mathbf{R}^n$ belongs to F.

Suppose that $x \in F$. Then by the definition of F for every k we have $E_{k,j}$ with x in $E_{k,j}$ and

$$(2.7)\qquad |E_{k,j} \cap Q| > \tfrac{2}{3}|Q|$$

for sufficiently small cubes Q which contain x. Fix such a cube Q. Then $m_{s_k}(Q) = \alpha_{k,j}$. Since $|\{y \in Q:\ m_f(Q) \leqslant f(y)\}| \geqslant |Q|/2$ there exists $y \in E_{k,j}$ such that $m_f(Q) \leqslant f(y)$. By $\alpha_{k,j} = s_k(x) = s_k(y)$ and (2.6) we have

$$\begin{aligned} |m_f(Q) - f(x)| &\leqslant |m_f(Q) - s_k(y)| + |s_k(y) - f(x)| \\ &\leqslant f(y) - s_k(y) + |s_k(x) - f(x)| \\ &\leqslant 2^{-k+1} \quad (\text{by } (2.5)). \end{aligned}$$

This implies (2.4) for x. ∎

Remark 2.1. If $f(x)$ is locally integrable, (2.4) is proved immediately from Lemma 2.1 and (2.1) by using Lebesgue's theorem on differentiating the integral.

Let $A > 1$ and set

$$(2.8)\qquad M_{A,f}(Q) = \inf\{\lambda \geqslant 0:\ |\{x \in Q:\ |f(x) - m_f(Q)| > \lambda\}| \leqslant |Q|/A\}$$

for a measurable function $f(x)$ and a cube Q. The sharp maximal function $M_A^{\sharp} f(x) = \sup_{Q \ni x} M_{A,f}(Q)$ was studied by Strömberg [14]. An observation on $M_A^{\sharp} f(x)$ by Jawerth and Torchinsky [10] yields the following lemmas:

LEMMA 2.3. *Suppose that $f(x)$ is an integrable function supported by a cube Q and that I and $\bar{I}$ are cubes such that $I \subset \bar{I}$, $|\bar{I}| = 2^n|I|$, and $2\bar{I} \subset Q$. Then*

$$|m_{T^*f}(I) - m_{T^*f}(\bar{I})| \leqslant C \sup_{I \subset J \subset Q,\, J \text{ cube}} |J|^{-1} \int_J |f(x)|\,dx$$

where T^* is a maximal singular integral operator with kernel $K(x)$ which satisfies (1.1)–(1.3) *and the constant C is independent of f, I and Q.*

Proof. Let x_0 be the center of I and $c_I = T^*(f\chi_{C2\bar{I}})(x_0)$. Then using

Lemma 2.1 we see that

$$|m_{T^*f}(I)-m_{T^*f}(\bar{I})| \leqslant |m_{T^*f}(I)-c_I|+|c_I-m_{T^*f}(\bar{I})| \leqslant m_{|T^*f-c_I|}(I)+m_{|T^*f-c_I|}(\bar{I}).$$

From the argument of [10, pp. 260–261] we obtain

$$m_{|T^*f-c_I|}(\bar{I}) \leqslant C \sup_{I\subset J\subset Q} |J|^{-1}\int_J |f(x)|\,dx.$$

The same estimate holds for $m_{|T^*f-c_I|}(I)$. ∎

LEMMA 2.4 (Jawerth and Torchinsky [10, pp. 260–261]). *Let a function $f(x)$ be supported by a cube Q. Then for $A>1$ and for a subcube I of Q*

$$M_{A,T^*f}(I) \leqslant CA \sup_{I\subset J\subset Q} |J|^{-1}\int_J |f(x)|\,dx.$$

Here the constant C is independent of f, I and A.

3. Some lemmas. In this section we prepare two lemmas for the proof of the Theorem. Lemma 3.1 is a refinement of Lemma 1 of [8] in a particular case; its proof is based on the argument of Carleson [1]. In Lemma 3.3 we shall use the hypothesis (W_p).

If a subcube I of a cube Q is obtained by bisecting Q a finite number of times we shall say that I is a *dyadic subcube* of Q.

LEMMA 3.1. *Let $f(x)$ be an integrable compactly supported function and let T^* be a maximal operator with kernel $K(x)$ which satisfies* (1.1)–(1.3). *If a cube Q contains the support of f, then for any $A>2^{n+1}$ there exist a measurable function $g(x)$, families $\mathscr{F}_j$, $j\geqslant 1$, of dyadic subcubes of Q and sequences $\{a_I^{(j)}\}$, $I\in\mathscr{F}_j$, $j\geqslant 1$, of numbers such that*

(i) $\mathscr{F}_j$ *is a disjoint family for each* j,

(ii) $$\bigcup_{I\in\mathscr{F}_{j+1}} I \subset \bigcup_{I\in\mathscr{F}_j} I,$$

(iii) *for all $j\geqslant 0$, $k\geqslant 1$ and $I\in\mathscr{F}_j$ ($\mathscr{F}_0=\{Q\}$)*

$$\sum_{J\in\mathscr{F}_{k+j},\,J\subset I} |J| \leqslant (2/A)^k |I|,$$

(iv) $$T^*f(x) = g(x)+\sum_{j=1}^{\infty}\sum_{I\in\mathscr{F}_j} a_I^{(j)}\chi_I(x) \quad \text{for a.e. } x \text{ in } Q,$$

(v) $$|g(x)| \leqslant CA \sum_{I\in\mathscr{F}_j} \chi_I(x) \sup_{I\subset J\subset Q} |J|^{-1}\int_J |f(x)|\,dx$$

$$\text{for a.e. } x \text{ in } \bigcup_{I\in\mathscr{F}_j} I \cap C\Big(\bigcup_{J\in\mathscr{F}_{j+1}} J\Big),\ j\geqslant 0,$$

(vi) $$|a_I^{(j)}| \leqslant CA \sup_{I\subset J\subset Q} |J|^{-1}\int_J |f(x)|\,dx, \quad I\in\mathscr{F}_j,\ j\geqslant 1.$$

Here the constant C depends only on $K(x)$ and n.

Proof. Put $(T^*f)_1(x) = T^*f(x)-m_{T^*f}(Q)$. Let $\mathscr{F}_1$ be the family of all maximal dyadic subcubes I of Q which satisfy

$$|m_{(T^*f)_1}(I)| > \max\Big\{M_{A,T^*f}(Q),\ |Q|^{-1}\int_Q |f(x)|\,dx\Big\}$$

and set $a_I^{(1)} = m_{(T^*f)_1}(I)$ for every I in $\mathscr{F}_1$. We put

$$(T^*f)_2(x) = (T^*f)_1(x) - \sum_{I\in\mathscr{F}_1} a_I^{(1)}\chi_I(x).$$

Inductively, we let $\mathscr{F}_j$ be the family of all maximal dyadic subcubes I of Q such that I is contained in a unique cube J in $\mathscr{F}_{j-1}$ and

$$|m_{(T^*f)_j}(I)| > \max\Big\{M_{A,T^*f}(J),\ |Q|^{-1}\int_Q |f(x)|\,dx\Big\}. \tag{3.1}$$

We also put

$$a_I^{(j)} = m_{(T^*f)_j}(I) \quad \text{for every } I \text{ in } \mathscr{F}_j, \tag{3.2}$$

$$(T^*f)_{j+1}(x) = (T^*f)_j(x) - \sum_{I\in\mathscr{F}_j} a_I^{(j)}\chi_I(x). \tag{3.3}$$

If $J\subset I$, $J\in\mathscr{F}_{j+1}$ and $I\in\mathscr{F}_j$, we get from the definition (3.2)

$$a_J^{(j+1)} = m_{(T^*f)_j}(J)-m_{(T^*f)_j}(I) = m_{T^*f}(J)-m_{T^*f}(I). \tag{3.4}$$

Hence we have from (3.1) and Lemma 2.1

$$M_{A,T^*f}(I) < |m_{T^*f}(J)-m_{T^*f}(I)| \leqslant m_{|T^*f-m_{T^*f}(I)|}(J).$$

Therefore we obtain from (2.1)

$$|J|/2 \leqslant |\{x\in J:\ |T^*f(x)-m_{T^*f}(I)| > M_{A,T^*f}(I)\}|.$$

Summing over all $J\subset I$ we get from the definition of $M_{A,T^*f}(I)$

$$\sum_{J\subset I} |J| \leqslant 2\,|\{x\in I:\ |T^*f(x)-m_{T^*f}(I)| > M_{A,T^*f}(I)\}| \leqslant \frac{2}{A}|I|.$$

Repeating this argument we get (iii). (i) and (ii) follow from the definition.

Now suppose that I is in $\mathscr{F}_j$, $j\geqslant 1$, and that $\bar{I}$ is the dyadic subcube of Q such that $I\subset\bar{I}$ and $|\bar{I}| = 2^n|I|$. Then from the definition of $a_I^{(j)}$ it follows that

$$|a_I^{(j)}| = |m_{(T^*f)_j}(I)| \leqslant |m_{(T^*f)_j}(I)-m_{(T^*f)_j}(\bar{I})|+|m_{(T^*f)_j}(\bar{I})| = |m_{T^*f}(I)-m_{T^*f}(\bar{I})|+|m_{(T^*f)_j}(\bar{I})|. \tag{3.5}$$

From the definition (3.1) we observe that

$$|m_{(T^*f)_j}(\bar{I})| \leqslant \max\Big\{M_{A,T^*f}(\bar{I}),\ |Q|^{-1}\int_Q |f(x)|\,dx\Big\}, \tag{3.6}$$

where $\tilde{I}$ is in $\mathscr{F}_{j-1}$ and $I \subset \tilde{I}$. From Lemmas 2.3 and 2.4 we have

$$|m_{T^*f}(I) - m_{T^*f}(\tilde{I})| \leqslant C \sup_{I \subset J \subset Q} |J|^{-1} \int_J |f(x)|\,dx,$$

$$M_{A,T^*f}(\tilde{I}) \leqslant CA \sup_{\tilde{I} \subset J \subset Q} |J|^{-1} \int_J |f(x)|\,dx.$$

Hence we obtain (vi) from (3.5) and (3.6).

Next we assert that $\lim_{j\to\infty}(T^*f)_j(x)$ converges for almost every x. We fix a point x in Q which satisfies $\lim_{|I|\to 0, I\ni x} m_{T^*f}(I) = T^*f(x)$. Then, if there exists $I \in \mathscr{F}_j$ which is a very small cube containing x, we see that

$$|m_{(T^*f)_{j+1}}(J)| = |m_{T^*f}(J) - m_{T^*f}(I)| \leqslant |Q|^{-1} \int_Q |f(x)|\,dx$$

for all cubes J such that $x \in J \subset I$. This implies that the number of cubes I containing x is finite. Thus if we put

$$g(x) = m_{T^*f}(Q) + \lim_{j\to\infty} (T^*f)_j(x),$$

Lemma 2.2 shows (iv).

Finally, for almost every x in $\bigcup_{I\in\mathscr{F}_j} I \cap C\big(\bigcup_{J\in\mathscr{F}_{j+1}} J\big)$ we have

$$g(x) = m_{T^*f}(Q) + (T^*f)_{j+1}(x).$$

Since $x \notin \bigcup_{J\in\mathscr{F}_{j+1}} J$, from the definition, if $x \in I$ and if I is in $\mathscr{F}_j$ then we get

$$(3.7) \qquad |m_{(T^*f)_{j+1}}(J)| \leqslant \max\{M_{A,T^*f}(I),\ |Q|^{-1} \int_Q |f(x)|\,dx\}$$

for all dyadic subcubes J such that $J \subsetneq I$ and $x \in J$. Hence Lemmas 2.2, 2.4 and the fact that $m_{T^*f}(Q) \leqslant C|Q|^{-1} \int_Q |f(x)|\,dx$ imply (v). ∎

If a cube I has the form $\prod_{j=1}^n [l_j 2^k, (l_j+1)2^k)$ where k and l_j are integers, the cube I is said to be a dyadic cube. According to Sawyer [12] we call I a *t-dyadic cube* if I coincides with some dyadic cube after translation by t, $t \in \mathbf{R}^n$. In this note, "*dyadic subcubes of a cube*" and "*t-dyadic cubes*" have different meanings. To state the next lemma, we introduce some notation. Let $\tilde{I}$ be a fixed cube, $0 < \varepsilon < 1$, and let J be a t-dyadic cube for some $t \in \mathbf{R}^n$. Then

$$\tilde{I} \cap J = \prod_{j=1}^n [m_j, m_j + l_j)$$

where $m_j \in \mathbf{R}$, $l_j \geqslant 0$. We define

$$l(J) = \min_{1\leqslant j\leqslant n} l_j, \qquad R(J) = \tilde{I} \cap \prod_{j=1}^n [m_j - \varepsilon l(J), m_j + l_j + \varepsilon l(J)).$$

LEMMA 3.2. *For a cube $\tilde{I}$, any $t \in \mathbf{R}^n$ and $\varepsilon \in (0, 1)$ there exists a family $\mathscr{F}$ of t-dyadic cubes, $1 \leqslant \operatorname{card}\mathscr{F} \leqslant 2^n$, such that*

(i) $\qquad 2^n|\tilde{I}| \leqslant |J| < 2^{2n}|\tilde{I}|$ *for all* $J \in \mathscr{F}$,

(ii) $\qquad \tilde{I} = \bigcup_{J\in\mathscr{F}} R(J)$,

(iii) $\qquad l(J) \geqslant c(n, \varepsilon)|\tilde{I}|^{1/n}$ *for all* $J \in \mathscr{F}$,

where the constant $c(n, \varepsilon)$ depends only on n and ε.

Proof. Let J be a t-dyadic cube which intersects $\tilde{I}$ and satisfies (i). We may assume that $l_1 \leqslant \ldots \leqslant l_n \leqslant |\tilde{I}|^{1/n}$.

Case 1: $l_n \leqslant (\varepsilon/2)|\tilde{I}|^{1/n}$. In this case there exists a unique t-dyadic cube J' such that the side lengths of $J' \cap \tilde{I}$ are larger than $(1-\varepsilon/2)|\tilde{I}|^{1/n}$ and $R(J') = \tilde{I}$.

Case 2: $l_n > (\varepsilon/2)|\tilde{I}|^{1/n}$ and $\varepsilon_1 l_{j+1} \leqslant l_j$, $1 \leqslant j \leqslant n-1$, where ε_1 is a positive number smaller than $\varepsilon/(1+\varepsilon)$. Then we have $l(J) \geqslant (\varepsilon\varepsilon_1^{n-1}/2)|\tilde{I}|^{1/n}$. Hence $l(J)$ satisfies (iii).

Case 3: $l_n > (\varepsilon/2)|\tilde{I}|^{1/n}$ and there exists j_0 such that $1 \leqslant j_0 \leqslant n-1$, $l_{j_0} < \varepsilon_1 l_{j_0+1}$ and $l_j \geqslant \varepsilon_1 l_{j+1}$ $(j \geqslant j_0+1)$, $n \geqslant 2$. Then there exists J' such that the side lengths of $J' \cap \tilde{I}$ are $|\tilde{I}|^{1/n} - l_1, \ldots, |\tilde{I}|^{1/n} - l_{j_0}, l_{j_0+1}, \ldots, l_n$, with $l(J') \geqslant \min\{(1-\varepsilon_1)|\tilde{I}|^{1/n}, l_{j_0+1}\}$.

If $l(J') \geqslant (1-\varepsilon_1)|\tilde{I}|^{1/n}$, then for $j \leqslant j_0$ we have

$$(|\tilde{I}|^{1/n} - l_j) + \varepsilon l(J') \geqslant \{1 + \varepsilon - (1+\varepsilon)\varepsilon_1\}|\tilde{I}|^{1/n} \geqslant |\tilde{I}|^{1/n}.$$

If $l(J') = l_{j_0+1}$, we have $l(J') \geqslant \frac{1}{2}\varepsilon\varepsilon_1^{n-1}|\tilde{I}|^{1/n}$ and for $j \leqslant j_0$

$$(|\tilde{I}|^{1/n} - l_j) + \varepsilon l(J') \geqslant |\tilde{I}|^{1/n} + (\varepsilon - \varepsilon_1) l_{j_0+1} \geqslant |\tilde{I}|^{1/n}.$$

Hence, in this case we get $R(J') \supset \tilde{I} \cap J$.

Thus, in either case, if $|\tilde{I} \cap J|$ is small, the rectangle $\tilde{I} \cap J$ is contained in another enlarged rectangle $R(J')$. ∎

Remark 3.1. If ε is sufficiently small with respect to $\alpha \in (0, 1)$, we have subcubes $\{J_\nu\}_\nu$ of $R(J)$ for $J \in \mathscr{F}$ such that

$$(3.8) \qquad |J_\nu| = l(J)^n \quad \text{for all } \nu,$$

$$(3.9) \qquad |J_\nu \cap J| \geqslant \frac{1+\alpha}{2}|J_\nu| \quad \text{for all } \nu,$$

$$(3.10) \qquad R(J) = \bigcup_\nu J_\nu \quad \text{and} \quad \sum_\nu \chi_{J_\nu}(x) \leqslant C_n \quad \text{for all } J \in \mathscr{F}.$$

Notice that there exists a constant $c(n, \alpha)$ depending only on n and α such that $J \subset c(n, \alpha)J_\nu$ for all ν.

Next we shall show a lemma which will reduce our argument to that for the dyadic maximal functions. We define the t-dyadic maximal function ${}^tM_\sigma^*(f)$ for a nonnegative function $\sigma(x)$ as follows:

$${}^tM_\sigma^*(f)(x) = \sup_{J\ni x, J\ t\text{-dyadic}} \frac{1}{\int_J \sigma(y)\,dy} \int_J |f(y)|\sigma(y)\,dy.$$

Fix a cube I. Let $\{I_\xi\}_\xi$ be a family of pairwise disjoint dyadic subcubes of I, let J_ξ be a cube with the same center as I_ξ with $I_\xi \subset J_\xi$ and $|J_\xi| \leqslant |I|$ for each ξ and let E_ξ be a measurable subset of I_ξ for every ξ. Then we have the following lemma:

LEMMA 3.3. *Suppose that a pair (w, v) satisfies the hypothesis (W_p). Then there exist positive constants $A(n, \alpha)$ and C depending only on α, β, C_0, n and p such that if $A > A(n, \alpha)$ and if each E_ξ satisfies $|E_\xi| \leqslant A^{-m}|I_\xi|$ where $m \geqslant 1$, then*

$$(3.11)\quad \sum_\xi \int_{E_\xi} w(x)\,dx\Big(|J_\xi|^{-1}\int_{J_\xi}|f(x)|\,dx\Big)^p \leqslant CN^{-n}A^{-\beta m}\int_{|t|\leqslant N} dt \int_{I\cap C(\bigcup_\xi E_\xi)} ({}^tM_\sigma^*(f/\sigma)(x))^p\sigma(x)\,dx,$$

$$(3.12)\quad \sum_\xi \int_{I_\xi} w(x)\,dx\Big(|J_\xi|^{-1}\int_{J_\xi}|f(x)|\,dx\Big)^p \leqslant CN^{-n}\int_{|t|\leqslant N} dt \int_{I\cap C(\bigcup_\xi E_\xi)} ({}^tM_\sigma^*(f/\sigma)(x))^p\sigma(x)\,dx,$$

for any function $f(x)$ which satisfies $\int |f(x)|^p v(x)\,dx < \infty$. Here $\sigma(x) = v^{1-p'}(x)$ and N is a positive number larger than $2^3 n^{1/2}|I|^{1/n}$.

Proof. For each ξ we can take a dyadic subcube $\tilde{I}_\xi$ of I such that $\tilde{I}_\xi \supset I_\xi$ and $|J_\xi| \leqslant |\tilde{I}_\xi| < 2^n|J_\xi|$. For every ξ and every t we have a family $\mathscr{F}_\xi$ of t-dyadic cubes, $1 \leqslant \operatorname{card}\mathscr{F}_\xi \leqslant 2^n$, and the associated rectangles defined as in Lemma 3.2 such that

$$(3.13)\quad 2^n|\tilde{I}_\xi| \leqslant |J| < 2^{2n}|\tilde{I}_\xi| \quad \text{for all } J \in \mathscr{F}_\xi,$$

$$(3.14)\quad \tilde{I}_\xi = \bigcup_{J\in\mathscr{F}_\xi} R(J).$$

Here the constant ε in Lemma 3.2 is determined only by the constant α in the hypothesis (W_p), see Remark 3.1.

If there exists $J \in \mathscr{F}_\xi$ such that $J_\xi \subset J$, we have

$$\Big(|J_\xi|^{-1}\int_{J_\xi}|f(y)|\,dy\Big)^p\chi_{E_\xi}(x) \leqslant C\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p\chi_{E_\xi\cap R(J)}(x).$$

Hence from the observation by C. Fefferman and Stein [6] we see that

$$\Big(|J_\xi|^{-1}\int_{J_\xi}|f(y)|\,dy\Big)^p\chi_{E_\xi}(x) \leqslant CN^{-n}\int_{|t|\leqslant N} dt \sum_{J\in\mathscr{F}_\xi}\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p\chi_{E_\xi\cap R(J)}(x).$$

Thus we get

$$(3.15)\quad \sum_\xi\int_{E_\xi} w(x)\,dx\Big(|J_\xi|^{-1}\int_{J_\xi}|f(y)|\,dy\Big)^p \leqslant C\sum_\xi\int_{E_\xi} w(x)\,dx\Big(N^{-n}\int_{|t|\leqslant N} dt\sum_{J\in\mathscr{F}_\xi}\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p\chi_{E_\xi\cap R(J)}(x)\Big)$$
$$= CN^{-n}\int_{|t|\leqslant N} dt\sum_\xi\sum_{J\in\mathscr{F}_\xi}\int_{E_\xi\cap R(J)} w(x)\,dx\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p.$$

Assume that $A_0 > 1$ and fix t, $|t| \leqslant N$. Let $\mathscr{G}_1$ be the subfamily of all maximal cubes of $\{\tilde{I}_\xi\}$ and let $\mathscr{R}_1 = \{R\colon R = R(J)$ for some $J \in \mathscr{F}_\xi$, there exist $\tilde{I}_\eta \in \mathscr{G}_1$ and $J' \in \mathscr{F}_\eta$ such that $\tilde{I}_\xi \subset \tilde{I}_\eta$, $J \subset J'$ and $|J|^{-1}\int_J|f(x)|\,dx \leqslant A_0|J'|^{-1}\int_{J'}|f(x)|\,dx\}$. Inductively we set $\mathscr{G}_k = \{\tilde{I}_\xi\colon \tilde{I}_\xi$ is the maximal cube for which there exists $R \notin \bigcup_{j=1}^{k-1}\mathscr{R}_j\}$ for $k \geqslant 2$ and $\mathscr{R}_k = \{R\colon R = R(J)$ for some $J \in \mathscr{F}_\xi$, $R \notin \bigcup_{j=1}^{k-1}\mathscr{R}_j$, there exist $\tilde{I}_\eta \in \mathscr{G}_k$ and $J' \in \mathscr{F}_\eta$ such that $\tilde{I}_\xi \subset \tilde{I}_\eta$, $J \subset J'$, $R(J') \notin \bigcup_{j=1}^{k-1}\mathscr{R}_j$ and $|J|^{-1}\int_J|f(x)|\,dx \leqslant A_0|J'|^{-1}\int_{J'}|f(x)|\,dx\}$.

Then $\mathscr{R}_k \cap \mathscr{R}_j = \emptyset$ if $k \neq j$ and $\bigcup_\xi\bigcup_{J\in\mathscr{F}_\xi}R(J) = \bigcup_{k=1}^\infty\bigcup_{R\in\mathscr{R}_k}R$. In fact, let R be associated with $\tilde{I}_\xi$; then if there exists $\tilde{I}_\eta \in \bigcup_k\mathscr{G}_k$ such that $\tilde{I}_\xi \subsetneq \tilde{I}_\eta$ (otherwise $\tilde{I}_\xi$ is in $\mathscr{G}_1$ and $R \in \mathscr{R}_1$), we take $k = \max\{j\colon \tilde{I}_\eta \supsetneq \tilde{I}_\xi$ with $\tilde{I}_\eta$ in $\mathscr{G}_j\}$. In this case, if $R \notin \bigcup_{j=1}^k\mathscr{R}_j$, we have $\tilde{I}_\xi \in \mathscr{G}_{k+1}$, thus $R \in \mathscr{R}_{k+1}$. (The above argument is due to Professor Yabuta.)

Therefore we have

$$(3.16)\quad \sum_\xi\sum_{J\in\mathscr{F}_\xi}\int_{E_\xi\cap R(J)} w(x)\,dx\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p$$
$$= \sum_{k=1}^\infty\sum_{R(J)\in\mathscr{R}_k, J\in\mathscr{F}_\xi}\int_{E_\xi\cap R(J)} w(x)\,dx\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p$$
$$= \sum_{k=1}^\infty\sum_{\tilde{I}_\eta\in\mathscr{G}_k}\sum_{R(J)\in\mathscr{R}_k, \tilde{I}_\xi\subset\tilde{I}_\eta}\int_{E_\xi\cap R(J)} w(x)\,dx\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p$$
$$= \sum_{k=1}^\infty\sum_{\tilde{I}_\eta\in\mathscr{G}_k}\sum_{R(J')\in\mathscr{R}_k, J'\in\mathscr{F}_\eta}\sum_{R(J)\in\mathscr{R}_k, J\in\mathscr{F}_\xi, J\subset J'}\int_{E_\xi\cap R(J)} w(x)\,dx\Big(|J|^{-1}\int_J|f(y)|\,dy\Big)^p.$$

Since $R(J) \subset R(J')$ and $\{E_\xi\}$ are pairwise disjoint, from the definition of $\mathscr{R}_k$ we majorize the above expression by

$$A_0^p\sum_{k=1}^\infty\sum_{\tilde{I}_\eta\in\mathscr{G}_k}\sum_{R(J')\in\mathscr{R}_k, J'\in\mathscr{F}_\eta}\int_{\bigcup_\xi E_\xi\cap R(J')} w(x)\,dx\Big(|J'|^{-1}\int_{J'}|f(y)|\,dy\Big)^p$$
$$\leqslant A_0^p\sum_{k=1}^\infty\sum_{\tilde{I}_\eta\in\mathscr{G}_k}\sum_{R(J')\in\mathscr{R}_k, J'\in\mathscr{F}_\eta}\int_{\bigcup_\xi E_\xi\cap R(J')} w(x)\,dx\Big(|J'|^{-1}\int_{J'}\sigma(y)\,dy\Big)^p\Big(\inf_{z\in J'}({}^tM_\sigma^*(f/\sigma)(z)\Big)^p.$$

For each $R(J') \in \mathscr{R}_k$ we have subcubes $\{J'_\nu\}$ which satisfy (3.8)–(3.10) of Remark 3.1 to Lemma 3.2. Hence we get

$$(3.17)\quad \int_{\bigcup_\xi E_\xi\cap R(J')} w(x)\,dx\Big(|J'|^{-1}\int_{J'}\sigma(y)\,dy\Big)^p \leqslant C\sum_\nu\int_{\bigcup_\xi E_\xi\cap J'_\nu} w(x)\,dx\Big(|J'_\nu|^{-1}\int_{c(n,\alpha)J'_\nu}\sigma(y)\,dy\Big)^p,$$

where we take $c(n, \alpha)$ to satisfy $J' \subset c(n, \alpha)J'_\nu$.

We set

$$F'_\nu = J'_\nu \cap J' \cap C\Big\{\Big(\bigcup_{|I_\xi|\leqslant|\tilde{I}_\eta|}E_\xi\Big)\cup\Big(\bigcup_{R(J)\in\bigcup_{j\geqslant k+1}\mathscr{R}_j}R(J)\cap J\Big)\Big\}.$$

Notice that there exist no I_ξ such that $|I_\xi| > |\tilde{I}_\eta|$ and $I_\xi \cap \tilde{I}_\eta \neq \emptyset$ because $\{I_\xi\}$ are pairwise disjoint dyadic subcubes. This observation implies that

$$F'_\nu \cap \bigcup_\xi (E_\xi \cap J'_\nu) = \emptyset. \tag{3.18}$$

For F'_ν we have

$$\Big|J'_\nu \cap J' \cap \Big(\bigcup_{R(J) \in \bigcup_{j \geq k+1} \mathscr{R}_j} R(J) \cap J\Big)\Big| \leqslant \sum_{J \subset J',\, J \text{ maximal}} |J|. \tag{3.19}$$

Since $|J|^{-1} \int_J |f(x)|\,dx > A_0 |J'|^{-1} \int_{J'} |f(x)|\,dx$, from the definition of $\mathscr{R}_j$ we see that the right-hand side of (3.19) is bounded by

$$|J'| \big(A_0 \int_{J'} |f(x)|\,dx\big)^{-1} \sum_{J \subset J'} \int_J |f(x)|\,dx.$$

Since $|J'| \approx |J'_\nu|$ from (iii) of Lemma 3.2 and (3.8), we have

$$\Big|J'_\nu \cap J' \cap \Big(\bigcup_{R(J) \in \bigcup_{j \geq k+1} \mathscr{R}_j} R(J) \cap J\Big)\Big| \leqslant C A_0^{-1} |J'_\nu|. \tag{3.20}$$

If $|E_\xi| \leqslant A^{-m} |I_\xi|$, we get

$$\sum_{|I_\xi| \leqslant |\tilde{I}_\eta|,\, I_\xi \cap J'_\nu \neq \emptyset} |E_\xi| \leqslant A^{-m} \sum_{|I_\xi| \leqslant |\tilde{I}_\eta|,\, I_\xi \cap J'_\nu \neq \emptyset} |I_\xi| \leqslant C A^{-m} |J'_\nu|. \tag{3.21}$$

Therefore from (3.20), (3.21) and (3.9) we can find positive numbers $A(n, \alpha)$ and A_0, depending only on n and α, such that $|F'_\nu| \geqslant \alpha |J'_\nu|$ for $A > A(n, \alpha)$. Then, from (3.18) the hypothesis (W_p) for (w, v) implies

$$\sum_\nu \int_{\bigcup_\xi E_\xi \cap J'_\nu} w(x)\,dx \Big(|J'_\nu|^{-1} \int_{c(n,\alpha) J'_\nu} \sigma(y)\,dy\Big)^p \tag{3.22}$$
$$\leqslant C A^{-m\beta} \sum_\nu \int_{F'_\nu} \sigma(x)\,dx \leqslant C A^{-m\beta} \int_{U_k} \sigma(x)\,dx \quad \text{(from (3.10))},$$

where

$$U_k = R(J') \cap J' \cap C\Big(\Big(\bigcup_\xi E_\xi\Big) \cup \Big(\bigcup_{R(J) \in \bigcup_{j \geq k+1} \mathscr{R}_j} R(J) \cap J\Big)\Big).$$

Thus we see from (3.17) and (3.22) that the right-hand side of (3.16) is majorized by

$$C A^{-m\beta} \sum_k \int_{V_k} ({}^t M^*_\sigma (f/\sigma)(x))^p \sigma(x)\,dx \leqslant C A^{-m\beta} \int_{I \cap C(\bigcup_\xi E_\xi)} ({}^t M^*_\sigma (f/\sigma)(x))^p \sigma(x)\,dx,$$

where

$$V_k = \Big(\bigcup_{I_\eta \in \mathscr{G}_k} \bigcup_{J \in \mathscr{F}_\eta} R(J) \cap J\Big) \cap C\Big(\Big(\bigcup_\xi E_\xi\Big) \cup \Big(\bigcup_{R(J) \in \bigcup_{j \geq k+1} \mathscr{R}_j} R(J) \cap J\Big)\Big).$$

Hence we have (3.11) from (3.15) and (3.16). Using (1.8) we also have (3.12) by the same argument. ∎

4. Proof of the Theorem. We now prove our Theorem. We apply the same method as in the proof of the Theorem of [8], and use the boundedness of the t-dyadic maximal functions as in Sawyer [12] and Jawerth [9]. The idea of the proof is partly due to Carleson [1] and Uchiyama [15].

Proof of the Theorem. First we assume that $f(x)$ is compactly supported and $\int |f(x)|^p v(x)\,dx < \infty$. ($f(x)$ is integrable). Let a cube Q contain the support of $f(x)$. Then we shall show under the hypothesis (W_p) that

$$\int_Q (T^* f(x))^p w(x)\,dx \leqslant C \int_Q |f(x)|^p v(x)\,dx. \tag{4.1}$$

Here the constant C is independent of $f(x)$ and Q.

For $f(x)$, Q and a number $A > 2^{n+1}$, there exist a measurable function $g(x)$, families $\mathscr{F}_j$, $j \geqslant 0$, of dyadic subcubes of Q and sequences $\{a_I^{(j)}\}$, $I \in \mathscr{F}_j$, $j \geqslant 1$, of numbers which satisfy (i)–(vi) of Lemma 3.1. From (iv) of Lemma 3.1 we obtain

$$(T^* f(x))^p \leqslant c_p |g(x)|^p + c_p \Big(\sum_{j=1}^\infty \sum_{I \in \mathscr{F}_j} |a_I^{(j)}| \chi_I(x)\Big)^p \tag{4.2}$$

for a.e. x in Q, where $c_p = 2^{p-1}$. For the second term of the right-hand side of (4.2) we see that

$$\Big(\sum_{j=1}^\infty \sum_{I \in \mathscr{F}_j} |a_I^{(j)}| \chi_I(x)\Big)^p$$
$$= \sum_{k=1}^\infty \chi_{(\bigcup_{I \in \mathscr{F}_k} I) \cap C(\bigcup_{I' \in \mathscr{F}_{k+1}} I')}(x) \Big(\sum_{j=1}^k \sum_{I \in \mathscr{F}_j} |a_I^{(j)}| \chi_I(x)\Big)^p$$
$$\leqslant c_p \sum_{k=1}^\infty \sum_{I \in \mathscr{F}_k} |a_I^{(k)}|^p \chi_{I \cap C(\bigcup_{I' \in \mathscr{F}_{k+1}} I')}(x)$$
$$+ c_p \sum_{k=2}^\infty \chi_{(\bigcup_{I \in \mathscr{F}_k} I) \cap C(\bigcup_{I' \in \mathscr{F}_{k+1}} I')}(x) \sum_{j=1}^{k-1} c_p^{k-j} \sum_{I \in \mathscr{F}_j} |a_I^{(j)}|^p \chi_I(x).$$

(See Lemma 2 in [8].) We put

$$a(x, Q, p) = |g(x)|^p + \sum_{k=1}^\infty \sum_{I \in \mathscr{F}_k} |a_I^{(k)}|^p \chi_{I \cap C(\bigcup_{I' \in \mathscr{F}_{k+1}} I')}(x),$$

$$b(x, Q, p) = \sum_{k=2}^\infty \chi_{(\bigcup_{I \in \mathscr{F}_k} I) \cap C(\bigcup_{I' \in \mathscr{F}_{k+1}} I')}(x) \sum_{j=1}^{k-1} c_p^{k-j} \sum_{I \in \mathscr{F}_j} |a_I^{(j)}|^p \chi_I(x).$$

Then from (4.2) we have

$$(T^* f(x))^p \leqslant c_p (a(x, Q, p) + b(x, Q, p)) \quad \text{for a.e. } x \text{ in } Q.$$

Therefore the inequalities (4.3) and (4.4) below suffice to prove (4.1):

$$\int_Q a(x, Q, p) w(x)\,dx \leqslant C \int_Q |f(x)|^p v(x)\,dx, \tag{4.3}$$

$$\int_Q b(x, Q, p) w(x)\,dx \leqslant C \int_Q |f(x)|^p v(x)\,dx. \tag{4.4}$$

We shall prove only (4.4), for (4.3) may be obtained by the same argument and it also follows from Sawyer's theorem [12] for the maximal functions because $a(x, Q, p) \leqslant C(M^* f(x))^p$ for a.e. x in Q.

For every $I_\xi \in \mathscr{F}_k$, $k \geqslant 1$, let $N(I_\xi)$ be the positive integer such that $|2^{N(I_\xi)} I_\xi| = |I'_\xi|$ where $I'_\xi \supset I_\xi$, $I'_\xi \in \mathscr{F}_{k-1}$, and let $N(Q) = 0$. Then, for every cube I such that $I_\nu \subset I \subset Q$, $I_\nu \in \mathscr{F}_j$ and $I \not\subset 2I_\nu$ there exists $I_\xi \in \mathscr{F}_k$, $1 \leqslant k \leqslant j$, and an integer l, $0 \leqslant l \leqslant N(I_\xi)$, such that $I \subset 2^l I_\xi$ and $I \not\subset 2^{l-1} I_\xi$, where we take $\frac{1}{2} I_\xi = 2^{N(I'_\xi)} I'_\xi$, $I'_\xi \in \mathscr{F}_{k+1}$. We also have $|I| \approx |2^l I_\xi|$. Hence

$$\sup_{I \supset I_\nu} |I|^{-1} \int_I |f(x)|\, dx \leqslant C \max_{0 \leqslant k \leqslant j, 0 \leqslant l \leqslant N(I_\xi)} |2^l I_\xi|^{-1} \int_{2^l I_\xi} |f(x)|\, dx,$$

where $I_\xi \supset I_\nu$, $I_\xi \in \mathscr{F}_k$.

Let J^k_ξ be a cube such that

$$|J^k_\xi|^{-1} \int_{J^k_\xi} |f(x)|\, dx = \max_{0 \leqslant l \leqslant N(I_\xi)} |2^l I_\xi|^{-1} \int_{2^l I_\xi} |f(x)|\, dx.$$

Then by (vi) of Lemma 3.1 we get

$$(4.5) \qquad |a^{(j)}_{I_\nu}| \leqslant CA \sum_{l=1}^{j} |J^l_\xi|^{-1} \int_{J^l_\xi} |f(x)|\, dx, \qquad J^l_\xi \supset I_\nu.$$

By (4.5) we have

$$(4.6) \qquad \int_Q b(x, Q, p) w(x)\, dx \leqslant \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \int_{\bigcup_{I \in \mathscr{F}_k} I} \sum_{I_\nu \in \mathscr{F}_j} |a^{(j)}_{I_\nu}|^p \chi_{I_\nu}(x) w(x)\, dx$$

$$\leqslant CA^p \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \sum_{I_\nu \in \mathscr{F}_j} \int_{\bigcup_{I \in \mathscr{F}_k} I \cap I_\nu} \Big(\sum_{l=1}^{j} |J^l_\xi|^{-1} \int_{J^l_\xi} |f(y)|\, dy\Big)^p w(x)\, dx$$

$$\leqslant CA^p \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \sum_{I_\nu \in \mathscr{F}_j} \int_{\bigcup_{I \in \mathscr{F}_k} I \cap I_\nu} \sum_{l=1}^{j} c_p^{j-l} \Big(|J^l_\xi|^{-1} \int_{J^l_\xi} |f(y)|\, dy\Big)^p w(x)\, dx$$

$$= CA^p \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \sum_{l=1}^{j} c_p^{j-l} \sum_{I_\xi \in \mathscr{F}_l} \int_{(\bigcup_{I \in \mathscr{F}_k} I) \cap (\bigcup_{I \in \mathscr{F}_j} I) \cap I_\xi} \Big(|J^l_\xi|^{-1} \int_{J^l_\xi} |f(y)|\, dy\Big)^p w(x)\, dx$$

$$= CA^p \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \sum_{l=1}^{j} c_p^{j-l} \sum_{I'_\xi \in \mathscr{F}_{l-1}} \sum_{I_\xi \subset I'_\xi, I_\xi \in \mathscr{F}_l} \int_{(\bigcup_{I \in \mathscr{F}_k} I) \cap I_\xi} \Big(|J^l_\xi|^{-1} \int_{J^l_\xi} |f(y)|\, dy\Big)^p w(x)\, dx.$$

If we put $E_\xi = (\bigcup_{I \in \mathscr{F}_k} I) \cap I_\xi$, from (iii) of Lemma 3.1 we have $|E_\xi| \leqslant (2/A)^{k-l} |I_\xi|$. Since $|J^l_\xi| \leqslant |I'_\xi|$, taking $I = I'_\xi$, $m = k-l$ and $N > C|I|^{1/n}$ we obtain from (3.11) of Lemma 3.3

$$\sum_{I_\xi \subset I'_\xi, I_\xi \in \mathscr{F}_l} \int_{(\bigcup_{I \in \mathscr{F}_k} I) \cap I_\xi} \Big(|J^l_\xi|^{-1} \int_{J^l_\xi} |f(y)|\, dy\Big)^p w(x)\, dx$$

$$\leqslant CN^{-n} \int_{|t| \leqslant N} dt\, (C/A)^{\beta(k-l)} \int_{I'_\xi \cap C(\bigcup_{I \in \mathscr{F}_k} I)} ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx.$$

Thus we majorize the right-hand side of (4.6) by

$$(4.7) \qquad CN^{-n} \int_{|t| \leqslant N} dt \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \sum_{l=1}^{j} c_p^{j-l} (C/A)^{\beta(k-l)}$$

$$\times \sum_{I'_\xi \in \mathscr{F}_{l-1}} \int_{I'_\xi \cap C(\bigcup_{I \in \mathscr{F}_k} I)} ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx.$$

If we put $i = k-j$ and $m = j-l$, and interchange the order of summation in the integrand of (4.7), then we see that

$$\sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_p^{k-j} \sum_{l=1}^{j} c_p^{j-l} (C/A)^{\beta(k-l)} \sum_{I'_\xi \in \mathscr{F}_{l-1}} \int_{I'_\xi \cap C(\bigcup_{I \in \mathscr{F}_k} I)} ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx$$

$$= \sum_{l=1}^{\infty} \sum_{i=1}^{\infty} c_p^i (C/A)^{\beta i} \sum_{m=0}^{\infty} c_p^m (C/A)^{\beta m}$$

$$\times \int_{\bigcup_{I'_\xi \in \mathscr{F}_{l-1}} I'_\xi \cap C(\bigcup_{I \in \mathscr{F}_{l+i+m}} I)} ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx$$

$$= \sum_{i=1}^{\infty} c_p^i (C/A)^{\beta i} \sum_{m=0}^{\infty} c_p^m (C/A)^{\beta m}$$

$$\times \sum_{l=1}^{\infty} \int_{\bigcup_{I'_\xi \in \mathscr{F}_{l-1}} I'_\xi \cap C(\bigcup_{I \in \mathscr{F}_{l+i+m}} I)} ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx$$

$$\leqslant \sum_{i=1}^{\infty} c_p^i (C/A)^{\beta i} \sum_{m=0}^{\infty} c_p^m (C/A)^{\beta m} (i+m+1) \int_Q ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx.$$

When we take A to be large enough, we conclude that (4.7) is bounded by

$$CN^{-n} \int_{|t| \leqslant N} dt \int_Q ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx.$$

Since ${}^t M^*_\sigma(f/\sigma)(x)$ is the dyadic maximal function of f/σ, we have

$$\int_Q ({}^t M^*_\sigma(f/\sigma)(x))^p \sigma(x)\, dx \leqslant C \int_Q |f(x)|^p v(x)\, dx,$$

where the constant C is independent of t. Therefore we obtain (4.4).

From (v) of Lemma 3.1, (3.11) and (3.12) of Lemma 3.3 the same argument is valid for (4.3). Thus we get (4.1).

Next for any $f(x)$ we may assume that $\int |f(x)|^p v(x)\,dx < \infty$. Then Carleson and Jones' condition (1.7) and Hölder's inequality show that $\int |f(x)|(1+|x|)^{-n}\,dx < \infty$ and $T^*f(x)$ is well defined.

Let $f_M(x) = f(x)\chi_{\{|x| \leqslant M\}}(x)$. Then from (4.1) we obtain

$$\int (T^* f_M(x))^p w(x)\,dx \leqslant C \int |f(x)|^p v(x)\,dx,$$

where the bound C is independent of M. By taking M to tend to infinity Fatou's lemma shows the conclusion (1.6) of our Theorem. ∎

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The structure of module derivations of $C^n[0, 1]$ into $L_p(0, 1)$

by

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Abstract. We completely determine the structure of all continuous and discontinuous module derivations $D\colon C^n[0, 1] \to L_p(0, 1)$, $n = 1, 2, \ldots$ and $1 \leqslant p < \infty$, where $C^n[0, 1]$ is the Banach algebra of n times continuously differentiable functions on the unit interval $[0, 1]$ and $L_p(0, 1)$ is considered as a $C^n[0, 1]$-module with module multiplication defined by the $C^n[0, 1]$-operational calculus for the operator $M - nJ$ where $M\colon f(t) \mapsto tf(t)$ and $J\colon f(t) \mapsto \int_0^t f(s)\,ds$.

1. Preliminaries. Let $C^n[0, 1]$ denote the algebra of all complex-valued functions on $[0, 1]$ which have n continuous derivatives. It is well known that $C^n[0, 1]$ is a Banach algebra under the norm

$$\|f\| = \max_{t \in [0,1]} \sum_{k=0}^{n} |f^{(k)}(t)|/k!,$$

and that its structure space is $[0, 1]$. We will need a characterization of the squares of the closed primary ideals with finite codimension in $C^n[0, 1]$. We use the notation

$$M_{n,k}(t_0) = \{f \in C^n[0, 1] \mid f^{(j)}(t_0) = 0,\ j = 0, 1, \ldots, k\}.$$

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,0}(t_0)$ which consists of functions vanishing at t_0. Throughout this paper we write $M_{n,k}$ for $M_{n,k}(0)$ and set $z(t) = t$, $0 \leqslant t \leqslant 1$. We have

1.1. THEOREM. *Let n be a positive integer. Then*

(i) $M_{n,0}^2 = zM_{n,0} = \{f \mid f(0) = f'(0) = 0$ *and* $f^{(n+1)}(0)$ *exists*$\}$,
(ii) $M_{n,k}^2 = z^{k+1} M_{n,k}$, $1 \leqslant k \leqslant n-1$,
(iii) $M_{n,n}^2 = z^n M_{n,n}$.

Part (i) is from [1, Example 3]. Part (ii) is due to Dales and McClure [3, Theorem 3.1.]. The proof of part (iii) can be found in [2].

The squares of the closed primary ideals $M_{n,k}(t_0)$ at other points t_0 in $[0, 1]$ are given by exactly similar formulas, where z is replaced by $z - t_0$.

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