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Computing summing norms and type constants on few vectors

by

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Abstract. We show that the (Gaussian) type p and cotype q constants of an n-dimensional Banach space can be computed, up to a universal (i.e. independent of the space, p or q) constant, on just n vectors. The same holds for the $(q, 2)$-summing norms of rank n operators. This unifies the results of N. Tomczak-Jaegermann and H. König. In the case of the 1-summing norm of an operator defined on an n-dimensional Banach space we have a similar result for $n \log n$ vectors.

In the local theory of Banach spaces the concepts of type and cotype of a space and that of a p-summing operator are of primary importance. In this note we present several results concerning computation of the constants associated with these concepts, in the case when some information about the dimension of the space or rank of the operator is available. Our first two results, concerning the (Gaussian) type p and cotype q constants of an n-dimensional Banach space and the $(q, 2)$-summing norms of rank n operators, unify the work of N. Tomczak-Jaegermann [TJ1] and H. König [Kön]; the third deals with the 1-summing norm of an operator defined on an n-dimensional Banach space. In addition to the two papers mentioned above, our approach relies on the methods of [BLM], [BT1], [BT2] and especially [Tal].

Recall that a Banach space X is said to be of (Gaussian) *type p* (resp. *cotype q*) iff there is a constant $K > 0$ such that, for any finite sequence (x_j) of elements of X,

$$(1)\qquad (\mathbf{E}\|\sum_j \gamma_j \mathbf{x}_j\|^2)^{1/2} \leqslant K(\sum_j \|x_j\|^p)^{1/p} \qquad (\text{resp. } \geqslant K^{-1}(\sum_j \|x_j\|^q)^{1/q}),$$

where γ_j's are independent $N(0, 1)$ Gaussian random variables and $\mathbf{E}$ stands for the expected value. The smallest constant K which works in the above is called the (Gaussian) *type p* (resp. *cotype q*) *constant* of X, and denoted by $T_p(X)$ (resp. $C_q(X)$). If we consider only sequences (x_j) of length not exceeding n, we denote the corresponding quantities by $T_p^{(n)}(X)$ and $C_q^{(n)}(X)$ respectively. We then have

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THEOREM 1. *There exists a universal constant C such that whenever E is a finite-dimensional Banach space, say* $\dim E = n$, *and* $p = [1, 2]$ (*resp.* $q \in [2, \infty]$), *then*

$$T_p(E) \leqslant C T_p^{(n)}(E) \quad (\textit{resp. } C_q(E) \leqslant C C_q^{(n)}(E)).$$

Let us remark here that the "operator version" of Theorem 1 also holds with the same proof (type p and cotype q constants of an operator u are defined by replacing x_j by ux_j at the "lower sides" of the inequalities in (1)). In another direction, with the Rademacher type and cotype defined by replacing γ_j's by r_j's (the Rademacher functions) in (1), the "type part" of Theorem 1 extends to the new setting since the ratio beween the corresponding constants is bounded by universal constants from above and below. For the cotype, the situation is somewhat more complicated; see [K-T] or [TJ2], §25, for details.

Theorem 1 follows in a standard way (we indicate the argument at the end of the proof of the theorem) from the following fact, which is of independent interest. Again, we recall some notation. An operator $u\colon X \to Y$ is called (p, q)-*summing* iff there is a constant $K \geqslant 0$ such that, for any finite sequence (x_j) of elements of X,

$$(2) \qquad (\sum_j \|u(x_j)\|^p)^{1/p} \leqslant K \sup\{(\sum_j |f(x_j)|^q)^{1/q}\colon f \in X^*, \ \|f\| \leqslant 1\}.$$

The smallest constant K which works is called the (p, q)-*summing norm* of u and denoted by $\pi_{p,q}(u)$; if $p = q$, we abbreviate these to "p-summing" and "$\pi_p(u)$". As before, if only sequences (x_j) of length not exceeding n are taken into account, we speak of $\pi_p^{(n)}(u)$ and $\pi_{p,q}^{(n)}(u)$. We have

PROPOSITION 2. *There exists a universal constant C such that whenever* $u\colon X \to Y$ *is a finite rank operator (say* $\operatorname{rank} u = n$; X, Y *Banach spaces) and* $q \in [2, \infty]$, *then*

$$\pi_{q,2}(u) \leqslant C \pi_{q,2}^{(n)}(u).$$

This was proved for $q = 2$ in [TJ1] (see [TJ2], § 18, for a proof giving, in that case, $C = 2^{1/2}$; our present methods give apparently much worse constants) and, for $q > 2$, *with C depending on q*, in [Kön].

For the 1-summing norm we have a similar, somewhat weaker result.

THEOREM 3. *There exists a universal constant C such that whenever E is a finite-dimensional Banach space, say* $\dim E = n$, *and u is an operator from E into another Banach space, then, for some* $m \leqslant n \log n$, *we have*

$$\pi_1(u) \leqslant C \pi_1^{(m)}(u).$$

In fact, in the assertion of Theorem 3 one can replace $n \log n$ by $\beta^2 n$, where $\beta \equiv \sup\{K(v)\colon v \in \mathscr{L}(l_\infty^n, E), \ \|v\| \leqslant 1\}$; in particular, $\beta \leqslant K(E)$ and $\beta \leqslant C(q, C_q(E))$ for $q \in [2, \infty)$ (see the remarks at the end of the proof of Theorem 3; $K(\cdot)$ denotes here the K-convexity constant of the space or operator, see any of the references mentioned below for definitions). We do not know if the $\log n$ factor is necessary. We do point out, however, that the hypothesis "$\dim E = n$" cannot be replaced by the weaker one "$\operatorname{rank} u = n$", as shown by T. Figiel and A. Pełczyński [Peł]. The number of vectors one may need in that case is, in general, exponential in the rank. To our knowledge, no better estimates were available until now also in our more restrictive setting, or even for the identity operator on a general n-dimensional space (cf. also [KTJ]).

We use the standard notation from the local theory of Banach spaces as can be found e.g. in [M-S] or [TJ2]. In particular, [TJ2] contains most of the facts we quote in this paper by referring to the original source; [Pi3] may be consulted for results concerning factorization of operators. Let us only mention the following: by $|\cdot|$ we denote the Euclidean norm on $\mathbf{R}^n$ (and also the cardinality of a set); $(e_j)_{1 \leqslant j \leqslant n}$ is the standard unit vector basis in $\mathbf{R}^n$. For Banach spaces E, F, $\mathscr{L}(E, F)$ is the space of bounded linear operators from E to F, usually endowed with the operator norm; if $E = l_p^m$ and $F = l_r^n$, we denote that norm by $\|\cdot\|_{p \to r}$, or by $\|\cdot\colon l_p^m \to l_r^n\|$ to emphasize the dimensions. We concentrate on the case of real spaces; however, the results and, with some minor modifications, the arguments carry over to the complex case. The letters C, c, c_0 etc. are reserved for universal numerical constants, whose exact values may, however, vary throughout the paper.

Proposition 2 (and hence Theorem 1) will follow easily from the following fact.

LEMMA 4. *Let n, N be positive integers and A an* $n \times N$ *matrix with* $\|A\|_{2 \to 2} = \|A\colon l_2^N \to l_2^n\| \leqslant 1$. *Let* A_1 *be the matrix obtained from A by normalizing the column vectors (in* l_2^n). *Then there exists* $\sigma \subset \{1, \ldots, N\}$ *satisfying*

(i) $c\,\mathrm{hs}(A)^2 \leqslant |\sigma| \leqslant \mathrm{hs}(A)^2$ $(\leqslant n)$,
(ii) $\|A_1 P_\sigma\|_{2 \to 2} \leqslant c^{-1}$,

where $c > 0$ *is a universal numerical constant and* P_σ *is the (orthogonal) projection of* $\mathbf{R}^n$ *onto* $\operatorname{span}\{e_j\colon j \in \sigma\}$. *Moreover, if* $(w_i)_{1 \leqslant i \leqslant N}$, $\sum_{1 \leqslant i \leqslant n} w_i = 1$, *is a sequence of weights, one can additionally require that*

(iii) $\sum_{i \in \sigma} w_i |Ae_i|^{-2} \geqslant c$.

We first show how to derive Proposition 2 (and Theorem 1) from the lemma.

Proof of Proposition 2. Observe first that the $(q, 2)$-summing norm of u may be as well defined via

$$(3) \qquad \pi_{q,2}(u) = \sup\{(\sum_{j \leqslant N} \|uAe_j\|^q)^{1/q}\}$$

with the supremum taken over all $A\colon l_2^N \to X$, $\|A\| \leqslant 1$, and all $N \in \mathbf{N}$ (resp.

with N fixed for $\pi_{q,2}^{(N)}(u)$). By a standard argument (see [TJ2], §§ 11, 18), it is enough to consider the case of $X = l_2^n$. Choose N and $A\colon l_2^N \to l_2^n\ (= X)$, $\|A\| \leqslant 1$, for which the supremum on the right hand side of (3) is achieved (this is possible, by [TJ2], § 18, with $N \leqslant n^2$; in fact, for our purposes it would be enough to consider N and A for which the expression from (3) is e.g. $\geqslant \frac{1}{2}\pi_{q,2}(u)$). Set, for $i \leqslant N$,

$$w_i = \|uAe_i\|^q/\pi_{q,2}(u)^q$$

and apply Lemma 4. Identifying the obtained σ with $\{1, \ldots, k\}$, $k = |\sigma| \leqslant n$, we get from (iii)

$$c\pi_{q,2}(u)^q \leqslant \sum_{i \leqslant k} \|uAe_i\|^q |Ae_i|^{-2} \leqslant \sum_{i \leqslant k} \|uAe_i\|^q |Ae_i|^{-q} = \sum_{i \leqslant k} \|uA_1 e_i\|^q$$

and so, substituting cA_1P_σ for A in (3) we get

$$\pi_{q,2}(u) \leqslant c^{-(q+1)/q}\pi_{q,2}^{(n)}(u) \leqslant c^{-3/2}\pi_{q,2}^{(n)}(u),$$

as required.

Proof of Theorem 1. We use the following observation from [TJ1]. For a Banach space X, $n \in \mathbf{N}$, $p \in [1, 2]$ and $q \in [2, \infty]$, one has

$$T_p^{(n)}(X) = \sup\{\pi_{q,2}^{(n)}(u)\colon u \in \mathscr{L}(l_2^n, X),\ l(u) \leqslant 1\}$$
$$C_q^{(n)}(X) = \sup\{l(v)\colon v \in \mathscr{L}(l_2^n, X),\ (\pi_{q,2}^{(n)})^*(v^*) \leqslant 1\},$$

where, for $u \in \mathscr{L}(l_2^n, X)$,

$$l(u) \equiv (\mathbf{E}\|\sum_{j \leqslant N} \gamma_j u e_j\|^2)^{1/2}$$

and, for an ideal norm α on $\mathscr{L}(E, F)$, α^* is the dual (in trace duality) norm on $\mathscr{L}(F, E)$; $\mathbf{E}$ and γ_j's have the same meaning as in (1). From this, Proposition 2, and the "defining formulae" (1) and (2), Theorem 1 readily follows.

Proof of Lemma 4. We start by taking care of some "trivial" cases. First, if $w_j|Ae_j|^{-2}$ is large (e.g. $\geqslant 2^{-4}$) for some j, one forces (iii) by adjoining j to any σ satisfying (i) and (ii), and adjusting the constant c if necessary. In particular, if $\mathrm{hs}(A)$ is small (e.g. $\leqslant 4$), we *must* be in the situation as above; then, in fact, the choice of $\sigma = \{j\}$ works. This means that we may as well assume that

(4) $$w_j|Ae_j|^{-2} < 2^{-4} \quad \text{for all } j \leqslant N, \qquad \mathrm{hs}(A)^2 > 2^4.$$

For $j \in \{1, \ldots, N\}$, set $x_i = Ae_i$ (the ith column of A), $\mu_i = |x_i|^{-1}$, $\delta_i = |x_i|^2$ and let $\xi_1, \ldots, \xi_N$ be independent (0, 1)-valued "selectors" with $\mathbf{E}\xi_i = \delta_i$ (defined on some probability space $(\Omega, \mathscr{P})$, $\mathbf{E}$ stands for the expected value). Consider the random set $\tau = \tau(\omega) = \{j\colon \xi_j(\omega) = 1\} \subset \{1, \ldots, N\}$. Then (4) implies that

(5) $$\mathscr{P}\big(\tfrac{1}{2}\mathrm{hs}(A)^2 \leqslant |\tau| \leqslant \tfrac{3}{2}\mathrm{hs}(A)^2\big) > 3/4, \qquad \mathscr{P}\big(\sum_{i \in \tau} w_i|Ae_i|^{-2} \geqslant 1/2\big) > 3/4.$$

For a sequence of scalars $\alpha = (\alpha_j)$ denote by M_α the diagonal matrix with α_j's on the diagonal; then $P_\tau = M_\xi$ and $A_1 = AM_\mu$. We are going to ensure first the conditions (i) and (ii). Consider the quantity

$$Q \equiv \mathbf{E}\|(A_1P_\tau)^*\|_{2\to 1} = \mathbf{E}\|M_{\xi\mu}A^*\|_{2\to 1}.$$

We have

$$\begin{aligned} Q &= \mathbf{E}\max\{\sum_{1 \leqslant j \leqslant N} \mu_j\xi_j|\langle x_j, u\rangle|\colon u \in l_2^n,\ |u| \leqslant 1\} \\ &= \mathbf{E}\max\{\sum_{1 \leqslant j \leqslant N} \mu_j(\xi_j - \delta_j)|\langle x_j, u\rangle| + \sum_{1 \leqslant j \leqslant N} \mu_j\delta_j|\langle x_j, u\rangle|\colon u \in l_2^n,\ |u| \leqslant 1\} \\ &= \mathbf{E}\max\{(*) + (**)\}. \end{aligned}$$

Now, by the Cauchy–Schwarz inequality,

$$(**) \leqslant (\sum_{j \leqslant N} \mu_j^2\delta_j^2)^{1/2}(\sum_{j \leqslant N} |\langle x_j, u\rangle|^2)^{1/2} = \mathrm{hs}(A)|A^*u| \leqslant \mathrm{hs}(A).$$

On the other hand,

$$\mathbf{E}\max\{(*)\} \leqslant (2\pi\mathbf{E}\max\{\sum_{j \leqslant N} \gamma_j\mu_j(\xi_j - \delta_j)|\langle x_j, u\rangle|\colon |u| \leqslant 1\},$$

where γ_j's are independent (also of ξ_j's) $N(0, 1)$ Gaussian random variables; this follows from the fact that if Z_j's are *mean zero, independent vector* random variables, then

$$\mathbf{E}\|\sum_j Z_j\| \leqslant (2\pi)^{1/2}\mathbf{E}\|\sum_j \gamma_j Z_j\|.$$

Combining the estimates and repeating the argument used above to estimate $(**)$ we obtain

(6) $$Q \leqslant (2\pi)^{1/2}\mathbf{E}\max\{\sum_{j \leqslant N} \gamma_j\mu_j\xi_j|\langle x_j, u\rangle|\colon |u| \leqslant 1\} + (1 + (2\pi)^{1/2})\mathrm{hs}(A).$$

Now, by the Slepian's lemma (say, in the version due to Fernique [Fer]; or see [M-S] or [Gro]) applied, for fixed (ξ_j), to the processes

$$X_u(\omega) \equiv \sum_{j \leqslant N} \gamma_j(\omega)\mu_j\xi_j|\langle x_j, u\rangle| \quad \text{and} \quad Y_u(\omega) \equiv \sum_{j \leqslant N} \gamma_j(\omega)\mu_j\xi_j\langle x_j, u\rangle,$$

the first term on the right hand side of (6) does not exceed

$$\begin{aligned} &(2\pi)^{1/2}\mathbf{E}\max\{\sum_{j \leqslant N} \gamma_j\mu_j\xi_j\langle x_j, u\rangle\colon |u| \leqslant 1\} \\ &\quad = (2\pi)^{1/2}\mathbf{E}|\sum_{j \leqslant N} \gamma_j\mu_j\xi_j x_j| \leqslant (2\pi)^{1/2}(\mathbf{E}_\xi\mathbf{E}_\gamma|\sum_{j \leqslant N} \gamma_j\mu_j\xi_j x_j|^2)^{1/2} \\ &\quad = (2\pi)^{1/2}(\mathbf{E}_\xi \sum_{j \leqslant N} \mu_j^2\xi_j|x_j|^2)^{1/2} = (2\pi)^{1/2}\mathrm{hs}(A), \end{aligned}$$

which shows that

$$Q = \mathbf{E}\|(A_1P_\tau)^*\|_{2\to 1} \leqslant (1 + (8\pi)^{1/2})\mathrm{hs}(A).$$

Consequently, setting $C_0 = 4(1+(8\pi)^{1/2})$, we conclude that

(7) $$\mathscr{P}(\|(A_1 P_\tau)^*\|_{2\to 1} \leqslant C_0 \operatorname{hs}(A)) > 3/4.$$

Now, for any given τ ($\equiv \{j:\ \xi_j(\omega) = 1\}$), by the "little" Grothendieck Theorem (applied in the same context in the proof of Thm. 1.2 in [BT1] or in [Kas]), $(A_1 P_\tau)^*$ "well factors" through a diagonal operator acting from l_2^N into l_1^N. More precisely, there exists a sequence $\alpha = (\alpha_j)_{j\leqslant N} \in \mathbf{R}_+^N$, $|\alpha| \leqslant 1$, such that $(A_1 P_\tau)^* = M_{\xi\mu} A^* = M_\alpha B$ with $\|B\|_{2\to 2} \leqslant (\pi/2)^{1/2} \|M_{\xi\mu} A^*\|_{2\to 1}$. In other words,

$$\|M_{\alpha^{-1}} M_{\xi\mu} A^*\|_{2\to 2} \leqslant (\pi/2)^{1/2} \|M_{\xi\mu} A^*\|_{2\to 1}.$$

Now, if τ satisfies simultaneously the conditions from (5) and (7) (which happens with probability $> 1/2$), we can pass to a subset $\sigma \equiv \{j\in\tau:\ \alpha_j \leqslant (2/|\tau|)^{1/2}\} \subset \tau$, $|\sigma| \geqslant |\tau|/2 \geqslant \frac{1}{4}\operatorname{hs}(A)^2$, for which

$$\|A_1 P_\sigma\|_{2\to 2} = \|M_\alpha P_\sigma M_{\alpha^{-1}} M_{\xi\mu} A^*\|_{2\to 2} \leqslant (2/\operatorname{hs}(A))(\pi/2)^{1/2} \cdot C_0 \operatorname{hs}(A) < 2^6;$$

this clearly implies the first two statements of the lemma. We note in passing that we could as well use the Bernoulli–Rademacher random variables instead of the Gaussian ones and, in place of the Slepian's lemma, the comparison theorem for Rademacher processes from [L-T] (the latter is simpler, but less known than the former one); this would in fact allow us to improve the constants somewhat.

To obtain (iii), we first show (using virtually the same argument as above) that

$$\|M_{w^{1/2}\mu^2\xi} A^*\|_{2\to 1} \leqslant (1+(8\pi)^{1/2}),$$

and so, if $C_1 = 4(1+(8\pi)^{1/2})$, then

(8) $$\mathscr{P}(\|M_{w^{1/2}\mu^2\xi} A^*\|_{2\to 1} \leqslant C_1) > 3/4.$$

Consequently, for any τ (i.e. ξ) satisfying the condition from (8), there exists a sequence $\alpha = \alpha(\tau) \in \mathbf{R}_+^N$, $|\alpha| \leqslant 1$, such that

$$\|M_{\alpha^{-1}} M_{w^{1/2}\mu^2\xi} A^*\|_{2\to 2} \leqslant (\pi/2)^{1/2} C_1.$$

Again, pass to a subset $\sigma_0 \equiv \{j\in\tau:\ \alpha_j (w_j^{1/2}\mu_j)^{-1} \leqslant 2\} \subset \tau$; then

$$1 \geqslant \sum_{j\in\tau\setminus\sigma_0} \alpha_j^2 \geqslant 4 \sum_{j\in\tau\setminus\sigma_0} w_j \mu_j^2 = 4 \sum_{j\in\tau\setminus\sigma_0} w_j |Ae_j|^{-2}$$

and so, if τ additionally satisfies (5), it follows that

$$\sum_{j\in\sigma_0} w_j |Ae_j|^{-2} \geqslant 1/4.$$

Now replacing σ by $\sigma \cup \sigma_0$ and passing if necessary (to conform to (ii)) to a twice as small subset still satisfying (iii), we get the assertion.

Let us mention here, without proof, the following variant of Lemma 4, which is also a "rectangular" version of Theorem 1.1 from [BT2] (or a "weighted" version of Theorem 2 from [Lun]).

PROPOSITION 5. *In the notation of Lemma 4 set* $\delta_0 = \sum_{j\leqslant N} w_j |Ae_j|^2$ *and let* $\delta = \max\{\delta_0, n/N\}$. *Then there exists* $\sigma \subset \{1, \ldots, N\}$ *satisfying*

(i) $|\sigma| \leqslant \delta N$,
(ii) $\|AP_\sigma\|_{2\to 2} \leqslant \delta^{1/2}$,
(iii) $\sum_{i\in\sigma} w_i \geqslant c\delta$,

where $c > 0$ *is a universal numerical constant.*

We now pass to the proof of Theorem 3. We need the following "weighted version" of the Theorem from [Tal] (which was, in turn, an "improved" Theorem 1.2 from [BLM]).

LEMMA 6. *Let* n, N *be positive integers,* E *an* n*-dimensional Banach space,* $A \in \mathscr{L}(l_\infty^N, E)$, $\|A\| \leqslant 1$, *and* $(w_i)_{1\leqslant i\leqslant N}$, $\sum_{1\leqslant j\leqslant N} w_i = 1$, *a sequence of weights. Then there exists* $\sigma \subset \{1, \ldots, N\}$ *and scalars* $(\mu_i)_{i\in\sigma}$ *satisfying*

(i) $|\sigma| \leqslant \beta^2 n$,
(ii) $\|\sum_{i\in\sigma} t_i \mu_i Ae_i\| \leqslant c^{-1} \max_{i\in\sigma} |t_i|$ *for any scalars* $(t_i)_{i\in\sigma}$,
(iii) $\sum_{i\in\sigma} w_i \mu_i \geqslant c$,

where $c > 0$ *is a universal numerical constant and* $\beta \equiv \sup\{K(v):\ v\in\mathscr{L}(l_\infty^N, E),\ \|v\| \leqslant 1\}$.

Once Lemma 6 is proved, Theorem 3 follows immediately. Indeed, the following variant of (3) is just a rephrasing of the definition (2) for the 1-summing norm:

(9) $$\pi_1(u) = \sup\{\sum_{j\leqslant N} \|uAe_j\|\}$$

with the supremum taken over all $A:\ l_\infty^N \to E$, $\|A\| \leqslant 1$, and all $N\in\mathbf{N}$ (resp. with N fixed for $\pi_1^{(N)}(u)$). We then argue as in the proof of Proposition 2, with Lemma 6 and (9) playing the role of Lemma 4 and (3). One just needs to observe that, for any E, the quantity β does not exceed $c_1(1+\log n)^{1/2}$; this is implied by the fact that any $v\in\mathscr{L}(l_\infty^N, E)$ factors through an n-dimensional quotient of l_∞^N, hence v^* factors through an n-dimensional subspace of l_1^N, and for such subspaces, as is known e.g. from [Pi1], the K-convexity constant is $\leqslant c_1(1+\log n)^{1/2}$. As far as the remarks following the statement of Theorem 3 are concerned, one clearly has $\beta \leqslant K(E)$; the fact that $\beta \leqslant C(q, C_q(E))$ for $q\in[2, \infty)$ follows from [Pi2], Theorem 10.

For the proof of Lemma 6 we need the following fact well known to specialists (recall that for $\beta = (\beta_j)$, M_β is the diagonal matrix with β_j's on the diagonal).

LEMMA 7. *Let* $n < N$ *be positive integers and let* F *be an* n*-dimensional subspace of* l_1^N *(or any* L_1*-space). Then there exists* $\alpha = (\alpha_j) \in l_2^N$, $|\alpha| \leqslant 1$, *such that*

$$\pi_2(M_{\alpha^{-1}} i_{1,2}|_F) \leqslant (2n)^{1/2},$$

where $i_{1,2}$: $l_1^N \to l_2^N$ *is the formal identity. Consequently, there exists* $\tau \subset \{1, \ldots, N\}$, $|\tau| \geqslant \frac{3}{4}N$, *such that*

$$\pi_2(P_\tau i_{1,2}|_F) \leqslant (8n/N)^{1/2}.$$

The first inequality is essentially the result of Lewis [Lew], for $p = 1$; the subset τ can be defined by $\tau = \{j: |\alpha_j| \leqslant (4/N)^{1/2}\}$, cf. also Lemma 4.5 from [BLM].

Proof of Lemma 6. The argument is very similar to that of [Tal]. Roughly speaking, having an operator A: $l_\infty^N \to E$, $\|A\| \leqslant 1$, $N \in \mathbf{N}$ arbitrary, we want to replace it by a "related" operator A_1: $l_\infty^m \to E$, $\|A_1\| \leqslant c^{-1}$, with m "not too big" and A_1 still "sufficiently large" in the sense of (iii); in [Tal] and [BLM] a very similar problem was considered, with (iii) replaced by the requirement that A_1 be "nearly a quotient map". The proof is based on an iteration procedure: in each step we reduce the dimension of l_∞^N by a fixed factor (say, $\leqslant 3/4$), while increasing the norm and decreasing the weight from (iii) only marginally. More precisely, we have

Claim. *In the notation of Lemma 6, there exist scalars* $(\mu_i)_{i \leqslant N}$ *with* $\mu_i \in \{0, 1, 2\}$ *and* $|\{i: \mu_i \neq 0\}| \leqslant \frac{3}{4}N$ *such that* (ii) *and* (iii) *hold with* $c^{-1} \leqslant 1 + 2^4\beta(n/N)^{1/2}$ *and* $c \geqslant 1 - 10N^{-1/2}$ *respectively (and, say,* $\sigma = \{i: \mu_i \neq 0\}$).

Since it is clear that, in view of the exponential decrease of N's corresponding to successive steps, one can iterate the procedure from the Claim as long as $1 + 2^4\beta(n/N)^{1/2}$ remains bounded, Lemma 6 follows. To prove the Claim, we choose first $\tau_1, \tau_2 \subset \{1, \ldots, N\}$, such that $|\tau_i| \geqslant \frac{3}{4}N$ for $i = 1, 2$ and

(10) $$|w_i| \leqslant 4/N \quad \text{for } i \in \tau_1,$$

(11) $$\pi_2(P_{\tau_2} i_{1,2}|_{\operatorname{ran} A^*}) \leqslant (8n/N)^{1/2},$$

where $i_{1,2}$: $l_1^N \to l_2^N$ is the formal identity. It is clear that one can achieve (10); (11) follows from Lemma 7.

We now set $\tau = \tau_1 \cap \tau_2$ and $\mu_i = 1$ for $i \notin \tau$. For $i \in \tau$ we choose μ_i's *at random* to be 0 or 2; i.e. we consider $(\mu_i)_{i \in \tau}$ to be a sequence of independent random variables with $\mathscr{P}(\mu_i = 0) = \mathscr{P}(\mu_i = 2) = 1/2$. Then clearly

(12) $$\mathscr{P}(|\{i \in \tau: \mu_i = 0\}| \geqslant \tfrac{1}{4}N) \geqslant 1/2$$

while, on the other hand,

(13) $$\mathscr{P}\Big(\sum_{i \in \tau} w_i \mu_i \geqslant \sum_{i \in \tau} w_i - 10N^{-1/2}\Big) > 3/4.$$

The latter inequality follows from the fact that the sequence $(\mu_i - 1)$ has Bernoulli distribution (i.e. the same as the Rademacher functions r_i), from the well known estimate

$$\mathscr{P}\Big(\sum_{i \leqslant m} c_i r_i \geqslant \alpha\Big) \leqslant \exp\big(-t^2/(4m\|(c_i)\|_\infty^2)\big)$$

and from (10). Thus it remains to show that the variant of (ii) required by the Claim holds with sufficiently large probability. We have, as in the proof of Lemma 4,

$$\begin{aligned}
&\mathbf{E}\max\Big\{\Big\|\sum_{1 \leqslant i \leqslant N} t_i \mu_i A e_i\Big\|: t = (t_i) \in l_\infty^N, \|t\|_\infty \leqslant 1\Big\}\\
&\leqslant \mathbf{E}\max\Big\{\sum_{1 \leqslant i \leqslant N} \mu_i |\langle A e_i, u\rangle|: u \in E, \|u\| \leqslant 1\Big\}\\
&\leqslant \mathbf{E}\max\Big\{\sum_{1 \leqslant i \leqslant N} (\mu_i - 1)|\langle A^* u, e_i\rangle| + \sum_{1 \leqslant i \leqslant N} |\langle A^* u, e_i\rangle|: u \in E, \|u\| \leqslant 1\Big\}\\
&\qquad \leqslant \mathbf{E}\max\Big\{\sum_{i \in \tau} |\langle A^* u, e_i\rangle| r_i: u \in E, \|u\| \leqslant 1\Big\} + 1\\
&\qquad \leqslant 1 + (\pi/2)^{1/2}\mathbf{E}\max\Big\{\sum_{i \in \tau} |\langle A^* u, e_i\rangle| \gamma_i: u \in E, \|u\| \leqslant 1\Big\}\\
&\qquad \leqslant 1 + (\pi/2)^{1/2}\mathbf{E}\Big\|\sum_{i \in \tau} \gamma_i A e_i\Big\| \leqslant 1 + \Big((\pi/2)\mathbf{E}\Big\|\sum_{i \in \tau} \gamma_i A e_i\Big\|^2\Big)^{1/2}\\
&\qquad = 1 + (\pi/2)^{1/2} l(A i_{2,\infty} P_\tau) \leqslant 1 + (\pi/2)^{1/2} l(A i_{2,\infty} P_{\tau_2}),
\end{aligned}$$

where $l(\cdot)$ was defined in the proof of Theorem 1 and $i_{2,\infty}$: $l_2^N \to l_\infty^N$ is the formal identity. Consider the canonical factorization $A = \tilde{A}q$, where q: $l_\infty^N \to l_\infty^N/\ker A$ is the quotient map. By [DMT], Lemma 1, one has

$$\begin{aligned}
l(AP_{\tau_2} i_{2,\infty}) &= l(\tilde{A}qP_{\tau_2} i_{2,\infty}) \leqslant \pi_2((qi_{2,\infty}P_{\tau_2})^*)T_2(\tilde{A}) = \pi_2(P_{\tau_2} i_{1,2}|_{\operatorname{ran} A^*})T_2(\tilde{A})\\
&\leqslant \pi_2(P_{\tau_2} i_{1,2}|_{\operatorname{ran} A^*})K(A)C_2(\operatorname{ran} A^*) \leqslant (8n/N)^{1/2}\beta
\end{aligned}$$

by (11) and, say, [Pi2], Prop. 3(v) (or see any of the general references mentioned after the statement of Theorem 3). Consequently,

$$\mathscr{P}\Big(\max\Big\{\Big\|\sum_{1 \leqslant i \leqslant N} t_i \mu_i A e_i\Big\|: t = (t_i) \in l_\infty^N, \|t\|_\infty \leqslant 1\Big\} \leqslant 1 + 4(\pi/2)^{1/2}\beta(8n/N)^{1/2}\Big) > 3/4.$$

This, together with (12) and (13), proves the Claim, hence Lemma 6, and concludes the proof of Theorem 3.

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Every Radon–Nikodym Corson compact space is Eberlein compact

by

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Abstract. We prove the result announced in the title. The Banach space version of this topological result reads as follows: A Banach space E whose dual unit ball is a weak* Corson compact and which is GSG (i.e. there is an Asplund space X and a continuous linear operator from X into E with dense range) is weakly compactly generated. We also analyze a relevant example of M. Talagrand and obtain solutions to three problems posed by I. Namioka.

1. Introduction. A compact topological space is called an *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space and is called *Radon–Nikodym compact* if it is homeomorphic to a weak* compact subset of the dual of an Asplund space. By the factorization result of [DFJP] every Eberlein compact space is homeomorphic to a weakly compact subset of a reflexive Banach space, therefore an Eberlein compact space is a Radon–Nikodym compact space. (For definitions and unexplained notation we refer to the end of the introduction.)

To see that these two notions are different, observe that for a compact scattered space K the Banach space $C(K)$ is Asplund as the dual $C(K)^*$ equals $l^1(K)$ and therefore has RNP. Hence K is a Radon–Nikodym compact space. For example the ordinal interval $[0, \omega_1]$ is a Radon–Nikodym compact but fails to be Eberlein (by Eberlein's theorem). Using the idea of dentability I. Namioka gave a topological characterization of Radon–Nikodym compacta [N] as those compact spaces which admit a lower semicontinuous fragmenting metric (see below). This gives rise to the notion of fragmented compacta ([JR], [N] where the lower semicontinuity assumption is dropped and we obtain the following chain of implications:

Eberlein compact ⇒ Radon–Nikodym compact ⇒ fragmented compact.

We shall prove in this paper that the second implication above also fails to be an equivalence.

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